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Beyond the Kirchhoff approximation

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The three most successful models for describing scattering from random rough surfaces are the Kirchhoff approximation (KA), the small perturbation method (SPM) and the two-scale roughness (or composite roughness) surface scattering (TSR) models. In this paper it will be shown how these three models can be derived rigorously from one perturbation expansion based on the extinction theorem for scalar waves scattering from perfectly rigid surfaces. It will also be shown how corrections to the Kirchhoff approximation proportional to the surface curvature and higher order derivatives may be obtained. Using these results, the scattering cross section will be derived for various surface models.

1. INTRODUCTION

When a field scatters from a surface which is smooth compared to the field's wavelength, most of the field is scattered in the specular direction. A more precise statement of this phenomenon is that the field momentum transferred by the scattering in the plane tangent to the surface is much smaller than the momentum transferred perpendicular to the surface. In this paper, we make use of this fact to construct a perturbation series for the scattered field in which we use the ratio of the tangent to perpendicular momentum transfers as the small ordering parameter.

We shall limit ourselves to the simple case of the scattering of a scalar wave which is constrained to vanish on the surface. This boundary condition is appropriate for scattering from a perfect conductor, if the scalar wave represents the component parallel to the mean surface plane of an electromagnetic wave, and if depolarization is negligible; from an impenetrable potential barrier, if it represents a solution to the Schroedinger equation; and from perfectly rigid surfaces, if the scalar wave represents a sound wave. We leave the generalization of the method presented here to full electromagnetic scattering to a future publication.

To develop the perturbation series, the extinction theorem [*Waterman*, 1975] for the scattered field will be used. Recently, Nieto-Vesperinas et al. (see *Nieto-Vesperinas and Garcia* [1981] for a summary of their work) have proposed a perturbation series (the "small perturbation method" (SPM) series) for the scattering problem which gives a recursive formula for computing the field perturbation to any order of approximation by using this theorem and the assumption that the surface roughness is much smaller than the field wavelength. Subsequently, *Marvin* [1980] showed that, in the small momentum transfer limit, this series could be partially resummed. This allowed him to give a rigorous derivation of the Kirchhoff approximation. The philosophy of this paper is similar to Marvin's. The main difference is that we do not make use of the SPM series as a starting point but proceed directly to the small momentum transfer approximation. This provides a very simple derivation of the Kirchhoff approximation: it appears as the first term in the perturbation expansion. It is then shown how higher order terms provide corrections due to surface curvature and higher order derivatives. Thus, a systematic way of obtaining corrections to the Kirchhoff, or tangent plane, approximation is obtained using this approach.

The second part of this paper shows that, for surfaces which may be decomposed into roughness smaller than the incident electromagnetic wavelength superimposed on a large scale "smooth" random surface, one can develop a perturbation expansion using two smallness parameters: the momentum transfer and the small roughness surface scale. This is the basic premise of the two-scale roughness scattering models which have appeared in the literature [e.g., *Brown*, 1978; *McDaniel and Gorman*, 1983; *McDaniel*, 1986]. We show that, in

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the appropriate limit, results similar to those in the literature are obtained. This approach contrasts with the more ad hoc methods used in deriving the usual results: in the past, it had to be assumed that the Kirchhoff approximation was a suitable first order approximation to the scattered field. No such assumption is necessary here. Furthermore, a framework is provided for taking curvature corrections into account, as well as higher order terms.

The plan for this paper is as follows: in the second section, notation conventions are established and the results of *Nieto-Vesperinas and Garcia* [1981] and *Marvin* [1980] are summarized. The third section introduces the momentum transfer perturbation expansion while the fourth section applies these results to "smooth" random rough surfaces. The fifth section introduces the two-scale perturbation expansion. Finally, in the sixth section, this expansion is applied to two-scale random rough surfaces and the composite surface roughness model scattering cross section is derived using this perturbation expansion. An appendix establishes conventions and derives some useful results for rough surfaces.

