Some Applications of Time Series Analysis to Atmospheric Turbulence and Oceanography*

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Summary

The basic relations for the periodogram, spectrum and correlogram are presented. The main features of atmospheric turbulence in micrometeorology and the general circulation of the atmosphere are discussed with examples. A stochastic process model for ocean waves is reported and examples of time-series analyses in oceanography are given. There are general comments and a selected bibliography of geophysical applications is appended.

1. *Introduction*

The purpose of this contribution is to review briefly some aspects of the application of time-series analysis to geophysical problems as a step to stimulating discussion, on the one hand from geophysicists in explaining to their statistical colleagues the special physical features of the problems they deal with and the analytical and interpretative questions which thereby arise, and on the other hand from statisticians in their task of exegesis concerning the nature, scope and limitations of analytical and inferential methods in the treatment of time series. We are not attempting a comprehensive review: our examples are taken from the fields of meteorology and oceanography only. A bibliography is appended which includes a selection of recent applications (mostly since 1950) of analytical methods in geophysical time-series, with the emphasis on spectral analysis. We regret that shortage of time in preparing this contribution has prevented our surveying the important Russian literature.

Time series occur in all branches of geophysics and the familiar aspects of analysis—estimation and removal of trend, the periodogram, correlation analysis, and spectral analysis—all play their part. The basic formulae defining the periodogram, the spectral function and the correlogram, and their inter-relations are probably too well known to need repetition here, but, because of the special nature of the meeting at which this contribution is presented, it seems advisable to have a general outline. An elementary, over-simplified and therefore incomplete account is given in the next section; for a consistent theoretical account in the general context of stochastic processes the reader is referred to Bartlett (1955). At a recent symposium held by the Research Section of the Royal Statistical Society, Jenkins and Priestley (1957) presented a general review of methods associated with spectral analysis—including tests of significance in the periodogram; the estimation of and confidence intervals for the spectral density, spectral bands and the integrated spectrum; and the estimation of mixed spectra. Papers on the estimation of the spectral density function and periodogram smoothing were also presented by Lomnicki and Zaremba (1957) and Whittle (1957).

* Fellows will have learned with regret that Dr. S. Rushton died on June 6th, 1957. The galley proofs of the present paper were corrected in September 1957 by Dr. Neumann who wishes to accept responsibility for any errors left in the paper.—*Ed.*

2. *Periodogram and Spectrum*

In the periodogram the "intensity" $I_\omega = A_\omega^2 + B_\omega^2$ is plotted against the angular frequency ω with

$$A_\omega = (2/N) \sum_{t=1}^{N} x_t \cos \omega t, \; B_\omega = (2/N) \sum_{t=1}^{N} x_t \sin \omega t, \quad . \quad . \quad . \quad (1)$$

for a discrete (equally spaced) sequence $x_1, x_2, \; . \; . \; . \; , x_N$, for

$$\omega = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \; . \; . \; . \; , \frac{(N-1)\pi}{N} \text{ when } N \text{ is odd,}$$

and

$$\omega = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \; . \; . \; . \; , \frac{(N-2)\pi}{N} \text{ when } N \text{ is even,}$$

and the factor 2 is replaced by $\sqrt{2}$ in the multiplier $(2/N)$ of A_ω (i) for $\omega = \pi$ when N is even and (ii) for $\omega = 0$ in all cases. Here we use "angular frequency" (or "wave number") $\omega = 2\pi/(\text{period})$; the "frequency" $= 1/(\text{period})$. There are just $N/2 + 1$ periodogram ordinates when N is even and $(N + 1)/2$ ordinates when N is odd, spaced at intervals of $2\pi/N$.

Then

$$\frac{1}{2} \sum_0 I_\omega = \frac{1}{N} \sum_{t=1}^{N} x_t^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\frac{1}{2} \sum_1 I_\omega = \frac{1}{N} \sum_{t=1}^{N} (x_t - \bar{x})^2,$$

where

$$\bar{x} = \sum_{t=1}^{N} x_t/N.$$

The most convenient form for computation is in terms of mean serial products

$$C_s = \frac{1}{N - |s|} \sum_{t=1}^{N-|s|} x_t x_{t+|s|}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

when, for $\omega > 0$,

$$I_\omega = \frac{4}{N} \sum_{s=-(N-1)}^{N-1} \left(1 - \frac{|s|}{N}\right) C_s \cos \omega s \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$= \frac{4}{N^2} \left\{ \sum_{t=1}^{N} x_t^2 + 2 \sum_{s=1}^{N-1} \left(\sum_{t=1}^{N-|s|} x_t x_{t+|s|} \right) \cos \omega s \right\} \quad . \quad . \quad . \quad (5)$$

and

$$\frac{1}{2} I_0 = \bar{x}^2 \text{ (all } N\text{)},$$

and, for N even and $\omega = \pi$,

$$I_\omega = \frac{2}{N^2} \{(x_1 + x_3 + \; . \; . \; . \;) - (x_2 + x_4 + \; . \; . \; . \;)\}^2.$$

The periodogram ordinates are just the least squares estimates of the amplitudes of periodic terms of the form $\sqrt{I_\omega} \, . \cos (\omega t + \varepsilon_\omega)$ and for a sequence of independent normal observations a complete decomposition of the total χ^2 on N degrees of freedom (if the population is normal with zero mean and variance σ^2) is achieved into independent component χ^2, either on 1 degree of freedom with expectation $2\sigma^2/N$ (I_0 always and the last I_ω if N is even, because in each case only one parameter is being estimated) or on 2 degrees of freedom with expectation $4\sigma^2/N$ (all other I_ω, because two parameters, the amplitude and phase, are being estimated)—see Fisher (1929). The expectation of $\frac{1}{2} \Sigma I_\omega$ is then σ^2.

