New Partial Differential Equations Governing the Joint, Response-Excitation, Probability Distributions of Nonlinear Systems, under General Stochastic Excitation. I: Derivation

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ABSTRACT: In the present work the problem of determining the probabilistic structure of the dynamical response of nonlinear systems subjected to general, external, stochastic excitation is considered. The starting point of our approach is a Hopf-type equation, governing the evolution of the joint, response-excitation, characteristic functional (see, e.g., R.M. Lewis & R.M. Kraichnan, "A space-time functional formalism for turbulence", *Come. Pure Appl. Math.*, Vol. XV, pp. 397-411, 1962, or M. J. Beran, *Statistical Continuum Mechanics*, Interscience Publishers, 1968). This is a linear functional differential equation, i.e. a differential equation containing Volterra functional derivatives. The method is applicable to any, state-space, differential system exhibiting polynomial nonlinearities, but in this paper it is illustrated through a detailed analysis of a simple, first-order, scalar equation, with a cubic nonlinearity. Emphasis is given to the case of excitation processes with correlation structure and continuous (or smoother) sample functions, which implies a non-Markovian character of the response. Exploiting the Hopf equation, we derive new partial differential equations governing the joint, response-excitation, characteristic function, which can be considered as an extension of the well-known Fokker-Planck-Kolmogorov equation to the case of a general, correlated excitation and, thus, non-Markovian response character. The validity of this new equation is also checked by showing its equivalence with the (nonlinear) infinite system of moment equations. It is also shown that the same approach, i.e. starting from the Hopf equation, is also able to derive the Fokker-Planck-Kolmogorov equation for the case of independent-increment excitation. Numerical solution of this new equation for the joint characteristic function will be presented in a companion paper (Sapsis & Athanassoulis 2006, *this Conference*). Extension to general, multidimensional, dynamical systems exhibiting any polynomial nonlinearity will be presented in a forthcoming paper (Athanassoulis & Sapsis 2006).

KEYWORDS: Stochastic Dynamics, Stochastic Differential Equations, Functional Differential Equations, Correlated Stochastic Excitation, Fokker-Planck-Kolmogorov Equation, Generalized Fokker-Planck-Kolmogorov Equation, Non-Markovian Responses, Characteristic Functional.

1 INTRODUCTION

Many problems occurring in applied sciences and engineering are successfully modelled as stochastic differential equations. A very important class of such problems are those modeled as stochastically excited, nonlinear, dynamical systems. Well-known examples include the dynamic responses of ships and other man-made structures and systems under the influence of wind-generated waves in the sea (Schlesinger & Swean 1998, Wilson 2002, Belenky & Sevastianov 2003, Arnold *et al.* 2004), the dynamic responses of buildings and bridges under the influence of earthquakes (Lin & Cai 1995, Deodatis 1996, Kafali & Grigoriu 2003), as well as the dynamic responses of structures and vehicles under the influence of wind forces (Simiu & Scanlan 1986, Kree & Soize 1986, Soong & Grigoriu 1993, Hemon & Santi 2006). In all these cases the excitation loads are assumed to be known stochastic processes, either Gaussian or non-Gaussian, as in the case of wind loads. Their probabilistic and correlation structure can be (and, usually, have been) inferred by means of statistical data analysis and, in most cases, have been conveniently parameterized for easy reference and use in calculations. Most of the foundational facts and aspects concerning the stochastic modeling philosophy in engineering and applied science, and the corresponding mathematical background can be found nowadays in book form; see, e.g., Kree & Soize 1986, Sobczyk 1991, Soong & Grigoriu 1993, Roberts & Spanos 2003.

The ultimate objective in the analysis of such problems is to obtain a complete probabilistic description of the response process, permitting to answer any important questions about the response dynamics. Examples of such questions concerns the distributions of local extrema, of upcrossing rates at certain levels, of the first passage time associated with a critical level value, etc. To make this possible we need, in principle, to know the whole Kolmogorov hierarchy of the *n*-fold, joint, probability distributions $F_{x(t_1)x(t_2)\ldots x(t_n)}(a_1, a_2, ..., a_n)$ of the *n*-variate response random variables $(x(t_1), x(t_2), ..., x(t_n))$ at any collection of time instances $(t_1, t_2, ..., t_n)$ or, equivalently and more concisely, the Characteristic Functional (Ch.Fl) of the response process. Because of the obvious difficulties of this general concept of solution of the probabilistic dynamics problems, there is a constant tendency –at least in the applied and engineering literature– to avoid such an approach, resorting to simpler (partial) solution concepts.

An important, and extensively studied, context, permitting a relatively easy, complete characterization of the probabilistic responses of a dynamical system, occurs if we assume that the excitation is a process with independent increments (see, e.g., Pugachev & Shinitsyn 1987, Soize 1994, Grigoriu 2004). The key feature in this context is that the response vector, in the state-space formalism, is a Markovian process and, thus, its probability density function is governed by the Fokker-Planck-Kolmogorov (FPK) equation (in the Gaussian case) or by reasonable extensions of the FPK equation (in the non Gaussian case). Interestingly enough, there have been identified broad classes of problems in which analytic solutions of the classical FPK equation are available (see, e.g., Soize 1994), making this approach even more attractive.

An approximate method dealing with nonlinear systems under general stochastic excitation is the Statistical Linearization Method (see, e.g., Roberts & Spanos 2003), which is based on the approximation of the full system by a 'statistically equivalent' linear one. Some variations of the method, concerning local linearization in the phase space, have been recently presented (Pradlwarter 2001), giving promising results.

Another well-known method that can be applied to any type of stochastic excitation and to any type of nonlinearity is the method of moments, which reduces the initial stochastic dynamics problem to an infinite system of deterministic differential equations for the moment functions (Beran 1968, Pugachev & Shinitsyn 1987). This infinite system should be truncated and becomes closed (in the case of nonlinear problems) by means of appropriate closure schemes. Then, it is solved numerically, providing us with restricted information about the probabilistic characterization of the response process.

Another method, in principle well-known but in very little use for solving practical problems, is the one based on the Ch.Fl of the full probability measure associated with the dynamic response. The first step in this direction was made by Hopf (1952) who derived a Functional Differential Equation (FDE) for the Ch.Fl associated with the probabilistic solution of the Navier-Stokes equations. This approach, known as *the statistical approach to turbulence*, has been developed further by many authors (see, e.g., Lewis & Kraichnan 1962, Monin and Yaglom 1971, 1975), with only restricted success as regards the production of useful solutions to practically interesting problems. In parallel, a simpler version of the same approach has been developed and applied to finite-dimensional dynamical systems, governed by Stochastic Ordinary Differential Equations (SODEs). See, e.g., Beran (1968). Such Hopf-type FDEs are always linear, and govern the Ch.Fl of the sought-for probability measure or –depending on the specific formulation– the joint Ch.Fl of the joint, response-excitation, probability measure. In recent years successful attempts have been reported towards the analytic determination of the response Ch.Fl for some classes of linear problems, even avoiding the explicit use of Hopf's FDE (Caseres & Budini 1997, Budini & Caseres 1999, 2004). For some non-linear problems, the Ch.Fl can be expressed as a formal infinite-dimensional (functional) integral (Monin & Yaglom 1975), which is of little (or no) practical use.