2. A RIGOROUS DERIVATION OF THE KIRCHHOFF APPROXIMATION

Consider a scalar wave ψ scattering from a rough surface whose mean is the x-y plane and whose height above the plane is given by $\xi(x, y)$. Assuming harmonic time dependence $e^{-i\omega t}$, the incident field is assumed to be a plane wave of wavelength $\lambda = 2\pi/k_0$ and wave vector $\mathbf{k}_0$:

$$\psi_0(\mathbf{r}) = \exp(i\mathbf{k}_0 \cdot \mathbf{r}) = \exp(i\boldsymbol{\kappa}_0 \cdot \boldsymbol{\rho} - ip_0 z) \tag{1}$$

where we have introduced the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\kappa}_0$, the projections onto the x-y plane of the position vector, $\mathbf{r} = \boldsymbol{\rho} + z\hat{\mathbf{z}}$, and the incident wave vector, $\mathbf{k}_0 = \boldsymbol{\kappa}_0 + p_0\hat{\mathbf{z}}$. This convention relating (Roman) three-vectors to (Greek) two-vectors in the x-y plane will be maintained throughout. The function $p_0 \equiv p(\boldsymbol{\kappa}_0)$ is defined by

$$p \equiv p(\boldsymbol{\kappa}) \equiv (k_0^2 - \boldsymbol{\kappa} \cdot \boldsymbol{\kappa})^{1/2} \qquad \mathrm{Im}\, p \geq 0 \tag{2}$$

The scalar field satisfies the boundary condition $\psi(\xi) = 0$.

Applying Green's theorem and using the free space Green's function

$$G_0(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\exp(ik_0|\mathbf{r}_1 - \mathbf{r}_2|)}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} \tag{3}$$

one obtains the equations

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \int dS\, G_0(\mathbf{r} - \mathbf{r}_s)\hat{\mathbf{n}} \cdot \nabla\psi(\mathbf{r}_s) \tag{4}$$

for $z \geq \xi(\boldsymbol{\rho})$, and

$$0 = \psi_0(\mathbf{r}) - \int dS\, G_0(\mathbf{r} - \mathbf{r}_s)\hat{\mathbf{n}} \cdot \nabla\psi(\mathbf{r}_s) \tag{5}$$

for $z \leq \xi(\boldsymbol{\rho})$. This last equation is called the "extinction theorem" because it shows that the incident field below the surface is extinguished by the scattered field. The vector $\hat{\mathbf{n}}$, the unit normal to the surface, is given by

$$\hat{\mathbf{n}} = \frac{\hat{\mathbf{z}} - \nabla\xi}{[1 + (\nabla\xi)^2]^{1/2}} \tag{6}$$

In order to solve (4) and (5), it is useful to introduce the Weyl plane wave expansion for G_0 [*Nieto-Vesperinas and Garcia*, 1981]

$$G_0(\mathbf{r}_1 - \mathbf{r}_2) = \frac{i}{2(2\pi)^2} \int d\boldsymbol{\kappa} \exp[i\boldsymbol{\kappa} \cdot (\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)] \cdot \frac{\exp(ip(\kappa)|z_1 - z_2|)}{p(\kappa)} \tag{7}$$

One can write, without loss of generality,

$$\hat{\mathbf{n}} \cdot \nabla\psi(\mathbf{r}_s) = 2if(\boldsymbol{\rho})[1 + (\nabla\xi)^2]^{-1/2} \exp(i\boldsymbol{\kappa}_0 \cdot \boldsymbol{\rho}) \tag{8}$$

where $f(\boldsymbol{\rho})$ is an unknown "source" function. Replacing these two equations into (4) and (5), one obtains a set of equations that can be solved perturbatively [*Nieto-Vesperinas and Garcia*, 1981]:

$$-p_0\delta_{\boldsymbol{\gamma},0} = \frac{1}{(2\pi)^2} \int d\boldsymbol{\rho} \exp[ip\xi(\boldsymbol{\rho})] \exp(-i\boldsymbol{\gamma} \cdot \boldsymbol{\rho}) f(\rho) \tag{9}$$