To consider the periodogram as a *power spectrum* it is useful to exploit the electrical analogy (although the analogy can also be expressed in physical terms using, e.g., the idea of power in the theory of light or the theory of sound). If a voltage V is applied directly across a unit resistance the instantaneous power dissipated in the resistance is just V^2, so that for a set $x_1, x_2, \ldots, x_N$ of instantaneous inputs over a time N (at unit intervals) the total energy produced is $\sum_1^N x_t^2$ and the *average power* is just $\Sigma x_t^2/N$. Now, the periodogram achieves a representation

$$\Sigma \sqrt{I_\omega} \,.\, \cos(\omega t + \varepsilon_\omega)$$

of this set of inputs over the "period" N, and, incidentally, therefore presupposes the exact reproduction of this representation over all successive periods of length N; the instantaneous power dissipated in the unit resistance is just $(\Sigma \sqrt{I_\omega} \,.\, \cos(\omega t + \varepsilon_\omega))^2$ and the *average power*, averaging over a full period N, is just

$$\frac{1}{N}\int_0^N \left\{\Sigma \sqrt{I_\omega} \cos\left(k\frac{2\pi}{N}t + \varepsilon_\omega\right)\right\}^2 dt = \frac{1}{2}\sum_0 I_\omega,$$

which, of course, must be the same as $\Sigma x_t^2/N$. This is just the relation (2). The average power is thus completely decomposed into (line) elements represented by the ordinates of the periodogram, each ordinate giving the contribution to the average power transmitted by the "signal" with the corresponding frequency. This is just another way of achieving the *analysis of variance* of a discrete sequence into *discrete* components due to a *discrete* Fourier representation.

For a continuous (real) function $x(t)$ defined on a (finite) time interval $(-h, h)$, the total energy is

$$\int_{-h}^{h} x^2(t)\, dt$$

and the *average power* is

$$\frac{1}{2h}\int_{-h}^{h} x^2(t)\, dt. \qquad (6)$$

If we now define a new function

$$\begin{aligned} y(t) &= \begin{cases} x(t) & \text{for } -h \leqslant t \leqslant h, \\ 0 & \text{for } |t| > h, \end{cases} \end{aligned} \qquad (7)$$

then, if $x(t)$ is in fact defined for all t, we see that $y(t)$ will become equal to $x(t)$ as h tends to infinity, and the average power is the limit of (6), if the limit exists. For a continuous Fourier (i.e. periodogram or spectral) representation of $x(t)$ it is natural to consider the Fourier transform

$$g(\omega) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} y(t)\, e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{-h}^{h} x(t)\, e^{i\omega t}\, dt, \qquad (8)$$

so that

$$y(t) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t}\, d\omega.$$

Then the *average power*

$$\lim_{h\to\infty} \frac{1}{2h}\int_{-h}^{h} x^2(t)\,dt = \lim_{h\to\infty}\int_{-\infty}^{\infty} \frac{y^2(t)}{2h}\,dt$$

$$= \lim_{h\to\infty}\int_{-\infty}^{\infty} \frac{|\,g(\omega)\,|^2}{2h}\,d\omega, \text{ by Parseval's Theorem,}$$

$$= \lim_{h\to\infty}\int_{0}^{\infty} \frac{|\,g(\omega)\,|^2}{h}\,d\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

because $|\,g(\omega)\,|^2 = g(\omega)\,g(-\,\omega)$ is an even function. The *power spectral density* (or *spectral function*, or *spectral density*) is the contribution $f(\omega)\,d\omega$ to the average power from a band of frequencies of width $d\omega$, that is

$$f(\omega) = \lim_{h\to\infty} |\,g(\omega)\,|^2/h$$

$$= \lim_{h\to\infty} \frac{1}{2\pi h}\left|\int_{-h}^{h} x(t)\,e^{i\omega t}\,dt\right|^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$= \lim_{h\to\infty}\left\{\frac{1}{2\pi h}\left(\int_{-h}^{h} x(t)\cos\omega t\,dt\right)^2 + \frac{1}{2\pi h}\left(\int_{-h}^{h} x(t)\sin\omega t\,dt\right)^2\right\}$$

if the limits of these expressions exist. The frequency ω is allowed to range over all positive values. This is the decomposition of the average power as a continuous spectrum $f(\omega)$, i.e. an analysis of variance into a continuous distribution of contributions from all frequencies, resting formally on Parseval's theorem, as all such Fourier decompositions of mean squares essentially do.

We may note here that the ordinary Fourier representation of $x(t)$ defined on $(-\,h,\,h)$ as a countable infinity of discrete periodic components, i.e.

$$x(t) = \sum_{0}^{\infty} c_r \cos(r\omega t + \varepsilon_r),$$

where $\omega = \pi/h$, gives for $r > 1$

$$c_r^2 = \frac{1}{h^2}\left|\int_{-h}^{h} x(t)\,e^{i\omega t}\,dt\right|^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and for $r = 0$ the divisor should be $1/4h^2$; and considering "average power" again (averaging over a full period of length $2h$, because this representation is exactly reproduced over all subsequent periods) we shall have

$$\frac{1}{2h}\int_{-h}^{h} x^2(t)\,dt = \frac{1}{2}\sum_{r=0}^{\infty} c_r^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

To compare (10) and (11), the factor 2 should be omitted in (10) since there is a similar factor in (12), and the change from $1/\pi h$ in (10) is accounted for because $f(\omega)$ in (10) is the contribution per unit band-width, whilst in (11) c_r^2 is a contribution over a band-width interval π/h.

Now the relation (4) shows that the periodogram ordinates I_ω are just cosine transforms of the sample serial products C_s, with more weight being attached to the C_s of short lag than to those of longer lags. To obtain the corresponding relation in the continuous case we consider the mean serial product of lag τ

$$C(\tau) = \lim_{h\to\infty} \frac{1}{2h} \int_{-h}^{h} x(t)\, x(t+\tau)\, dt, \quad \text{(13)}$$

$$= \lim_{h\to\infty} \frac{1}{2h} \int_{-\infty}^{\infty} y(t)\, y(t+\tau)\, dt,$$

$$= \lim_{h\to\infty} \frac{1}{2h} \int_{-\infty}^{\infty} g(\omega)\, g^*(\omega)\, e^{-i\omega\tau}\, d\omega, \text{ by the convolution theorem,}$$

$$= \lim_{h\to\infty} \frac{1}{2} \int_{-\infty}^{\infty} \frac{|\, g(\omega)\,|^2}{h}\, e^{-i\omega\tau}\, d\omega, \quad \text{(14)}$$

$$= \int_{0}^{\infty} f(\omega) \cos \omega\tau\, d\omega, \quad \text{(15)}$$

because $f(\omega) = \lim_{n\to\infty} |\, g(\omega)\,|^2/h$ is an even function.