In this paper, Hopf's FDE is taken as the starting point of the probabilistic analysis of the considered stochastic dynamics problem. Because of the generality of Hopf's approach, the method is applicable to any (at least) polynomially non-linear system and any kind of stochastic excitation. Nevertheless, for reasons of simplicity and clarity, our study will be carried out on a specific, first-order, dynamical system, with cubic nonlinearity. The excitation process will be assumed, in principle, completely known, with a given correlation structure and continuous (or smoother) sample functions. This implies a non-Markovian character of the response, making the approach based on the FPK equation inapplicable. Exploiting the Hopf FDE, new Partial Differential Equations (PDEs) governing the joint, response-excitation, characteristic functions (ch.f.), are derived. The corresponding equations for the joint pdfs are also obtained, by applying a Fourier transformation. These new PDEs, which are always linear, can be considered as a systematic and rigorous generalization of the FPK-type equations to the case of correlated excitation and non-Markovian responses. As an additional test of validity of these new PDEs, we show that they produce the correct infinite system of the moment equations. The same approach, i.e. starting from the Hopf FDE, is also applied to derive extended FPK equations, for the case of independent-increment excitation. In this connection, the results recently obtained by Grigoriu (Grigoriu 2004), concerning various cases of non-Gaussian, independent-increment forcing, are derived as special cases of our new extended FPK equations. We also show the consistency of our new PDE (for the joint, response-excitation, pdfs) with the usual (or extended) FPK equation, by deriving the latter as a limiting case of the former. A lack of rigor occurs here, when the sample functions of the response process are not continuous. It is conjectured that this derivation may be reformulated in a rigorous manner by invoking the dual of the space of cadlag (or regulated) functions, recently studied by Tvrdy (Tvrdy 2002). The question of the numerical solution of our new PDEs (for the joint, response-excitation, pdfs) will be considered in a companion paper, presented also in the same Conference, which will be referred to as [II].

Abbreviations

The following abbreviations –some of which have already been introduced above– will be consistently used in the sequel:

B-space	Banach space
ch.f(s)	characteristic function(s)
Ch.Fl(s)	characteristic functional(s)
F-derivative	Frechet derivative
FDE(s)	functional differential equation(s)
FPK	Fokker – Planck – Kolmogorov
ODE(s)	ordinary differential equation(s)
PDE(s)	partial differential equation(s)
pdf(s)	probability density function(s)
SODE(s)	stochastic ODE(s)

2 PRELIMINARIES AND NOTATION

In this work we consider ODEs (systems) of the form (in state space formulation):

$$\dot{x}(t) = G\big(x(t)\big) + y(t), \quad x(t_0) = x_0\,, \tag{2.1}$$

where x and y are scalar-valued or *N*-vector-valued, continuous (or smoother) functions, defined at least on an interval $I \equiv [t_0, T]$ (that is, $x, y : [t_0, T] \equiv I \to \mathbb{R}^N$), and $G : \mathbb{R}^N \to \mathbb{R}^N$, $N = 1$ or $N > 1$, is also a continuous (or smoother) function. Both the excitation $y(\bullet)$ and the initial conditions x_0 will be assumed known stochastic elements (function and variable, respectively). In contrast with the standard approach, followed in the case of an Ito SODE, the forcing $y(\bullet)$ is allowed to be smooth (e.g., *k*-times continuously differentiable), exhibiting any type of correlation structure in time. Thus, the sample functions $x(\bullet)$ and $y(\bullet)$ are considered as elements of smooth-function *B*-spaces, denoted by $\mathscr{X}$ and $\mathscr{Y}$, respectively. Our main results will refer to the case $N = 1$, $\mathscr{Y} = C^k(I)$, $I \subseteq \mathbb{R}$, $k = 0$ or $k > 0$, and $\mathscr{X}$ a similar space with smoother elements. The whole methodology can be extended to the vector case $N > 1$ with the usual trouble (see Athanassoulis & Sapsis 2006 for a detail analysis of a second-order system).

The topological dual spaces of $\mathscr{X}$ and $\mathscr{Y}$ are also *B*-spaces and will be denoted by $\mathscr{X}' = \mathscr{U}$ and $\mathscr{Y}' = \mathscr{V}$. The symbols $\langle u, x \rangle$ and $\langle v, y \rangle$ denote the standard duality pairings between $\mathscr{X}$ and $\mathscr{U}$, and $\mathscr{Y}$ and $\mathscr{V}$, respectively.

The underlying probability space is denoted by $\big(\Omega, \mathscr{B}(\Omega), \mathscr{P}_\Omega\big)$, where Ω is an abstract version of the sample (trial) space, $\mathscr{B}(\Omega)$ is the family of Borel sets of Ω, and $\mathscr{P}_\Omega$ is the corresponding probability measure over Ω. The stochastic processes X and Y are measurable maps $X, Y : \Omega \to \mathscr{X}, \mathscr{Y}$, which define the induced probability spaces $\big(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mathscr{P}_{\mathscr{X}}\big)$ and $\big(\mathscr{Y}, \mathscr{B}(\mathscr{Y}), \mathscr{P}_{\mathscr{Y}}\big)$, respectively. We shall also need and consider the joint process $X \times Y : \Omega \to \mathscr{X} \times \mathscr{Y}$ with induced probability space $\big(\mathscr{X}\times\mathscr{Y}, \mathscr{B}(\mathscr{X}\times\mathscr{Y}), \mathscr{P}_{\mathscr{X}\times\mathscr{Y}}\big)$. In the sequel we shall use the notation x or $x(\bullet)$ or $x(\bullet;\omega)$, and similarly for y, for the random element, and $x(t;\omega)$, $t \in [t_0, T] \equiv I \subseteq \mathbb{R}$, $\omega \in \Omega$, and similarly for y, for the sample functions, in accordance with the needs of the discussion.