$$T(\boldsymbol{\kappa}, \boldsymbol{\kappa}_0) = \frac{1}{p(2\pi)^2} \int d\boldsymbol{\rho} \exp[-ip\xi(\boldsymbol{\rho})] \exp(-i\boldsymbol{\gamma} \cdot \boldsymbol{\rho}) f(\boldsymbol{\rho}) \tag{10}$$

where $T(\boldsymbol{\kappa}, \boldsymbol{\kappa}_0)$, the "transition amplitude (T) matrix", is the coefficient in the plane wave expansion of the scattered field

$$\psi(\mathbf{r}) = \psi_0 + \int d\boldsymbol{\kappa} \exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\rho} + ipz)T(\boldsymbol{\kappa}, \boldsymbol{\kappa}_0) \quad (11)$$

We have introduced the vector $\boldsymbol{\gamma} \equiv \boldsymbol{\kappa} - \boldsymbol{\kappa}_0$, which we will call the "momentum transfer in the x-y plane". For a perfectly flat reflecting plane, its magnitude will be zero. For a smoothly varying surface of small slope, or for incidence angle very close to the mean surface normal, the magnitude of $\boldsymbol{\gamma}$ divided by the magnitude of the momentum transfer in the z direction, $(p_0 + p_s)$, will be small.

The scattering problem can be solved perturbatively. Assuming that the product $p(\boldsymbol{\kappa})\xi(\boldsymbol{\rho}) \sim O(\varepsilon) \ll 1$, and that the source function can be expanded in powers of ε:

$$f(\boldsymbol{\rho}) = \sum_{n=0}^{\infty} f^{(n)}(\boldsymbol{\rho}) \frac{\varepsilon^n}{n!} \quad (12)$$

one can obtain iterative solutions for $f^{(n)}$ [*Nieto-Vesperinas and Garcia*, 1981; *Marvin*, 1980],

$$f_{\boldsymbol{\gamma}}^{(0)} = -p_0 \delta_{\boldsymbol{\gamma},0} \quad (13)$$

$$f_{\boldsymbol{\gamma}}^{(n)} = -\sum_{m=1}^{n} \binom{n}{m} (ip)^m [f^{(n-m)}(\boldsymbol{\rho})\xi^m(\rho)]_{\boldsymbol{\gamma}} \qquad n \geq 1 \quad (14)$$

where $[\ \ ]_{\boldsymbol{\gamma}}$ means Fourier transform with respect to $\boldsymbol{\gamma}$; i.e.,

$$f_{\boldsymbol{\gamma}}^{(n)} \equiv [f^{(n)}(\rho)]_{\boldsymbol{\gamma}} = \int d^2\rho \ f^{(n)}(\rho) \exp(-i\boldsymbol{\gamma}\cdot\boldsymbol{\rho}) \quad (15)$$

By following a similar procedure, one can determine the T matrix to the same order of approximation as the source function and, thus, complete the solution to the problem to that order of approximation.

By keeping terms of order up to $n = 2$, the above equations provide the Bragg scattering approximation [*Winebrenner and Ishimaru*, 1985*b*]. However, it is not possible to explicitly invert the Fourier transform for the source function to obtain analytic expressions for it in the spatial domain. *Marvin* [1980] showed that this limitation could be overcome in the small momentum transfer limit. In the first order approximation, p is expanded in powers of the momentum transfer in the x-y plane and only the first two terms are kept

$$p = (p_0^2 - 2\boldsymbol{\kappa}_0\cdot\boldsymbol{\gamma} - \gamma^2)^{1/2} \quad (16)$$

$$p \approx p_0\left(1 - \frac{\boldsymbol{\kappa}_0\cdot\boldsymbol{\gamma}}{p_0^2}\right) \quad (17)$$

The series for $f(\rho)$ can now be summed and, to this order of approximation, the result is given by

$$f(\rho) = -(p_0 + \boldsymbol{\kappa}_0\cdot\nabla\xi)\exp(-ip_0\xi) \quad (18)$$

or, equivalently,

$$\hat{\mathbf{n}}\cdot\nabla\psi(\xi) = 2\hat{\mathbf{n}}\cdot\nabla\psi_0(\xi) \quad (19)$$

which is the Kirchhoff approximation.