By simple inversion

$$f(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} C(\tau)\, e^{i\omega\tau}\, d\tau. \quad \text{(16)}$$

$$= \frac{2}{\pi} \int_{0}^{\infty} C(\tau) \cos \omega\tau\, d\tau. \quad \text{(17)}$$

There are complications arising from considerations of ergodicity and stationarity, but we will merely note here that if both conditions are satisfied then the function $C(\tau)$ may be identified with the *autocovariance function* of $x(t)$ and is a function of τ only. A further complication which it is useful to mention arises if $x(t)$ actually possesses discrete periodic components in addition to a non-periodic component: in this case, $C(\tau)$ will also contain corresponding periodic components (cf. the well-known property of the correlogram), and, since the Fourier transform of a sinusoidal component is a delta function, discrete components appear in the spectral function at stage (17) the spectrum is quite properly a "mixed spectrum". Normalization with respect to mean and variance allows us to write the autocorrelation function $\rho(\tau)$ for $C(\tau)$ in (17).

The *integrated spectrum* $F(\omega)$ is given by

$$F(\omega) = \int_{0}^{\omega} f(\omega')\, d\omega' = \frac{2}{\pi} \int_{0}^{\infty} C(\tau) \frac{\sin \omega\tau}{\tau}\, d\tau \quad \text{(18)}$$

with $F(0) = 0$, $F(\infty) =$ "average power" or *mean square* and the contribution to the average power from a band width (ω_1, ω_2) is $F(\omega_2) - F(\omega_1)$; with normalization and $\rho(\tau)$ for $C(\tau)$ in (18), $F(\infty) = 1$.

Without going into details, we shall merely state about the case not so far considered of a discrete (in time) sequence defined for all time that the spectral function is

$$f(\omega) = \frac{1}{\pi} \sum_{s=-\infty}^{+\infty} C_s \cos \omega s, \quad \text{(19)}$$

with integrated spectrum

$$F(\omega) = C_0 \frac{\omega}{\pi} + \frac{2}{\pi} \sum_{s=1}^{\infty} C_s \frac{\sin s\omega}{s}, \quad \text{(20)}$$

where C_s = "mean serial product" of lag s and C_0 = mean square; and $f(\omega)$ is defined only on the interval $(0, \omega)$. For a complete treatment see Wold (1954). It is well known that if the sequence considered in fact consists of discrete observations (at unit intervals, say) on a continuous process then the spectral functions will coincide only if that of the continuous process is really zero outside the interval $(0, \pi)$: this is just a re-statement of the familiar fact that no information can be extracted about periodic components having a half-period less than the interval between observations; it gives rise to Shannon's sampling theorem in the theory of communication that "if a function contains no frequencies higher than W cycles/second it is completely determined by giving its ordinates at a series of points $1/2W$ seconds apart".

The periodogram has acquired a quite unenviable reputation amongst numerical techniques as having led many good men astray. Its often wildly fluctuating appearance and its ability to indicate "spurious" cyclical periods, e.g. when applied to series generated by known autoregressive schemes, are very familiar. Yet, the "failure" of classical periodogram analysis can be attributed: to its use in isolation from other aspects of analysis, to attempts to *describe* phenomena in terms of a discrete spectrum (i.e. discrete cyclical components of fixed periods) when this was inappropriate and the attaching of physical significance where this was unjustified—and failures in *description* inevitably lead to failures in *prediction*. It is now recognized that the different aspects of time-series analysis form an essential unity. The extensive computational work involved in calculating periodogram ordinates, correlations and spectral estimates has been enormously facilitated by the advent of electrical computers of the analogue and digital type; it may, however, not be out of place to repeat the warning that "forgetting the physics" is just as disastrous in the present as it has been in the past.

The peaks of the periodogram give the least squares estimates of the frequencies of discrete harmonic components and the corresponding ordinates provide the least squares estimates of the amplitudes which have minimum variance among all second order functions of the observations. This justifies the classical approach. When we come to consider the periodogram ordinates calculated in the usual way in relation to a continuous spectral function, it still remains true that the periodogram ordinate I_ω provides an estimate of $f(\omega)$ at a continuous part of the spectrum and this estimate is asymptotically unbiassed, i.e. the bias vanishes with increasing length of the record. Further, if it is assumed that the observations are normally distributed then the equidistant periodogram ordinates are independent and furnish virtually all the information in the sample about the spectral ordinates derivable from second order functions of the observations, although the first property is lost if the assumption of normality is relaxed, at least for finite samples (it is still true asymptotically)—see Whittle (1957). The well-known wild fluctuations of the periodogram ordinates as estimates of the spectral ordinates arise because the standard deviation of the estimate is of the same order of magnitude as the spectral ordinate being estimated and does not diminish with increasing length of record, and also because of the asymptotic independence mentioned above. Hence the development of "smoothing" techniques. In cases where the model fitted consists of both discrete and continuous components, Whittle (1952) has devised an iterative procedure for separating them: the periodogram ordinates, however, still provide estimates of the discrete harmonic components. There is an obvious point about "smoothing", viz., that it may be quite unjustified since the parent spectrum may be itself irregular. A full discussion of smoothing procedures (with references) is given in the symposium papers cited above (see also Daniels (1957) and Whittle (1954*b*)).

3. *Correlogram*

For a discrete sequence the sample autocorrelations

$$r_s = \frac{\sum\limits_{j=1}^{N-|s|} x_j x_{j+|s|} - (N-|s|)\left[\dfrac{\sum\limits_{j=1}^{N-|s|} x_j}{N-|s|}\right]\left[\dfrac{\sum\limits_{j=|s|+1}^{N} x_j}{N-|s|}\right]}{\left\{\sum\limits_{j=1}^{N-|s|} x_j^2 - (N-|s|)\left[\dfrac{\sum\limits_{j=1}^{N-|s|} x_j}{N-|s|}\right]^2\right\}^{\frac{1}{2}}\left\{\sum\limits_{j=|s|+1}^{N} x_j^2 - (N-|s|)\left[\dfrac{\sum\limits_{j=|s|+1}^{N} x_j}{N-|s|}\right]^2\right\}^{\frac{1}{2}}}$$

where $r_0 = 1$, $r_{-k} = r_k$, are plotted against lag s; or the approximate form $r_s = C_s/C_0$ is used.