The finite-dimensional distributions, densities and characteristic functions of the random element $x(\bullet;\omega)$ will be denoted by $F_{x(t_1)\ldots x(t_M)}(\alpha_1,\ldots,\alpha_M)$, $f_{x(t_1)\ldots x(t_M)}(\alpha_1,\ldots,\alpha_M)$ and $\phi_{x(t_1)\ldots x(t_M)}(\upsilon_1,\ldots,\upsilon_M)$, respectively. This implies a convenient notation for the joint random element $\big(x(\bullet;\omega), y(\bullet;\omega)\big)$; for example $f_{x(t_1)x(t_2)y(t_1)y(t_2)y(t_3)}(\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3)$ for the 2–*x*, $3-y$ density, and $\phi_{x(t_1)x(t_2)y(t_1)y(t_2)y(t_3)}(\upsilon_1,\upsilon_2,\nu_1,\nu_2,\nu_3)$ for the corresponding characteristic function. The usual (finite-dimensional) mean value operator (ensample average) will be denoted by $\mathbf{E}^{\omega}[\bullet]$. For example, the mean value function of the random element $x(\bullet;\omega)$ will be written as $m_{x(t)} = \mathbf{E}^{\omega}\big[x(t;\omega)\big]$. Slight variations (simplifications) of this notation will be introduced later, in accordance with the needs of the presentation.

Infinite-dimensional (global) moments, are defined by integrating over the whole sample space $\mathscr{X}$ with respect to the probability measure $\mathscr{P}_{\mathscr{X}}$ (See, e.g., Kree & Soize 1986, Vakhania *et al*. 1987, Egorov *et al*. 1993). For example, the mean (first moment) $m_{\mathscr{X}}$ is defined to be this element of $\mathscr{X}$, for which the following scalar equation holds true:

$$\langle u, m_{\mathscr{X}} \rangle = \int_{\mathscr{X}} \langle u, x \rangle \mathscr{P}(dx), \quad \forall u \in \mathscr{U}, \tag{2.2a}$$

where $\mathscr{U} \equiv \mathscr{X}'$. Furthermore, the correlation operator (second moment) is defined to be this linear operator $R_{\mathscr{X}\mathscr{X}} : \mathscr{U} \to \mathscr{X}$, for which the following scalar equation is valid $\forall u, w \in \mathscr{U}$:

$$\langle w, R_{\mathscr{X}\mathscr{X}} u \rangle = \int_{\mathscr{X}} \langle w, x \rangle \langle u, x \rangle \mathscr{P}(dx). \tag{2.2b}$$

The integrals appearing in the right-hand side of equs. (2.2) are infinite-dimensional (functional) integrals over *B*-spaces. (See references stated above or Dalecky & Fomin 1991). In general, the functional integral of any bounded, measurable, continuous functional $\mathscr{G} : \mathscr{X} \to \mathbb{C}$, with respect to a probability measure $\mathscr{P}$, is well defined, and will be denoted by $\int_{\mathscr{X}} \mathscr{G}(x) \mathscr{P}(dx)$.

Measures and integrals over infinite-dimensional vector spaces are related with the corresponding finite-dimensional ones through the concepts of cylinder sets, cylinder measures and cylinder functionals. Let $\mathscr{X}$ be a separable *B*-space, $\mathscr{U}$ be the dual of $\mathscr{X}$, and $u_1, \ldots, u_Q$, be Q linearly independent elements of $\mathscr{U}$. Then, to any element $x \in \mathscr{X}$ we associate the $Q-$dimensional projection $\Pi_{u_1,\ldots,u_Q} : \mathscr{X} \to \mathbb{R}^Q$, defined by

$$\Pi_{u_1,\ldots,u_Q}[x] = \left(\langle u_1, x \rangle, \ldots, \langle u_Q, x \rangle\right). \tag{2.3}$$

The inverse of $\Pi_{u_1,\ldots,u_Q}[\bullet]$, applied to the Borel sets $\mathscr{B}\left(\mathbb{R}^Q\right)$, defines the cylinder sets of $\mathscr{X}$. The existence of a probability measure $\mathscr{P}_{\mathscr{X}}$ on $\mathscr{X}$ implies the existence of *Q*-dimensional (marginal) measures $P_{u_1,\ldots,u_Q}$ on $\mathbb{R}^Q$, associated with the random vectors $\left(\langle u_1, x(\bullet;\omega) \rangle, \ldots, \langle u_Q, x(\bullet;\omega) \rangle\right)$ by means of the relation

$$P_{u_1,\ldots,u_Q}\left(E_Q\right) = \mathscr{P}_{\mathscr{X}}\left(\Pi^{-1}_{u_1,\ldots,u_Q}\left[E_Q\right]\right). \tag{2.4}$$

for any $E_Q \in \mathscr{B}\left(\mathbb{R}^Q\right)$.

Consider now an arbitrary cylinder functional $\mathscr{G} : \mathscr{X} \to \mathbb{C}$, that is a functional of the form

$$\mathscr{G}(x) = g\left(\langle u_1, x \rangle, \ldots, \langle u_Q, x \rangle\right), \qquad x \in \mathscr{X}, \tag{2.5}$$

where $g : \mathbb{R}^Q \to \mathbb{C}$ is an arbitrary, measurable, integrable function. In this case, the infinite-dimensional integral of $\mathscr{G}(x)$ with respect to the probability measure $\mathscr{P}$ over the space $\mathscr{X}$, can be expressed as a *Q*-dimensional integral by means of the formula:

$$\int_{\mathscr{X}} \mathscr{G}(x) \mathscr{P}(dx) = \int_{\mathbb{R}^Q} g(\boldsymbol{a})\, P_{u_1,\ldots,u_Q}(d\boldsymbol{a}). \tag{2.6}$$

Equs. (2.5) and (2.6) provide us with a powerful method for evaluating integrals over infinite-dimensional (function) spaces. They will be referred to as the (Q-dimensional) *Projection Theorem*.

3 A BRIEF REVIEW ON THE CHARACTERISTIC FUNCTIONAL AND ITS BASIC PROPERTIES

3.1 *Definition of the Characteristic Functional*

In this section we recall the definition and some basic properties of the Ch.Fl for probability measures defined on separable *B*-spaces.

Definition 3.1: Let $\mathscr{X}$ be a separable *B*-space and $\mathscr{P} = \mathscr{P}_{\mathscr{X}}$ be a probability measure defined on it. The Ch.Fl $\mathscr{F}$ of $\mathscr{P}$ is a cylinder functional defined on the dual space $\mathscr{X}' = \mathscr{U}$ by the formula

$$\mathscr{F}(u) = \int_{\mathscr{X}} e^{i\langle u, x \rangle} \mathscr{P}(dx), \qquad u \in \mathscr{U}. \tag{3.1}$$

This integral always exists provided that the corresponding probability measure is well defined.