Given Marvin's result, an obvious question is whether, by expanding p in powers of the momentum transfer to an arbitrary order m, it might not be possible to obtain a similar closed form expression for the source function. The answer to this conjecture is affirmative, but the procedure is not elegant. In the next section, we will rederive Marvin's result and extend it to higher derivatives by using a different expansion scheme.

3. THE MOMENTUM TRANSFER PERTURBATION EXPANSION

While the method outlined in the previous section provides a systematic way to obtain corrections proportional to curvature and higher order derivatives to Kirchhoff scattering, it is awkward. One assumes that the surface height is small and then, by assuming that the momentum transfer is small, obtains results which are applicable to arbitrary surface heights. It makes more sense to assume initially that the momentum transfer is small and form a perturbation expansion using it as the order parameter. We will do that in this section.

We proceed in a similar fashion to the previous section. However, instead of a source function of the form

$$f(\boldsymbol{\rho}) = -p_0\exp(-ip_0\xi)\sum_0^{\infty}\frac{f^{(n)}(\boldsymbol{\rho})\alpha^n}{n!} \quad (20)$$

we have chosen the alternate ("cumulant expansion") form

$$f(\boldsymbol{\rho}) = -p_0\exp(-ip_0\xi)\exp\left(\sum_1^{\infty}\frac{g^{(n)}(\boldsymbol{\rho})\alpha^n}{n!}\right) \quad (21)$$

where we have introduced the smallness parameter α proportional to the ratio of the momentum transfer in the x-y plane to the momentum transfer in the z direction. Notice that the factor $\exp(-ip_0\xi)$ is explicitly included in both forms of the source functions. That this is correct and necessary for homogeneous rough surfaces has been shown by *Brown* [1982].

There are two main reasons for studying the "cumulant expansion" form for the source function. First, *Lynch* [1970] has shown by using a variational principle that the first corrections to Kirchhoff scattering can be written as the exponential of a phase factor as in (21). In addition, several authors [*Winebrenner and Ishimaru*, 1985*a*, *b*; *Shen and Maradudin*, 1980], have pointed out that the cumulant approach has several advantages, including a clear separation between the coherent and incoherent scattered fields, and a higher rate of convergence than the approach of *Nieto-Vesperinas and Garcia* [1981]. It is our hope that these advantages will also be present in our approach.

Since both expansions must agree to each order in α, the expansion coefficients must be related. Indeed, their relation is the same as that of the moments of a distribution to its cumulants; i.e.,

$$f^{(1)} = g^{(1)} \tag{22}$$

$$f^{(2)} = g^{(2)} + (g^{(1)})^2 \tag{23}$$

$$f^{(3)} = g^{(3)} + 3g^{(2)}g^{(1)} + (g^{(1)})^3 \tag{24}$$

and so on [*Shen and Maradudin*, 1980].

To ease bookkeeping, introduce the function

$$q(\boldsymbol{\gamma}, \alpha) \equiv \left(1 - 2\alpha \frac{\boldsymbol{\kappa}_0}{p_0} \cdot \frac{\boldsymbol{\gamma}}{p_0} - \alpha^2 \frac{\gamma^2}{p_0^2}\right)^{1/2} - 1 \tag{25}$$

Proceeding as in the previous section, we solve term by term for $g^{(n)}$ by equating equal powers of α. This can be done by solving the generating equation

$$\frac{d^n}{d\alpha^n}\left\{\delta_{\boldsymbol{\gamma},0} - \left(\frac{1}{2\pi}\right)^2 \int d^2\rho \exp(-i\boldsymbol{\gamma}\cdot\boldsymbol{\rho}) \cdot \exp\left[(i\xi p_0 q(\boldsymbol{\gamma}, \alpha)\right] \exp\left(\sum_{n=1}^{\infty} \frac{g^{(n)}\alpha^n}{n!}\right)\right\}_{\alpha=0} = 0 \tag{26}$$

for $0 \le n \le \infty$.