4. *Atmospheric Turbulence: Micrometeorology*

Before citing some examples of spectral analysis in meteorology, we shall explain the basic background features of atmospheric turbulence.

The covariances of a wind-component and a property such as sensible heat, latent heat or water vapour, and "momentum", play a major role in studies of both extremes of atmospheric motion, viz., in micrometeorology (the meteorology of the air layer nearest to the earth's surface) and in investigations of the general circulation of the atmosphere (the study of air motions on the largest scale). In this section we consider some micrometeorological applications.

Consider a steady horizontal mean wind near the earth's surface, and denote by u and w the respective velocity components in the direction of the mean wind and in the direction of the vertical. These instantaneous values can be expressed

$$u = \bar{u} + u', \quad w = \bar{w} + w',$$

where $\bar{u}$ and $\bar{w}$ are the mean values of the wind components (i.e. time-averages at a point), and u' and w' are the turbulent fluctuations. In the Reynolds' modification of the Navier-Stokes equations of motion (see equation (2) in the paper by Charnock at this symposium), there occur terms of the form $\overline{u'w'}$ (Reynolds stresses, or eddy shearing stresses, or momentum fluxes). Here and in similar expressions below we omit certain multipliers which for all practical purposes are constants when we consider a given point in the atmosphere. The particular term $\overline{u'w'}$, which is obviously a covariance, represents the transfer, or flux, of horizontal momentum along the vertical, i.e. the transfer of fast moving air towards the surface and the simultaneous upward transfer of frictionally retarded air from near the surface. The physical significance of the term is that no sustained motion of the air could be maintained without such a transfer, or flux, since the air would be brought to a standstill by friction without a re-arrangement in the motion.

Further examples of fluxes are the fluxes of heat (sensible heat) and of water vapour (latent heat). The fluxes of heat and water vapour along the vertical have so far received most attention. They are defined by the covariances $\overline{w'T'}$ and $\overline{w'q'}$, where T' and q' are the fluctuations from their respective means of air temperature and absolute humidity. In the surface layer, $\overline{w'q'}$ in the steady state is just the evaporation or condensation (dew) of water vapour. According as evaporation or condensation is taking place, so the sign of the covariance changes.

When fluxes are measured by an instrument, the latter acts as a filter, suppressing usually the high frequency eddies of the relevant variables. Spectral analysis of the fluxes shows which range or set of ranges are important in effecting the flux. If, for example, a flux is primarily effected by high frequency eddies, it becomes necessary to use or develop instruments of appropriate rapid response.

5. *Examples in Micrometeorology*

Example I.—Broadly speaking, small-scale atmospheric turbulence may be divided into two classes with regard to its generation: mechanical (frictional) turbulence and heat convection. Heat convection in any fluid such as the atmosphere is intimately connected with (amongst other things) the variation of temperature with height, but, since the latter depends on radiation conditions, so also does heat convection, at least in a general manner.

By measuring and evaluating the spectrum of vertical velocity in the lower atmosphere, Van der Hoven and Panofsky (1954) were able to demonstrate that the low frequency part of the spectrum depends strongly on the intensity of solar radiation. They found that with increasing solar radiation the maximum of the spectral curve of vertical velocity moves to the lower frequencies and that the "energy" in the low-frequency part of the spectrum increases, whereas the high-frequency part of the spectrum remains practically unchanged (see their Fig. 36), and they therefore suggest that this high-frequency region represents mechanical turbulence, whilst the radiation-dependent low-frequency region represents turbulence connected with heat convection. Their findings also suggest that as convection intensifies, the convection cells become taller relative to their width, an inference which is supported by the work of Gifford (1955), who tracked the motion of no-lift balloons. By studying the spectrum of the product of horizontal and vertical wind speeds, Van der Hoven and Panofsky found that with increasing intensity of solar radiation, more and more of the vertical transfer of momentum is supplied by the low-frequency eddies.

Example II.—Deacon (1955) measured the vertical flux of momentum $\overline{u'w'}$ over a level grassland near Melbourne at a height of 7 m. in conditions of light wind and a moderately strong lapse of temperature with height. To get a reliable estimate of the mean flux, whose standard error would not differ by more than a given fraction from the mean for a very large sample, he considered the maximum permissible time interval between successive observations. The autocorrelation coefficients were first computed from a record of 300 readings at 1 second intervals and the observed correlogram approximated by $\rho(t) = e^{-\mu t}$, with $\mu = 0 \cdot 38$ (see Deacon (1955), Fig. 10).

For a sample of size n from an exponentially autocorrelated time-series, with observations taken at equal intervals of time over a total period of duration D, the variance of the sample mean is easily determined, hence the "effective" number of observations and so the required interval of observation over the duration D to achieve the necessary precision. The spectral function $f(\omega) = 4\mu/(\mu^2 + \omega^2)$ associated with an exponential autocorrelation function is closely approximated by the empirical spectrum, with some fluctuation in the high frequency range.

6. *General Circulation of the Atmosphere*

Meteorologists are particularly interested in the fluxes vu, vT, and vq, where u and v are respectively the westerly and poleward components of the wind, T is air temperature and q is absolute humidity. The flux vu denotes the meridional flux of "westerly momentum"; vT and vq respectively denote the meridional fluxes of heat and water vapour. These fluxes are of interest for the following reasons:

(i) The surface winds of the middle latitudes (about 35° to 60°) are westerlies; those of the tropical regions are easterlies. The surface frictional retardation of the low-latitude easterlies has the effect of injecting westerly momentum in these winds; the frictional retardation of the mid-latitude westerlies involves a loss of westerly momentum. Unless westerly momentum were continually being transferred poleward from the low-latitude easterlies, the general circulation of the atmosphere could not be maintained. This meridional transfer or flux of momentum is vu.

(ii) It is calculated that if in any latitude belt the sole source of heat were radiation, then the high latitudes of both hemispheres would be much colder and the tropical belts would be much hotter than they are at present. In fact, a transfer of heat (sensible heat) takes place from the low latitudes, where there is a net gain of radiation, to the high latitudes, where there is a net loss of radiation. This meridional flux of heat is vT.