3.2 *Infinite-Dimensional (Global) Moments*

Let the Ch.Fl be differentiable in the sense of Frechet. In order to calculate the *F*-derivative $D\mathscr{F}(u)$, we make use of the Gateaux derivative (which always exists for a *F*-differentiable map). Thus, we have

$$\begin{aligned} D\mathscr{F}(u)[z] &= \left.\frac{d\mathscr{F}(u+\varepsilon z)}{d\varepsilon}\right|_{\varepsilon=0} = \\ &= i \cdot \int_{\mathscr{X}} \langle z, x \rangle e^{i\langle u, x \rangle} \mathscr{P}(dx), \quad u, z \in \mathscr{U}. \end{aligned} \tag{3.2}$$

Setting $u = 0$, we obtain

$$D\mathscr{F}(0)[z] = i \cdot \int_{\mathscr{X}} \langle z, x\rangle \mathscr{P}(dx), \; z \in \mathscr{U}. \tag{3.3}$$

Since $D\mathscr{F}(0)[z]$ is a continuous, linear functional with respect to z, there should exists an element $m \in \mathscr{X}$, such that

$$\langle z, m\rangle = -i \cdot D\mathscr{F}(0)[z] = \int_{\mathscr{X}} \langle z, x\rangle \mathscr{P}(dx). \tag{3.3'}$$

Comparing the above equation with the equ. (2.2a), it easily seen that the element $m \in \mathscr{X}$ of equ. (3.3') coincides with the mean value $m_{\mathscr{X}}$ of the probability measure $\mathscr{P}$. The correlation operator can be associated in a similar way with the second *F*-derivative of the Ch.Fl.

3.3 *Finite-Dimensional (Point) Moments*

In the case where the space $\mathscr{X}$ is a function space, apart from infinite-dimensional (global) moments, we are also interesting in finite-dimensional moments associated with finite-dimensional projections $\left(x(t_1;\omega), x(t_2;\omega), \ldots, x(t_n;\omega)\right)$, for any set of time instances $(t_1, t_2, \ldots, t_n)$. This kind of moments can be obtained also by differentiating the Ch.Fl, this time using Volterra functional derivatives. (See, e.g., Volterra 1927/1959/2002 or Beran 1968). Volterra derivatives, e.g. the first-order one $\delta\mathscr{F}(u)/\delta u(t)$, can be calculated either by applying the original definition to the functional, or by applying the Frechet derivative $D\mathscr{F}(u)[z]$ at $z(\bullet) = \delta(\bullet - t)$. Following the second approach, and using equs. (3.3) and (2.6), we obtain

$$\frac{\delta\mathscr{F}(0)}{\delta u(t)} \overset{def}{=} D\mathscr{F}(0)[\delta(\bullet - t)] = i\int_{\mathscr{X}} x(t)\mathscr{P}(dx) = $$
$$= i\int_{\mathbb{R}} a\, dF_{x(t)}(a) = i\; \mathbf{E}^{\omega}\left[x(t;\omega)\right],$$

and thus

$$\mathbf{E}^{\omega}\left[x(t;\omega)\right] = \frac{1}{i}\frac{\delta\mathscr{F}(0)}{\delta x(t)}. \tag{3.4a}$$

Similarly we obtain

$$\mathbf{E}^{\omega}\left[x(t_1;\omega)\,x(t_2;\omega)\right] = \frac{1}{i^2}\cdot D^2\mathscr{F}(0)[\delta(\bullet - t_1), \delta(\bullet - t_2)] =$$
$$= \frac{1}{i^2}\frac{\delta^2\mathscr{F}(0)}{\delta u(t_1)\,\delta u(t_2)}, \tag{3.4b}$$

as well as analogous expressions for higher-order moments. Working similarly, and using appropriate generalized functions, we can derive equations for higher-order moments involving both the values of the random element at some time instances, as well as the values of its derivatives either at the same or at different time instances. As an example we give the formula:

$$\mathbf{E}^{\omega}\left[x'(t_1;\omega)\,x(t_2;\omega)\right] = \frac{1}{(-i)\,i}\cdot D^2\mathscr{F}(0)[\delta'(\bullet - t_1), \delta(\bullet - t_2)].$$

4 HOPF-TYPE EQUATION FOR THE CHARACTERISTIC FUNCTIONAL

In order to illustrate the derivation of Hopf-type FDEs for nonlinear dynamical systems, and pave the way to the next section, where these equations will be exploited to produce new PDEs for finite-dimensional ch.fs, we shall restrict ourselves here to a specific case of a simple (scalar, first-order) dynamical system having a cubic nonlinearity, which is described by the following SODE:

$$\dot{x}(t;\omega) + k\,x(t;\omega) + a\,x^3(t;\omega) = y(t;\omega), \tag{4.1a}$$
$$x(t_0;\omega) = x_0(\omega), \tag{4.1b}$$

where k, a are deterministic constants, $x_0(\omega)$ is a random variable with known ch.f $\phi_0(\upsilon)$, $\upsilon \in \mathbb{R}$, and the forcing $y(\bullet,\omega)$ is a real-valued random function, with sample space $\mathscr{Y}$, probability measure $\mathscr{P}_y$, and Ch.Fl $\mathscr{F}_y(v)$, $v \in \mathscr{Y}' = \mathscr{V}$. The sample space $\mathscr{Y}$ can be taken to be a quite general, separable, *B*-space. In the present work, it will be taken as a space $\mathscr{Y} = C^k(I)$, $I \subseteq \mathbb{R}$, for some $k \in \mathbb{N} \cup \{0\}$.

Standard existence and uniqness theory (see, e.g., Bunke 1972, or Sobczyk 1991) assure that there is a stochastic process $x(\bullet;\omega)$, with sample space $\mathscr{X} = C^{k+1}(I)$ and probability measure $\mathscr{P}_x$, and a joint probability space $\left(\mathscr{X}\times\mathscr{Y}, \mathscr{B}(\mathscr{X}\times\mathscr{Y}), \mathscr{P}_{xy}\right)$, such that the joint process $\left(x(\bullet;\omega), y(\bullet;\omega)\right)$ verifies the SODE (4.1).

The joint, response-excitation, probability measure $\mathscr{P}_{xy}$ is equivalently described by the joint Ch.Fl

$$\mathscr{F}_{xy}(u,v) = \int_{\mathscr{Y}}\int_{\mathscr{X}} e^{i(\langle u,x\rangle + \langle v,y\rangle)}\mathscr{P}_{xy}(dx,dy). \tag{4.2}$$

We shall now use the SODE (4.1) in order to obtain an FDE for $\mathscr{F}_{xy}(u,v)$. Let us consider the Volterra *u*-partial derivative of $\mathscr{F}_{xy}$ at time t:

$$\frac{\delta\mathscr{F}_{xy}(u,v)}{\delta u(t)} = \int_{\mathscr{Y}}\int_{\mathscr{X}} ix(t)\,e^{i(\langle u,x\rangle + \langle v,y\rangle)}\mathscr{P}_{xy}(dx,dy). \tag{4.3}$$

Since the sample space $\mathscr{X}$ consists of smooth functions, we can differentiate (4.3) with respect to t, obtaining:

$$\frac{d}{dt}\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta u(t)} = \int_{\mathscr{Y}}\int_{\mathscr{X}} i\, x'(t)\, e^{i(\langle u,x\rangle+\langle v,y\rangle)} \mathscr{P}_{xy}(dx,dy)\,. \quad (4.4)$$