The resulting integral equations can be inverted by performing integration by parts and applying the boundary conditions. Since the integrands will only contain polynomials in $\boldsymbol{\kappa}_0 \cdot \boldsymbol{\gamma}$ and γ^2, the inversion can always be accomplished explicitly by replacing $\boldsymbol{\kappa}_0 \cdot \boldsymbol{\gamma}$ by $-i\boldsymbol{\kappa}_0 \cdot \nabla$ and γ^2 by $-\nabla^2$, and carefully taking into account operator ordering. This is in contrast to the SPM where noninteger powers of $\boldsymbol{\gamma}$ are involved and the Fourier transforms cannot be inverted analytically. Solving for the first three terms, we have

$$g^{(1)} = \frac{\boldsymbol{\kappa}_0}{p_0} \cdot \nabla\xi \tag{27}$$

$$g^{(2)} = -\frac{i}{p_0}\left[\nabla^2\xi + \left(\frac{\boldsymbol{\kappa}_0}{p_0} \cdot \nabla\right)^2 \xi\right] - \left(\frac{\boldsymbol{\kappa}_0}{p_0} \cdot \nabla\xi\right)^2 \tag{28}$$

$$\begin{aligned} g^{(3)} = {} & 3\left[\left(\frac{\boldsymbol{\kappa}_0 \cdot \nabla}{p_0^2}\right)\left(\frac{\nabla^2}{p_0^2}\right) p_0\xi\right] + 3\left[\left(\frac{\boldsymbol{\kappa}_0 \cdot \nabla}{p_0^2}\right)^3 p_0\xi\right] \\ & + 2\left[\frac{\boldsymbol{\kappa}_0 \cdot \nabla\xi}{p_0^2}\right]^3 + 3i\left[\frac{\boldsymbol{\kappa}_0 \cdot \nabla\xi}{p_0}\left(\frac{\nabla^2}{p_0}\xi\right)\right] \\ & + 3i\left[\left(\nabla\xi \cdot \frac{\nabla}{p_0}\right)\left(\frac{\boldsymbol{\kappa}_0 \cdot \nabla\xi}{p_0}\right)\right] \\ & + 9i\left[\left(\frac{\boldsymbol{\kappa}_0 \cdot \nabla}{p_0^2} p_0\xi\right)\left(\frac{\boldsymbol{\kappa}_0 \cdot \nabla}{p_0^2}\right)^2 p_0\xi\right] \end{aligned} \tag{29}$$

If we recall that $|\boldsymbol{\kappa}_0/p_0| = \tan\theta_0$, where θ_0 is the incidence angle with respect to the z direction, we see that, to order α, we obtain the Kirchhoff approximation by stopping at the $n = 1$ term.

The $n = 2$ term can be cast in a form more amenable to geometrical interpretation. Let us assume for the moment that the surface slope is small, so that we may ignore terms proportional to the slope squared. By choosing our coordinate system to lie along the principal directions, $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ of the surface [*O'Neill*, 1966], we will have that $\partial_{uv}\xi(u, v) = 0$. In this coordinate, the $n = 2$ term can be written as

$$\begin{aligned} \frac{-i}{p_0}\left[\nabla^2\xi + \left(\frac{\boldsymbol{\kappa}_0}{p_0} \cdot \nabla\right)^2 \xi\right] = {} & \frac{-i}{k\cos\theta_0} \cdot [(1 + \tan^2\theta_0 \cos^2\phi)\partial_u^2 \\ & + (1 + \tan^2\theta_0 \sin^2\phi)\partial_v^2]\xi(u, v) \end{aligned} \tag{30}$$

where ϕ is the angle between the direction $\boldsymbol{\kappa}_0/\kappa_0$ and the direction $\hat{\mathbf{u}}$; and we have also used $p_0 = k_0\cos\theta_0$. For small slopes, however, the principal radii of curvature of the surface are given by $1/R_u =$