(iii) In some latitudes precipitation exceeds evaporation whilst in other latitude belts the opposite holds, and a transfer of water vapour takes place from regions of net gain of water vapour to regions of net loss. This meridional flux is vq.

(Throughout the above certain multipliers of the fluxes have been omitted.)

If we consider the average flux for a given time period at some pressure level at a point over the Earth's surface, then, if Reynolds axioms are satisfied,

$$\left.\begin{aligned} \overline{vu} &= \bar{v}\bar{u} + \overline{v'u'}, \\ \overline{vT} &= \bar{v}\bar{T} + \overline{v'T'}, \\ \overline{vq} &= \bar{v}\bar{q} + \overline{v'q'}, \end{aligned}\right\} \qquad (21)$$

where the first term on the right in each equation represents an advective flux by a mean meridional circulation. The second term is the "eddy flux".

If instead of the flux at one station, we consider the average flux for a number of different stations (e.g. stations along a latitude circle), then the eddy fluxes $\overline{v'u'}$, $\overline{v'T'}$ and $\overline{v'q'}$ will in fact each be the sums of two terms. Denoting a space-average by { . . . }, and letting u'_t and v'_t be deviations from a time-average, and similarly u_x' and v_x', deviations from a space-average, then

$$\{\overline{v'u'}\} = \{\overline{v_t'u_t'}\} + \{\bar{v}_x'\bar{u}_x'\} \qquad (22)$$

and similar equations hold for the heat and water-vapour fluxes. The first term on the right-hand side is simply the space average of the term previously denoted $\overline{v'u'}$. The second term on the right arises from a possible correlation in space between the time means of v and u, and is called the "standing" eddy flux. It is due (when it exists) to semi-permanent features of the atmosphere such as large eddies set up by major mountain ranges.

Several attempts have been made to compute the magnitude of such terms as are involved in (21) and it is generally agreed that in the case of momentum and sensible heat the eddy fluxes considerably outweigh in importance the terms involving the mean for meridional circulation.

The terms $\overline{v'u'}$, $\overline{v'T'}$ and $\overline{v'q'}$ are just covariances and a spectral analysis of the terms vu, vT and vq helps to throw light on the frequencies, or periods, of the eddies responsible for the major part of the flux.

The correlations involved in studies of the fluxes are Eulerian time-correlations and the information derived from the corresponding spectral analysis is in terms of frequencies or periods. Eulerian space correlations which would supply information on the *wave numbers*, instead of frequencies or periods, of the most important eddies would be much more valuable as they would yield information on the *spatial structure* of atmospheric disturbances. Taylor (1938) showed that if the magnitude of the turbulent fluctuations of the wind are small compared with the mean wind, then the form of the time correlation function for a point in the turbulent fluid will be virtually equal to the spatial correlation function, i.e. the variables times and distance are interchangeable (see also Ogura (1953)). MacCready (1953*b*) presented observational data indicating that the equivalence of the Eulerian time and space correlations is, at least on occasions, approximately valid even when the turbulent fluctuations of the wind are not small compared with the mean wind.

In 1941, Kolmogoroff presented a model of turbulence, called the *similarity hypothesis* (for an account of the model and of subsequent work by other authors, see Batchelor (1953); for a summary see Priestley and Sheppard (1952)). This model is capable of predicting on the basis of physical-dimensional considerations statistical characteristics such as the form of the correlation function for a certain range of eddy "sizes" (wave numbers) called the inertial or equilibrium sub-range. This range will be well developed when the Reynolds number of the motion of the fluid is large, such as is usually the case in the atmosphere. Some of the predictions of the model have been verified in practice. The unexpected thing is that the statistical laws predicted for the isotropic range have in several cases been found to hold, at least approximately, for wave numbers where the turbulence is certainly not isotropic. Examples of this are given by McCready (1953*b*); further examples relating to large-scale flow are given by Hutchings (1955).

7. *General Circulation of the Atmosphere*: *Example*

Estoque (1955) considers the turbulent flux of momentum and heat at the 850 mb. pressure level over the south-eastern United States for the year 1949. Annual means and harmonic components corresponding to the "annual eddy" were initially subtracted from u, v and T. Covariances of the form

$$\tfrac{1}{2}\,\overline{(v'(t)\,u'(t+\tau)+v'(t)\,u'(t-\tau))}$$

and

$$\tfrac{1}{2}\,\overline{(v'(t)\,T'(t+\tau)+v'(t)\,T'(t-\tau))}$$

were computed and analysed by spectral analysis (see Estoque (1955), Fig. 4).

It has been pointed out above that computations of meridional fluxes of momentum, heat and water vapour indicate that the most important mechanism for the transfer of momentum and heat are the eddy fluxes ($\overline{v'u'}$ and $\overline{v'T'}$), i.e. the turbulent "elements" of the general circulation—presumably related with the migratory cyclones and anticyclones. Since the passage of cyclones and anticyclones over any station takes but a few days, it is to be expected that the peak of the flux spectrum will be at a period of a few days. This anticipation is fully confirmed by Estoque's results.

8. *Stochastic Process Model for Ocean Waves*

A theoretical study of a model for surface waves in the open ocean as a stationary normal stochastic process with a two-dimensional spectrum has recently been made by Longuet-Higgins

(1957) and he has suggested that the model may be applicable to other geophysical phenomena, e.g. to microseisms or perturbations of the ionosphere. To take account of the fact that waves in the open sea are highly irregular and lack any apparent single direction or orientation, a two-dimensional Fourier synthesis in terms of a continuous spectrum involving wave-numbers in two horizontal directions is considered appropriate. The surface elevation z is represented in terms of the horizontal coordinates, x and y, and time t as

$$z(x, y, t) = \Sigma \cos (u_n x + v_n y + \sigma_n t + e_n),$$

where it is supposed that the wave numbers (u_n, v_n) are densely distributed throughout the u, v plane, i.e. there are an infinite number in any elementary area $du \,.\, dv$. The phases e_n are taken to be uniformly distributed in the interval $(0, 2\pi)$; σ_n is regarded as a function of u_n and v_n, i.e.