Further, we compute the three-fold *u*-partial Volterra derivative of $\mathscr{F}_{xy}(u,v)$ at time instants $t_1, t_2, t_3 \in I$:

$$\begin{aligned}\frac{\delta^3 \mathscr{F}_{xy}(u,v)}{\delta u(t_1)\delta u(t_2)\delta u(t_3)} &= \\ &= \int_{\mathscr{Y}}\int_{\mathscr{X}} ix(t_1)ix(t_2)ix(t_3)\, e^{i(\langle u,x\rangle+\langle v,y\rangle)}\mathscr{P}_{xy}(dx,dy).\end{aligned} \quad (4.5)$$

Setting $t_1 = t_2 = t_3 = t$ in the latter, and combining with equs. (4.3) and (4.5), we get

$$\begin{aligned}&\frac{d}{dt}\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta u(t)} + k\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta u(t)} - a\frac{\delta^3 \mathscr{F}_{xy}(u,v)}{\delta u(t)^3} = \\ &= i\int_{\mathscr{Y}}\int_{\mathscr{X}} \left[\, x'(t) + kx(t) + ax^3(t)\,\right] e^{i(\langle u,x\rangle+\langle v,y\rangle)}\mathscr{P}_{xy}(dx,dy) \overset{(4.1a)}{=} \\ &= i\int_{\mathscr{Y}}\int_{\mathscr{X}} y(t)\, e^{i(\langle u,x\rangle+\langle v,y\rangle)}\mathscr{P}_{xy}(dx,dy)\,. \end{aligned} \quad (4.6)$$

Clearly, the last double functional integral can be expressed as a *v*-partial Volterra derivative:

$$\int_{\mathscr{Y}}\int_{\mathscr{X}} iy(t) e^{i(\langle u,x\rangle+\langle v,y\rangle)}\mathscr{P}_{xy}(dx,dy) = \frac{\delta \mathscr{F}_{xy}(u,v)}{\delta v(t)}. \quad (4.7)$$

Combining (4.6) and (4.7) we derive the sought-for, Hopf-type, FDE that governs the joint Ch.Fl $\mathscr{F}_{xy}(u,v)$:

$$\begin{aligned}&\frac{d}{dt}\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta u(t)} + k\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta u(t)} - a\frac{\delta^3 \mathscr{F}_{xy}(u,v)}{\delta u(t)^3} =. \\ &= \frac{\delta \mathscr{F}_{xy}(u,v)}{\delta v(t)}\,.\end{aligned} \quad (4.8a)$$

Equ. (4.8a) is a linear FDE involving Volterra functional derivatives, as well as ordinary time derivatives. The cubic nonlinearity of the initial SODE corresponds to the 3-fold Volterra derivative $\delta^3\mathscr{F}_{xy}/\delta u(t)^3$. From the above derivation it is clear that any n^{th}-order polynomial nonlinearity of the initial differential equation is transformed to an *n*-fold Volterra derivative in the corresponding Hopf-type FDE. Another important feature of equation (4.8a) is that it holds true for any continuous functionals $u \in \mathscr{U},\ v \in \mathscr{V}$.

Equ. (4.8a) has to be supplemented by an appropriate initial condition, expressing that the probability measure associated with the initial value $x(t_0,\omega)$ is given. This condition can be implemented by means of the joint Ch.Fl $\mathscr{F}_{xy}(u,v)$ as follows. Setting $v=0$ (to restrict ourselves to the response process only) and $u = \upsilon\cdot\delta(\bullet - t_0)$, $\upsilon \in \mathbb{R}$, (to concentrate only at the initial time instant), will result in

$$\begin{aligned}\mathscr{F}_{xy}(\upsilon\delta(\bullet - t_0),0) &= \int_{\mathscr{Y}}\int_{\mathscr{X}} e^{i(\langle \upsilon\delta(\bullet-t_0),x\rangle+\langle 0,y\rangle)}\mathscr{P}_{xy}(dx,dy) = \\ &= \int_{\mathscr{X}} e^{i\langle \upsilon\delta(\bullet-t_0),x\rangle}\mathscr{P}_{x}(dx) \;=\; \phi_0(\upsilon),\end{aligned}$$

where $\phi_0(\upsilon)$ is the ch.f of $x(t_0,\omega) = x_0(\omega)$. Hence, the initial condition can be expressed as

$$\mathscr{F}_{xy}(\upsilon\delta(\bullet - t_0),0) = \phi_0(\upsilon)\,, \qquad \upsilon \in \mathbb{R}\,. \quad (4.8b)$$

5 DERIVATION OF NEW PDEs FOR JOINT RESPONSE-EXCITATION CHARACTERISTIC FUNCTIONS

In this section we shall exploit the Hopf-type FDE (4.8), obtained above, to derive new PDEs for the joint, response-excitation, ch.f when the excitation is a known stochastic process either with a.e. continuous sample functions or smoother. In contrast with the case of an independent-increment excitation process, where the randomness of the excitation "regenerates" every time instant and allows us to write explicitly an equation involving only the response density (the well-known FPK equation), in the case of a stochastic excitation with smooth sample functions, the randomness evolves, in general, in a smoother way, as a result of the finite correlation time, making necessary to consider response and excitation jointly.

The causality principle dictates that the current value $x(t;\omega)$ of the response, depends only on the history of the excitation $y(t_0 \le s < t;\omega)$. However, this does not prevent the stochastic dependence between $x(t;\omega)$ and $y(t+\varepsilon;\omega)$, $\varepsilon > 0$, which is a natural result of the smoothness and the finite correlation time of the excitation, $C_{yy}(t+\varepsilon,t) \neq 0$.

We shall proceed to derive a PDE for the joint ch.f $\phi_{x(t)y(t)}(\upsilon,\nu)$ corresponding to the pair of random variables $(x(t;\omega),y(t;\omega))$, $t = \mathit{fixed}$. To this end we apply equ. (4.8a), above, to the pair

$$u \;=\; \upsilon\cdot\delta(\bullet - t), \quad v \;=\; \nu\cdot\delta(\bullet - s)\,, \quad (5.1)$$

($\upsilon, \nu \in \mathbb{R}$) and take the limit $s \to t$, after some manipulations. For the first term of equ. (4.8a) (see also equ. (4.4)), we obtain

$$\frac{d}{dt}\left(\frac{\delta \mathscr{F}_{xy}\left(\upsilon \cdot \delta(\bullet - t),\, \nu \cdot \delta(\bullet - s)\right)}{\delta u(t)}\right) =$$

$$= \iint_{\mathscr{X}\times\mathscr{Y}} i\frac{dx(t)}{dt}\exp\{i\upsilon x(t) + i\nu y(s)\}\mathscr{P}_{xy}(dx,dy) =$$

$$= \frac{1}{\upsilon}\frac{\partial}{\partial t}\left[\iint_{\mathscr{X}\times\mathscr{Y}} \exp\{i\upsilon x(t) + i\nu y(s)\}\mathscr{P}_{xy}(dx,dy)\right] =$$

$$= \frac{1}{\upsilon}\frac{\partial}{\partial t}\left[\iint_{\mathbb{R}\times\mathbb{R}} \exp\{i\upsilon x + i\nu y\}\, f_{x(t)y(s)}(x,y)\,dxdy\right] =$$

$$= \frac{1}{\upsilon}\frac{\partial \phi_{x(t)y(s)}(\upsilon,\nu)}{\partial t}.$$