$\partial_u^2 \xi(u, v)$ and $1/R_v = \partial_v^2 \xi(u, v)$. Therefore, we can write

$$g^{(2)} = \frac{-i}{k_0 \cos^3 \theta_0} \left(\frac{\cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \phi}{R_u} + \frac{\cos^2 \theta_0 + \sin^2 \theta_0 \sin^2 \phi}{R_v} \right) \tag{31}$$

For $\xi = \xi(u)$ and $\phi = 0$ the above expression reduces exactly to the one obtained by *Lynch* [1970] for the one-dimensional surface.

In the past, there has been some discussion in the literature concerning the regime of applicability of the Kirchhoff approximation. *Brekhovskikh* [1952], for instance, takes $2k_0 R \cos \theta_0 \gg 1$ to be a sufficient condition. *Lynch* (1970], on the other hand, proposes the less stringent condition $2k_0 R \cos^3 \theta_0 \gg 1$. It can be seen from (31) that both of these conditions are special cases of a more general condition, namely, $g^{(2)}/2 \ll 1$.

In general, a necessary condition for stopping the perturbation expansion at the nth term is that $g^{(m)} \ll 1$ for $m > n$. If the angle of incidence is not too large, the nth order term will be of the same order of magnitude as the ratio of the nth derivative of the surface height to the wavelength raised to the power $n - 1$. Thus, the magnitude of this ratio should provide a good rule of thumb for terminating the series. For near-nadir incidence, one can ignore the odd terms of the expansion since they are proportional to $\kappa_0/p_0 = \tan \theta_0$, which vanishes for nadir incidence. A necessary criterion for stopping at $g^{(2)}$ for near nadir incidence is then given by

$$|g^{(4)}(\theta_0 = 0)| = \left| 3\left(\frac{\nabla^2}{p_0} \xi\right)^2 + 3i\left[\left(\frac{\nabla^2}{p_0^2}\right)^2 p_0 \xi\right] + 6 \frac{\nabla^2}{p_0^2} (\nabla \xi)^2 \right| \ll 1 \tag{32}$$

For angles of incidence which are very far from the nadir, on the other hand, the perturbation expansion will be dominated by terms of the form $1/\cos^n \theta_0$ which become singular as θ_0 approaches grazing incidence. Therefore, one does not expect this perturbation expansion to be applicable for these incidence angles. Physically, this is reasonable since the assumption behind this perturbation scheme is that the momentum transfer is small. For near-grazing incidence, however, this is seldom the case. Most surfaces present a strong diffuse component due to the emergence of other scattering processes such as multiple scattering and shadowing. The perturbation expansion based on the extinction theorem does theoretically include all these contributions [*Nieto-Vesperinas and Garcia*, 1981], if enough terms are retained. However, for this perturbation expansion, it is not expected that retaining a sufficient number of terms is a practical possibility.

4. APPLICATION TO ROUGH SURFACES

In this section, the expansion developed in the previous chapter will be applied to random rough surfaces. The appendix presents in detail the rough surface models which will be used. The major assumptions will be the following: both the surface and its derivatives will be zero mean random variables, and the correlation between surface height and slope is small enough to be ignored. This last assumption is true of Gaussian surfaces, but may not always hold for non-Gaussian surfaces; for example, due to nonlinear interactions, the ocean surface height and slope correlation may not be ignored for certain sea states.