$$\sigma_n = \sigma(u_n, v_n)$$

and the amplitudes c_n are random variables such that in any element $du\, dv$

$$\begin{aligned}\sum_n \tfrac{1}{2} c_n^2 &= E(u, v)\, du \,.\, dv \\ &= \text{the contribution to the mean energy from an element } du \,.\, dv.\end{aligned}$$

The function $E(u, v)$ is the spectrum of z, and the mean-square value of z per unit area of the sea surface per unit time is given by

$$\lim_{X,Y,Z,\to\infty} \frac{1}{8XYT} \int_{-X}^{X} \int_{-Y}^{Y} \int_{-T}^{T} z^2\, dx\, dy\, dz = \sum_n \tfrac{1}{2} c_n^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(u, v)\, du\, dv.$$

The correlation function $\psi(x, y, t)$ is just

$$\begin{aligned}\psi(x, y, t) &= \lim_{X,Y,T\to\infty} \frac{1}{8XYT} \int_{-X}^{X} \int_{-Y}^{Y} \int_{-T}^{T} z(x', y', t')\, z(x' + x, y' + y, t + t')\, dx'\, dy'\, dt' \\ &= \Sigma \tfrac{1}{2} c_n^2 \cos (u_n x + v_n y + \sigma_n t) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(u, v) \cos (ux + vy + \sigma t)\, du\, dv,\end{aligned}$$

so that ψ is the cosine transform of $E(u, v)$. This is just the three-dimensional analogue of the usual (one-dimensional) Fourier series representation of a normal stochastic process, used especially in communication theory. Such a model for ocean waves rests on the principle of superposition of disturbances due to different generating mechanisms, i.e. the mechanical system is assumed to be linear. This appears to be largely valid for low waves in deep water, but it is recognized that it would not apply to waves approaching the maximum height, or to surf, or to the rolling motion of a ship when this is large enough for non-linear terms to become important. The effect of non-linearity in the mechanism of waves is to make crests higher and troughs shallower than for a surface described by the above model.

If m_{rs} is the $(r, s)^{\text{th}}$ moment of the spectral distribution $E(u, v)$, i.e.

$$m_{rs} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(u, v)\, u^r v^s\, du\, dv,$$

then the conditions for the sea surface to degenerate in the following ways can be expressed solely in terms of the moments m_{rs}: (i) the surface to be "long-crested", i.e. for the energy to travel always in the same direction—in this case E degenerates into a single one-dimensional spectrum;

(ii) the surface to consist of not more than n long-crested systems—so E degenerates into n one-dimensional systems; (iii) the energy to correspond to wave components of the same length but possibly different directions, i.e. E to degenerate into a "ring" spectrum; (iv) the surface to have a "standing-wave" pattern with the energy situated at two diametrically opposite points of the spectrum; (v) energy uniform in both wavelength and direction, i.e. the spectrum is *narrow* with the energy concentrated at a single point; (vi) energy concentrated in not more than two narrow bands, i.e. the spectrum consisting solely of two opposite corners of a rectangle u = constant, v = constant.

The spectrum E^θ of a curve of intersection of the wave surface by a vertical plane in an arbitrary direction can be found and hence the *principal direction*, i.e. the direction in which the second moment of E^θ (and therefore the root-mean-square wave number) is a maximum, with the minimum r.m.s. wave number in the direction perpendicular to this principal direction. The ratio $(1/\gamma)$ of the r.m.s. wave numbers in these two directions is taken by Longuet-Higgins as a convenient measure of the long-crestedness of the surface, since for a long-crested system of waves the r.m.s. wave number is a maximum perpendicular to the crests and a minimum parallel to the crests. For a perfect long-crested system, i.e. a single one-dimensional spectrum, $\gamma = 0$ and for very short-crested waves $\gamma = 1$. The properties of the "carrier wave" and "wave envelope" are also derived.

The distribution of the elevation z is normal and that of the components of slope, $\partial z/\partial x$ and $\partial z/\partial y$ is bivariate normal with the greatest r.m.s. slope in the principal direction of the surface.

The distribution of the magnitude of the slope regardless of its direction can be reduced to a Rayleigh distribution (χ^2 on two degrees of freedom) for very short-crested waves ($\gamma = 1$) and it tends to a normal distribution for very long-crested waves (with a certain anomalous behaviour for very small slopes). A number of statistical properties, such as the average number of zero-crossings of z along a line in an arbitrary direction, are determined and an approximation sequence is derived for determining the spectrum given its statistical properties in terms of moments of the spectral distribution in any direction θ, it being supposed that the height and slope of the surface can be measured along such a line, e.g. by an aircraft flying on a fixed course over the sea surface at high speed and constant altitude.

The distribution of wave height (the difference in level between a crest and the preceding trough) had also been previously discussed by Longuet-Higgins (1952) for a narrow spectrum as a Rayleigh distribution and it had been shown that the values of the r.m.s. wave-height, the mean height of the highest one-third waves (defined by Sverdup and Munk (1947) as "significant" waves), and the most probable height of the largest wave in a given time-interval were in close agreement with observation. General agreement with the Rayleigh distribution in this case had already been found by Barber (1950).

It was suggested that such a situation for wind-generated waves with a narrow spectrum could arise from a single storm area at a great distance (compared with the dimensions of the storm); so that in the course of propagation, different frequencies in the spectrum, being propagated with different velocities, would become spread out in space, and over a short interval of time only a narrow range would be present. The analysis would then only be appropriate as long as the recording time was neither too long to allow a significant change in the frequency and amplitude of the waves nor too short to allow a representative sample of wave heights to be recorded. Further, it was suggested that the model would be more appropriate for a pressure wave record on the bottom than to actual surface elevation, since in the former case the high frequencies are damped out relative to the lower frequencies and the corresponding frequency spectrum is narrower; or, alternatively, to the angular deflection of a ship's mast to the vertical, since a ship acts as a resonant filter, amplifying those components of the spectrum close to its natural period.