Taking now the limit $s \to t$, we get

$$\lim_{s\to t}\frac{d}{dt}\left(\frac{\delta \mathscr{F}_{xy}\left(\upsilon \cdot \delta(\bullet - t),\, \nu \cdot \delta(\bullet - s)\right)}{\delta u(t)}\right) = \frac{1}{\upsilon}\left.\frac{\partial \phi_{x(t)y(s)}(\upsilon,\nu)}{\partial t}\right|_{s=t}. \qquad (5.2)$$

Working similarly, we readily obtain the following results concerning the remaining terms appearing in equ. (4.8a):

$$\frac{\delta \mathscr{F}\left(\upsilon \cdot \delta(\bullet - t), \nu \cdot \delta(\bullet - t)\right)}{\delta u(t)} = \frac{\partial \phi_{x(t)y(t)}(\upsilon,\nu)}{\partial \upsilon}, \quad (5.3)$$

$$\frac{\delta^3 \mathscr{F}\left(\upsilon \cdot \delta(\bullet - t), \nu \cdot \delta(\bullet - t)\right)}{\delta u(t)^3} = \frac{\partial^3 \phi_{x(t)y(t)}(\upsilon,\nu)}{\partial \upsilon^3}, \quad (5.4)$$

$$\frac{\delta \mathscr{F}\left(\upsilon \cdot \delta(\bullet - t), \nu \cdot \delta(\bullet - t)\right)}{\delta v(t)} = \frac{\partial \phi_{x(t)y(t)}(\upsilon,\nu)}{\partial \nu}. \quad (5.5)$$

Combining equs. (5.2)-(5.5) with the FDE (4.8a), we obtain the following PDE for the joint ch.f $\phi_{x(t)y(t)}(\upsilon,\nu)$, of the pair of random variables $\left(x(t;\omega), y(t;\omega)\right)$, for every $t > t_0$:

$$\frac{1}{\upsilon}\left.\frac{\partial \phi_{x(t)y(s)}(\upsilon,\nu)}{\partial t}\right|_{s=t} + k\frac{\partial \phi_{x(t)y(t)}(\upsilon,\nu)}{\partial \upsilon} - a\frac{\partial^3 \phi_{x(t)y(t)}(\upsilon,\nu)}{\partial \upsilon^3} = \frac{\partial \phi_{x(t)y(t)}(\upsilon,\nu)}{\partial \nu}. \quad (5.6a)$$

Now, since the stochastic process $y(\bullet,\omega)$ is given, its ch.f $\phi_{y(t)}(\nu)$ is known. Hence, the *y*-marginal of the joint ch.f $\phi_{x(t)y(t)}(\upsilon,\nu)$ has to coincide with $\phi_{y(t)}(\nu)$, resulting in the following ***marginal compatibility condition***:

$$\phi_{x(t)y(t)}(0,\nu) = \phi_{y(t)}(\nu), \qquad \nu \in \mathbb{R}, \quad t \ge t_0. \qquad (5.6b)$$

In addition, the initial condition (4.8b) implies the following ***initial condition*** to $\phi_{x(t)y(t)}(\upsilon,\nu)$:

$$\phi_{x(t_0)y(t_0)}(\upsilon,0) = \phi_{x(t_0)}(\upsilon) = \phi_0(\upsilon), \quad \upsilon \in \mathbb{R}. \quad (5.6c)$$

The problem (5.6) can also be reformulated in terms of the joint pdf $f_{x(t)y(t)}(\alpha,\beta)$. This reformulation, which can be readily obtained by means of Fourier transformation, will be explicitly given in [II].

To the best of our knowledge, equs. (5.6a,b,c), governing the evolution of the joint, response-excitation, ch.f $\phi_{x(t)y(t)}(\upsilon,\nu)$, appear here for the first time. They can be considered as a new kind of mathematical model, providing us with the probabilistic characterization of the response $x(t,\omega)$, $\omega \in \Omega$, for each $t \in I$, obtained by taking the marginal of the joint ch.f: $\phi_{x(t)}(\upsilon) = \phi_{x(t)y(t)}(\upsilon,0)$, $\upsilon \in \mathbb{R}$. This mathematical model is valid for any kind of stochastic excitation with a.e. continuous (or smoother) sample functions, having any (known) probabilistic structure.

Although the mathematical analysis (solvability theory) of problem (5.6a,b,c) is an open problem, existing numerical evidence of the authors (Sapsis & Athanassoulis 2006) suggests that it might be well-posed under reasonable assumptions.

In concluding this section we should emphasize that the above approach can be generalized in order to obtain similar, linear, PDEs for the joint, *N-x* and *M-y*, ch.f

$$\phi_{x(t_1)\ldots x(t_N)y(s_1)\ldots y(s_M)}(\upsilon_1,\ldots\upsilon_N,\nu_1,\ldots,\nu_M),$$

along with appropriate (marginal compatibility and initial) conditions. This point will be further discussed in another work (Athanassoulis & Sapsis 2006). It seems that in this way it is possible to construct a closed (finitely-solvable) hierarchy of linear problems providing us with the full hierarchy of the finite-dimensional probabilities of the stochastic response $x(\bullet;\omega)$.

6 DERIVATION OF THE FPK EQUATION FOR THE CASE OF INDEPENDENT INCREMENT EXCITATION

Equs. (5.6) –involving the joint, response-excitation, ch.f. – hold true for any kind of stochastic excitation process, provided that the latter has at least a.e. continuous sample functions. We shall now turn to the most commonly studied case, those of an Ito SODE, where $y(t;\omega)$ represents the generalized derivative of an independent-increment process. In this case the response $x(t;\omega)$ is continuous but not differentiable. Thus, the treatment based on the Hopf equation, developed in Section 5, is not valid, since the duality pairings (5.1) are not applicable. The question arises if it is possible to treat this case also by a similar method, starting from the Hopf equation and obtaining the usual FPK equation –which involves only the response ch.f (or pdf). In the present section we shall show how this is possible, by resorting back to the FDE for a finite-difference version of the SODE (4.1). The crucial property, to be exploited in this case, is the independence of the current value $x(t;\omega)$ of the response from the future increment $\Delta_\tau z(t;\omega)= z(t+\tau;\omega)-z(t;\omega)$, $\tau>0$, of the excitation. Everything presented in this Section can be generalized to multidimensional nonlinear dynamical systems.