The scattered field in the far region is given by

$$\psi_s = \frac{-2ie^{ikr_s}}{4\pi r_s} \int d^2\rho \exp\left[-i\gamma_d \cdot \rho\right] \exp\left[-ip_s \xi\right] f(\rho) \tag{33}$$

$$\gamma_d = \kappa_s - \kappa_0 \tag{34}$$

where r_s is the distance from the scattering center to the field point, and the subscript s refers to the scattered field. The coherent scattered field is then given by

$$\langle \psi_s \rangle = \frac{-2ie^{ikr_s}}{4\pi r_s} \int d^2\rho \exp\left[-i\gamma_d \cdot \rho\right] \langle \exp\left[-ip_s \xi\right] f(\rho) \rangle \tag{35}$$

where angular brackets denote ensemble averaging. The normalized bistatic scattering cross section is given by

$$\sigma_0(\kappa_0, \kappa_s) = \frac{4\pi r_s^2}{A} \left(\langle \psi_s \psi_s^* \rangle - \langle \psi_s \rangle \langle \psi_s^* \rangle \right) \tag{36}$$

where A is the area of the scattering surface. The first average in this expression, the mean scattered intensity, can be written as

$$\langle\psi_s\psi_s^*\rangle = \frac{1}{4\pi^2 r_s^2}\int d^2\rho_1 \int d^2\rho_2 \exp\left[-i\boldsymbol{\gamma}_d\cdot\boldsymbol{\rho}_d\right]$$

$$\cdot\langle\exp\{-ip_s[\xi(\boldsymbol{\rho}_1)-\xi(\boldsymbol{\rho}_2)]\}f(\boldsymbol{\rho}_1)f^*(\boldsymbol{\rho}_2)\rangle \quad (37)$$

where $\boldsymbol{\rho}_d = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$.

To perform these averages, we make use of the following property of cumulant expansions [*Shen and Maradudin*, 1980; *Winebrenner and Ishimaru*, 1985*b*]: if G is a random variable, then

$$\langle\exp(iG)\rangle = \exp\left[\sum_{n=1}^{\infty}\frac{i^n}{n!}\lambda_n\right] \quad (38)$$

where λ_n is the nth order cumulant of G. For two random variables, $G^{(1)}$ and $G^{(2)}$, the analogous expression is given by

$$\langle\exp[i(G^{(1)}+G^{(2)})]\rangle = \exp\left[\sum_{n=1}^{\infty} i^n \sum_{m=0}^{n}\frac{\lambda_{n-m,m}}{(n-m)!n!}\right] \quad (39)$$

where $\lambda_{n,m}$ represents the joint cumulant of order n, m.

To calculate the coherent field, let

$$G \approx -(p_0+p_s)\xi - ig^{(1)} - i\frac{g^{(2)}}{2}$$

To maintain consistency in the level of approximation, one must retain cumulants which are of order α^2 or less. These are given by

$$\lambda_1 = \langle G\rangle \approx \frac{i}{2}\left\langle\left(\frac{\boldsymbol{\kappa}_0}{p_0}\cdot\nabla\xi\right)^2\right\rangle \quad (40)$$

$$\lambda_2 = \langle G^2\rangle - \langle G\rangle^2 \approx \langle G^2\rangle \approx \Bigg\langle (p_0+p_s)^2\xi^2 + \frac{(p_0+p_s)\xi}{p_0}$$

$$\cdot\left[\nabla^2\xi + \left(\frac{\boldsymbol{\kappa}_0}{p_0}\cdot\nabla\right)^2\xi - \left(\frac{\boldsymbol{\kappa}_0}{p_0}\cdot\nabla\xi\right)^2\right]\Bigg\rangle \quad (41)$$

A straightforward calculation then yields

$$\langle\exp[-ip_s\xi]f(\boldsymbol{\rho})\rangle$$

$$= \exp\left[\frac{-1}{2}\sigma^2(p_0+p_s)^2\left(1+\frac{\Gamma(0)}{p_0(p_0+p_s)}\right)\right] \quad (42)$$

where σ is the standard deviation of the surface, and

$$\Gamma(\boldsymbol{\rho}_d) = \left[\nabla^2 + \left(\frac{\boldsymbol{\kappa}_0}{p_0}\cdot\nabla\right)^2\right]C(\boldsymbol{\rho}_d) \quad (43)$$

where $C(\boldsymbol{\rho}_d)$ is the correlation function for the surface.