Incidental results related to the above have also been given by Cartwright and Longuet-Higgins (1956). They derived the distribution of the height z between a wave-crest and the mean level by applying directly the results of Rice (1944, 1945) in communication theory on the distribution of the maximum of a stationary normal (one-dimensional) stochastic process represented by its Fourier synthesis. The distribution of z is found to depend only on the r.m.s. height and on a parameter $\varepsilon^2 = (m_0 m_4 - m_2^2)/m_0 m_4$, where the m_r are the moments of the spectral distribution. The parameter ε is obviously a measure of the relative width of the spectrum; when ε is small (an infinitely narrow spectrum) the distribution of z tends as expected to a Rayleigh distribution and when ε tends to its maximum value of unity (which can occur when one wave of high frequency and small amplitude is superposed on another disturbance of lower frequency) the distribution tends to a normal distribution. Statistical properties are derived: moments, mean values of the highest $1/n^{\text{th}}$ of the crest heights and the expectation of the highest in a sample of N crest heights. The distribution of crest heights as measured from records of ocean wave phenomena was found to agree with the theoretical distribution. The records examined comprised one of pressure on the sea-bed (off Cornwall), two records of wave-heights in deep water (Bay of Biscay), and two records of the pitching and rolling motion of *R.R.S. Discovery II* in a seaway in the North Atlantic. On the other hand, *crest-to-trough* heights were found to depart significantly, as expected, from the Rayleigh distribution for reasonably broad spectra.

The three-dimensional stationary normal stochastic process model of ocean waves with a two-dimensional spectrum has also been treated by Pierson (1955) and applied to the study of the motion of a ship by St. Denis and Pierson (1953). Regarding the appropriateness of the model, Pierson (1955) concludes that, depending on the meteorological conditions, areas of many hundreds of square miles for many hours, and wave records as a function of time lasting from twenty minutes to many hours, are so closely represented by treating them as samples from particular realizations of a stationary normal process that the departures are unimportant. In fact, of course, the spectra involved are slowly varying functions of time and space, so that the assumptions of stationarity hold only if the translations in time and space that are considered are not too large.

9. *Oceanographic Examples*

Example I: That the correlogram is an effective method of detecting a periodic "signal" in random "noise" is well illustrated by analyses by Seiwell and Wadsworth (1949) and by Seiwell (1949) of records of pressure pulses at the sea bottom. A typical (unsmoothed) periodogram from these records showed a characteristic fluctuating appearance with many well-defined peaks, whilst the corresponding correlogram showed a remarkably smooth curve maintaining a constant period and damping slowly to a terminal amplitude, which was thereafter maintained by smooth cycles. The correlogram of residuals after the single cyclical component had been estimated by least squares and subtracted from the record damped in a fashion typical of an autoregressive sequence. The analysis showed, therefore, a spectrum consisting of a single discrete component and a continuous component representing a wide band of frequencies with an average frequency equal to that of the single cyclical componnent but damping out rather rapidly. The physical characteristics of the sea surface wave pattern corresponding to such a record, it was suggested, comprised a fundamental wave (sea swell—the "signal"), on which was superimposed a series of local impulses (local sea—"noise") of the same average frequency, probably generated by local winds, which died out rather rapidly. One record analysed by Seiwell showed that in the complete absence of sea swell, with local influences dominating the pattern, the correlogram damped almost to zero, indicating almost completely autoregressive data. The author's interest in correlogram analysis led him (Seiwell (1950)) to carry out a number of experimental analyses which parallel the work of Professor M. G. Kendall (see Kendall (1946*a*, 1946*b*).)

Further analyses of ocean waves have also been given by Rudnick (1951). The actual computation of correlograms was carried out on an electronic analogue computer developed for the purpose. Signals were furnished to the correlator through a simple resistance-capacity high pass filter whose limit corresponded to a period equal to the longest correlation interval to be used, so that the correlograms were designed to be comparatively free of the effect of slow oscillation, or trend in the data. The correlograms were roughly described by the author as a curve oscillating under a diminishing envelope, with the two first-order parameters of the correlogram being the period for the oscillation (the dominant period of the wave spectrum) and the decay time of the envelope, i.e. $\rho(\tau) = e^{-|\tau|/aT} \cos 2\pi\tau/T$, with a corresponding spectrum of the form

$$\frac{1}{1 + \dfrac{(f - f_0)^2}{(bf_0)^2}},$$

where $f_0 = 1/T$. Broad agreement with this spectrum was obtained by determining one wave spectrum directly with a wave analyser. Repetition of the records at six-hourly intervals showed that only the first three periods of each correlogram were at all informative, the rest of the correlogram mainly constituting sampling errors, which the author discussed in the light of the results of Bartlett (1946) for the sampling errors of autocorrelations. The fact that the broad smoothed form of the spectrum is given by the initial part of the correlogram (i.e. for small lags) is the basis of the smoothing procedure of Bartlett (1950).

Another conclusion from Rudnick's work was that the overall distribution of displacements was normal, so that the record of waves as they pass a fixed point could be regarded as a realization of a normal stochastic process.

Example II: The analysis of a seiche record of 660 observations at 15-second intervals of water level in a rock channel on the coast of New Zealand has been presented by Whittle (1954*a*). The analysis required the development of a method for estimating the components of the mixed spectrum involved (for discussion of this, see Jenkins and Priestley (1957)). The observations were taken at the turn of the tide in order to minimize its effect. In spite of this, the data showed a definite trend which was eliminated by fitting a second-order polynomial.

The correlogram of actual deviations from trend showed oscillations damping away until about lag 30, after which they maintained a fairly constant amplitude; the tail showed a period of about 11 units (unit = 15 sec.) with an oscillation that was not simple. The first estimate of the spectrum was made using the smoothing method of Bartlett (1950) with autocovariances C_s up to lag 114. Sections of the periodogram were calculated in the usual way in the immediate neighbourhood of the maxima of the smoothed spectrum. When the smoothed spectrum was replotted as a function of period instead of the angular frequency ω, peaks were at $T = 11$ (narrow, well defined), 14 (strong, broad), 25 and 36 (subsidiary), and about 48 (faint suggestion). From the relations $25 = 11 + 14$, $36 = 2\ (11) + 14$, $48 = 3\ (11) + 14$ (nearly)—notice that these relations are for periods, not frequencies—Whittle suggested that a non-linear mechanism might have been present. A fourth order autoregressive scheme was then fitted to the data, the presence of wave-trains with periods $T = 10{\cdot}87$ and 123 established by a significance test, with amplitudes estimated from the periodogram; modified autocovariances computed and finally a further (fifth-order) autoregressive scheme fitted. Since no further periods were found to be significant, the spectral estimates consisted of two lines corresponding to the sustained waves of periods $T = 10{\cdot}87$ and $T = 123$ and a continuous spectrum corresponding to this autoregressive residual component. The possible subsidiary oscillations of periods $T = 25$ and $T = 36$ were not sufficiently pronounced to be explained by the final autoregressive scheme, which could in any case only reproduce the broader spectral peaks. The sustained sets of waves were identified as being due to reflection at the open end of the channel. The non-linear phenomenon giving unforced oscillations subject to damping was identified with intermittent washing in of water at the top (shore) end of the channel.