Let us rewrite the SODE (4.1a,b) in a finite-difference form:

$$\frac{\Delta_\tau x(t;\omega)}{\tau}+k\,x(t;\omega)+a\,x^3(t;\omega)=\frac{\Delta_\tau z(t;\omega)}{\tau}, \quad (6.1a)$$

$$x(t_0;\omega)=x_0(\omega), \quad (6.1b)$$

where $z(\bullet;\omega)$ is a known, real-valued process with independent increments, and $x_0(\omega)$ is a known random variable. The time increment τ is assumed to be positive, $\tau>0$, and this is essential in what follows.

The sample functions of the stochastic process $z(\bullet;\omega)$ may be either continuous functions (as in the case of normally-distributed, independent-increment processes) or non-decreasing, piecewise-constant functions (as in the case of Poisson distributed independent-increment processes). In the first case (continuous sample functions), it is clear that the previously developed approach can be applied to equ. (6.1). In the second case (*cadlag* sample functions) the applicability of the same arguments is not directly justifiable. Nevertheless, we shall take the liberty not to be completely rigorous, and apply the same approach to the general case as well. It seems to us quite remarkable and fascinating that the obtained PDE for the ch.f of the response $x(\bullet;\omega)$ coincides with the known one in all examined cases. Thus, the results of the present section can be considered as a rigorous rederivation of the classical FPK equation from the Hopf FDE, in the case of Gaussian forcing, and as a heuristic method to derive analogous equations in the case of a Poissonian or an $\alpha-$stable or a general Levy process forcing.

Working similarly as in Section 4, we obtain the following Hopf-type FDE that governs the evolution of the Ch.Fl $\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)$, parametrically dependent on $\tau>0$:

$$\tau^{-1}\Delta_\tau\left(\frac{\delta\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)}{\delta u(t)}\right)+k\,\frac{\delta\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)}{\delta u(t)}-$$
$$a\,\frac{\delta^3\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)}{\delta u(t)^3}=\frac{\delta\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)}{\delta v(t)}. \quad (6.2a)$$

$$\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}\left(\upsilon\delta(\bullet-t_0),0\right)=\phi_0(\upsilon), \quad \upsilon\in\mathbb{R}. \quad (6.2b)$$

Note that the $\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)$ is the finite-difference version of $\mathscr{F}_{xy}(u,v)=\lim_{\tau\to 0}\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)$.

Using again the arguments $u(\bullet)$, $v(\bullet)$, given by (5.1), and applying the same treatment as in Section 5, we obtain the following PDE that governs the joint ch.f $\phi_{x(\tau^{-1}\Delta_\tau z)}$:

$$\frac{1}{\upsilon}\tau^{-1}\Delta_\tau\left(\phi_{x(t)(\tau^{-1}\Delta_\tau z(s))}(\upsilon,\nu)\right)\Big|_{s=t}+k\,\frac{\partial\phi_{x(t)(\tau^{-1}\Delta_\tau z(t))}(\upsilon,\nu)}{\partial\upsilon}-$$
$$a\,\frac{\partial^3\phi_{x(t)(\tau^{-1}\Delta_\tau z(t))}(\upsilon,\nu)}{\partial\upsilon^3}=\frac{\partial\phi_{x(t)(\tau^{-1}\Delta_\tau z(t))}(\upsilon,\nu)}{\partial\nu}. \quad (6.3)$$

Setting $\nu=0$ in the equ. (6.3) and taking the limit as $\tau\to 0^+$, we obtain:

$$\frac{1}{\upsilon}\frac{\partial\phi_{x(t)}(\upsilon)}{\partial t}+k\,\frac{\partial\phi_{x(t)}(\upsilon)}{\partial\upsilon}-a\,\frac{\partial^3\phi_{x(t)}(\upsilon)}{\partial\upsilon^3}=$$
$$=\lim_{\tau\to 0^+}\frac{\partial\phi_{x(t)(\tau^{-1}\Delta_\tau z(t))}(\upsilon,\nu)}{\partial\nu}\Bigg|_{\nu=0}. \quad (6.4)$$

In the left-hand side of equ. (6.4) we can already recognize the sought-for result. In the right-hand side, because of the $\nu-$derivative, the situation is more complicated and should be studied further. In analogy with equs (4.8a) and (4.7), the “source” term $\partial\phi_{x(t)(\tau^{-1}\Delta_\tau z(t))}(\upsilon,\nu)/\partial\nu$, appearing in the right-hand side of equs. (6.3) and (6.4), comes from the following functional integral

$$\frac{\partial\phi_{x(t)(\tau^{-1}\Delta_\tau z(t))}(\upsilon,\nu)}{\partial\nu}\equiv I_\tau(\upsilon,\nu)=$$
$$=\int_{\mathscr{Z}}\int_{\mathscr{X}} i\left(\tau^{-1}\Delta_\tau z(t)\right)e^{i\left(\langle u,\,x\rangle+\langle v,\,\tau^{-1}\Delta_\tau z\rangle\right)}\times$$
$$\times\mathscr{P}_{x(\tau^{-1}\Delta_\tau z(t))}\left(dx,d\left(\tau^{-1}\Delta_\tau z\right)\right),$$

where $\mathscr{Z}$ is an appropriate space of continuous functions. Now, using the identity

$$\mathscr{P}_{x,(\tau^{-1}\Delta_\tau z)}\left(dx, d\left(\tau^{-1}\Delta_\tau z\right)\right) = \mathscr{P}_{x,\Delta_\tau z}\left(dx, d\left(\Delta_\tau z\right)\right),$$

we obtain

$$I_\tau = \int_{\mathscr{Z}}\int_{\mathscr{X}} i\left(\tau^{-1}\Delta_\tau z(t)\right) e^{i\left(\langle u,x\rangle + \langle v, \tau^{-1}\Delta_\tau z\rangle\right)} \mathscr{P}_{x,\Delta_\tau z}\left(dx, d\left(\Delta_\tau z\right)\right).$$

Let us now evaluate the above functional integral (for $\tau = fixed > 0$), under the specific choice of arguments $u(\bullet) = \upsilon\cdot\delta(\bullet - t)$ and $v(\bullet) = \nu\cdot\delta(\bullet - t)$:

$$I_\tau\left(\upsilon\delta(\bullet - t),\ \nu\cdot\delta(\bullet - t)\right) =$$

$$= \int_{\mathscr{Z}}\int_{\mathscr{X}} i\left(\frac{\Delta_\tau z(t)}{\tau}\right) e^{i\left(\upsilon x(t) + i\nu\frac{\Delta_\tau z(t)}{\tau}\right)} \mathscr{P}_{x,\Delta_\tau z}\left(dx, d\left(\Delta_\tau z\right)\right). \quad (6.5)$$

Because of the specific form of the excitation (independent-increment process), the response $x(t;\omega)$ is stochastically independent from the future increment of the forcing $\Delta_\tau z(t;\omega) = z(t+\tau;\omega) - z(t;\omega)$. (at this point we make use of the assumption $\tau > 0$). As a consequence, the joint probability measure $\mathscr{P}_{x,\Delta_\tau z}\left(dx, d\left(\Delta_\tau z\right)\right)$ can be written in multiplicative form

$$\mathscr{P}_{x,\Delta_\tau z}\left(dx, d\left(\Delta_\tau z\right)\right) = \mathscr{P}_x(dx)\cdot\mathscr{P}_{\Delta_\tau z}\left(d\left(\Delta_\tau z\right)\right). \quad (6.6)$$

Taking this into account, and making the substitution $\nu = \upsilon\tau$ (note that we are interested in the double limit $\nu \to 0$ and $\tau \to 0^+$), the double functional integral in the right-hand side of (6.5) can be factored out as follows:

$$I_s\left(\upsilon\delta(\bullet - t),\ \upsilon\tau\cdot\delta(\bullet - t)\right) =$$

$$= \int_{\mathscr{X}} \exp\{i\upsilon x(t)\}\mathscr{P}_x(dx)\ \times$$

$$\times \int_{\mathscr{Z}} i\left(\frac{\Delta_\tau z(t)}{\tau}\right)\exp\{i\upsilon\Delta_\tau z(t)\}\ \mathscr{P}_{\Delta_\tau z}\left(d\left(\Delta_\tau z\right)\right). \quad (6.7)$$

On the basis of the Projection Theorem (equ. (2.6)), the first functional integral of the right-hand side of the above equation is simply the ch.f of $x(t;\omega)$:

$$\int_{\mathscr{X}} \exp\{i\upsilon x(t)\}\mathscr{P}_x(dx) = \phi_{x(t)}(\upsilon), \quad (6.8)$$

To calculate the second functional integral in the right-hand side of equ. (6.7), we start by considering the functional integral:

$$J_\tau(\upsilon) = \int_{\mathscr{Z}} \exp\left\{i\upsilon\left(\Delta_\tau z(t)\right)\right\}\ \mathscr{P}_{\Delta_\tau z}\left(d\left(\Delta_\tau z\right)\right).$$

By the same token as above, $J_\tau(\upsilon)$ is the ch.f of the increment $\Delta_\tau z(t;\omega)$: $J_\tau(\upsilon) = \phi_{\Delta_\tau z(t)}(\upsilon)$. Assuming the latter is τ–differentiable in the vicinity of $\tau = 0^+$, we get

$$\left.\frac{\partial\phi_{\Delta_\tau z(t)}(\upsilon)}{\partial\tau}\right|_{\tau=0^+} =$$

$$= \lim_{\tau\to 0^+}\int_{\mathscr{Z}} i\upsilon\frac{\partial\left(\Delta_\tau z(t)\right)}{\partial\tau} e^{i\upsilon\left(\Delta_\tau z(t)\right)}\mathscr{P}_{\Delta_\tau z}\left(d\left(\Delta_\tau z\right)\right) =$$

$$= \upsilon\lim_{\tau\to 0^+}\int_{\mathscr{Z}} i\left(\frac{\Delta_\tau z(t)}{\tau}\right) e^{i\upsilon\left(\Delta_\tau z(t)\right)}\mathscr{P}_{\Delta_\tau z}\left(d\left(\Delta_\tau z\right)\right). \quad (6.9)$$

The last term in (6.9) coincides –apart from the factor υ– with the second integral in the right-hand side of equ. (6.7). Thus, on the basis of (6.8) and (6.9), we can rewrite equ. (6.7) as follows:

$$\lim_{\tau\to 0^+} I_\tau\left(\upsilon\delta(\bullet - t),\ \upsilon\tau\cdot\delta(\bullet - t)\right) =$$

$$= \frac{1}{\upsilon}\,\phi_{x(t)}(\upsilon)\cdot\lim_{\tau\to 0^+}\frac{\partial\phi_{\Delta_\tau z(t)}(\upsilon)}{\partial\tau}. \quad (6.10)$$

Combining now (6.4) and (6.10), we obtain

$$\frac{\partial\phi_{x(t)}(\upsilon)}{\partial t} + k\upsilon\frac{\partial\phi_{x(t)}(\upsilon)}{\partial\upsilon} - a\upsilon\frac{\partial^3\phi_{x(t)}(\upsilon)}{\partial\upsilon^3} =$$

$$= \phi_{x(t)}(\upsilon)\lim_{\tau\to 0^+}\frac{\partial\phi_{\Delta_\tau z(t)}(\upsilon)}{\partial\tau}, \quad \upsilon\in\mathbb{R}. \quad (6.11)$$

This is in fact the generalized FPK equation for general, independent-increment, excitation, written in terms of the ch.f of the response process. The corresponding FPK equation, in terms of the pdf, is easily derived by applying a Fourier transformation (see [II]). It can be shown (Athanassoulis & Sapsis 2006) that the above equation includes as special cases various generalized FPK equations, recently obtained by Grigoriu (2004).

7 MOMENT EQUATIONS FROM THE NEW PDE (5.6a)

It is worth noticing that the PDE (5.6a), derived at Section 5, can reproduce the infinite set of moment equations corresponding to the dynamical system equation (4.1a). Indeed, by direct integration of the SODE (4.1a), it is easily seen that infinite system of moment equations has the form

$$\frac{1}{n+1}\cdot\left.\frac{dM_{n+1,m}(t,s)}{dt}\right|_{s=t} + kM_{n+1,m}(t,t) =$$

$$= -a\,M_{n+3,m}(t,s) + M_{n,m+1}(t,s), \qquad (7.1)$$

where $M_{nm}(t,s) = \mathbf{E}^{\omega}\left(\left[x(t;\omega)\right]^{n}\left[y(s;\omega)\right]^{m}\right)$. Without going into details, we only state that the moment system (7.1) can also be derived from the PDE (5.6a), using simple properties connecting the ch.f of a stochastic process and its moments.

Thus, the new linear PDE (5.6a) can be viewed as an "integrating scheme" for the nonlinear infinite system of ODEs (7.1), permitting its replacement by a single linear PDE.

8 CONCLUSIONS

In this paper a new PDE (5.6a) governing the joint, response-excitation, ch.f has been derived. This equation supplemented with the marginal compatibility condition (5.6b) and the initial condition (5.6c) can provide us with the evolution of the joint ch.f. Its numerical solution is studied in a companion paper [II]. The method outlined above can be extended to general multidimensional dynamical systems exhibiting any polynomial nonlinearity. A detailed account of this approach applied to a more general, second-order, system will be presented in a forthcoming paper (Athanassoulis & Sapsis 2006).

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