Example III: Aerial photographs of the glitter of the sun on the sea surface were used by Cox and Munk (1954*b*) to derive the statistics of the slope distribution. It was found that the joint distribution of the up-wind and cross-wind components of slope was very nearly normal, with a slight negative skewness in the up-wind direction which increased with increasing wind speed. This negative skewness implied a modal slope of a few degrees rather than zero, the azimuth of ascent pointing downwind, so that small slopes were slightly more frequent than would have arisen from a purely normal distribution. A slight positive kurtosis was also found, which did not vary with wind speed. It was suggested that the superposition of non-linear effects accounted for the slight departure from a strictly normal slope distribution.

A number of other aspects was studied, including the effect of oil "slicks": it was found that such slicks reduced the mean square slope by a factor of two or three, eliminated skewness but left kurtosis unchanged.

The justification for studying slope and curvature of the sea surface rests partly on the simple fact that, if the elevation spectrum is $f(\omega)$, then the slope and curvature spectra are respectively $\omega^2 f(\omega)$ and $\omega^4 f(\omega)$, so that the latter are weighted progressively in favour of higher frequencies. The areas under the three spectral curves are respectively mean squares of elevation, slope and curvature. It follows that studies of slope and curvature provide direct knowledge of the higher frequency waves which play an important part in considerations of the momentum transfer from wind to water (i.e. wind stress) and the generation of the more important parts of the wave spectrum (i.e. larger waves). The action of a slick is to remove only high frequency components: in the experiments mentioned above, the mean square elevation was unchanged, the mean square slope was substantially reduced and the mean square curvature almost eliminated.

10. *Some General Remarks*

(i) It will be clear from the examples quoted in the foregoing sections that many of the time series occurring in geophysical work, particularly in meteorology, are multivariate (or multiple) series. Considerations of the "co-spectrum" (the in-phase relationship) and the "quadrature spectrum" (the out-of-phase relationship) occur in treating, for example, the turbulent flux of momentum and heat in the general circulation of the atmosphere, although we have omitted these complications (see Estoque (1955)). From the statistical point of view, some sampling results have been obtained (see Bartlett (1955) and references there), but the formulae involved become complicated; work, particularly in the field of econometrics, has been published on the simultaneous estimation of a number of autogreressive models, where the complication of "identification" arises; methods for testing the goodness of fit of the correlogram can also be extended to multivariate series; and a general treatment of such series has been given by Whittle (1953).

(ii) The over-riding feature of geophysical time-series is their essential non-stationarity. The examples quoted in previous sections show that stationarity or quasi-stationarity is only a reasonable assumption over relatively short time-scales, and obvious observational devices have to be adopted to take account of this physical fact.

(iii) The purely statistical problems involved in the analysis of geophysical time series are not different in kind from those arising in other fields.

(iv) It has been said of oceanography that the purpose of the science is "to understand the present, to interpret the past and to predict the future". This applies equally well to other branches of geophysics. On prediction, or forecasting, it appears that purely statistical techniques without regard to physical "determinants", whilst of incidental interest, are likely to have only limited success.

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DISCUSSION ON THE PAPERS

Mr. R. P. WALDO LEWIS: One can often approximately assess the significance of fluctuations in a time series, when mathematical analysis is difficult or impossible, by constructing a set of comparable series, using random numbers. A practical example is considered where an apparently plausible result is shown not to differ in any obvious way from results produced by random processes.

Dr. G. M. JENKINS: I should like to thank the authors of these three papers for a comprehensive account of the numerous applications of spectral analysis now being made in geophysics.

The use of moving average formulae for eliminating trend is not new to statisticians—what is perhaps new is the interpretation in terms of frequency and the use of an almost orthogonal set of moving averages to separate out the contributions into various band widths.

One of the many uses to which spectral analysis is put in the physical sciences is to provide a check on analogue equipment, e.g., a spectrum analyser or harmonic analyser. In this sort of instrument, the power in a frequency range is determined by precisely this method of filtering. Up to the present, results from a harmonic analyser have been compared with a power spectrum analysis whereas it would seem more logical to use a numerical filter with the same frequency response as the instrument itself.

As it has been already pointed out, the advantages of the filtering procedure are:

(i) It gives information about the distribution of power in a band width as opposed to the spectrum approach which gives the average power in a band; in addition it preserves information about phase relationships which the spectrum does not.

(ii) What is more important, the relation between the power at different frequencies may be investigated. In linear oscillations, these are uncorrelated but in non-linear disturbances, there is correlation between the power at the various frequencies which is of considerable interest in a practical problem.

Whilst I think that numerical filtering is a useful addition to the armoury of the time-series analyst, there are a number of points which require theoretical investigation first. In this connection, it is necessary to add a word of warning about interpretation. It is well known that if there is a random component in the series, moving average smoothing results in a residual with spurious periodicities. Since several smoothing operations have been performed before we are left with the contribution due to the higher frequencies, it may be that the effect on this end of the spectrum is quite serious.

Another disconcerting feature of the filters as they stand is that the total ppwer of the output is approximately 0·9 that of the input—this anomaly also creeps into the degrees of freedom where the sums of the degrees of freedom of the filtered outputs is not equal to the total degrees of freedom, although this is not essential since the outputs are correlated. Although the point raised by Mr. Craddock about the testing of a number of variances is well known to statisticians, it is worth pointing out that a test due to Bartlett is available for testing the homogeneity of several variances.

Another question which requires further work is the optimum choice of filter; also how efficient is the method as an estimate of the average power when compared with the periodogram. Another question not raised at this meeting is what would happen if the band width of the filters were narrowed. In the case of harmonic analysers, one obtains surprisingly different answers by narrowing the band width.