The integral in (35) can now be performed to yield the following expression for the coherent scattered field

$$\langle\psi_s\rangle = -i2\pi\frac{e^{ikr_s}}{r_s}p_0\delta(\boldsymbol{\gamma}_d)\exp\left[-2(\sigma p_0)^2\left(1+\frac{\Gamma(0)}{2p_0^2}\right)\right] \quad (44)$$

so that we may identify

$$R = \exp\left[-2(\sigma p_0)^2\left(1+\frac{\Gamma(0)}{2p_0^2}\right)\right] \quad (45)$$

as the coherent reflection coefficient.

Since Γ is proportional to the second derivative of the surface height, $\Gamma(0)$ must be negative for homogeneous surfaces (see the appendix). Recalling that the coherent reflection coefficient predicted by the Kirchhoff approximation is given by $\exp[-2(\sigma p_0)^2]$, one sees that the effect of including curvature terms in the scattered field is to increase its coherency. Physically, this is a sensible result. One expects that the effect of the finite resolution of the impinging field is to "smooth" the actual surface [*Hagfors*, 1966; *Tyler*, 1976]. Specular points which are separated by distances smaller than one wavelength must merge into one larger effective specular point.

As a special case of the previous formula, consider homogeneous, isotropic surfaces. In this case, the correlation function is a function of the magnitude of the separation vector, ρ_d, alone. In the appendix it is shown that for this class of surfaces, one can express $\Gamma(0)$ as

$$\Gamma(0) = \frac{C^{(2)}(0)}{L^2}\left(\frac{1}{\cos^2\theta_0}+1\right) \quad (46)$$

where L is a length characteristic of the surface spectrum and $C^{(2)}$ is the second derivative of the correlation function with respect to the dimensionless parameter $\delta = \rho_d/L$. Explicit expressions for $C^{(2)}$ and L for Gaussian and power law spectra are presented in the appendix.

To calculate the bistatic scattering cross section, let

$$G^{(1)} \approx -(p_0 + p_s)\xi(\boldsymbol{\rho}_1) - ig^{(1)}(\boldsymbol{\rho}_1) - i\,\frac{g^{(2)}(\boldsymbol{\rho}_1)}{2}$$

$$G^{(2)} \approx (p_0 + p_s)\xi(\boldsymbol{\rho}_2) - ig^{(1)*}(\boldsymbol{\rho}_2) - i\,\frac{g^{(2)*}(\boldsymbol{\rho}_2)}{2}$$

To calculate the average in (37) consistent with the level of approximation chosen, one needs to calculate the cumulants

$$\lambda_{10} = \lambda_{01}^* = \langle G^{(1)}\rangle = \langle (G^{(2)})^*\rangle = \lambda_1 \tag{47}$$

$$\lambda_{20} = \lambda_{02}^* = \langle (G^{(1)})^2\rangle - \langle (G^{(1)})\rangle^2 = \lambda_2 \tag{48}$$

$$\lambda_{11} = \langle G^{(1)}G^{(2)}\rangle - \langle G^{(1)}\rangle\langle G^{(2)}\rangle \approx \langle G^{(1)}G^{(2)}\rangle \tag{49}$$

The first two sets of cumulants have already been calculated. The last cumulant can be shown to be given by

$$\lambda_{11} = -(p_0 + p_s)^2\sigma^2 C(\boldsymbol{\rho}_d) - \frac{(p_0 + p_s)\sigma^2}{p_0}\,\Gamma(\boldsymbol{\rho}_d) + \sigma^2\Lambda(\boldsymbol{\rho}_d) \tag{50}$$

where

$$\Lambda(\boldsymbol{\rho}_d) = -\left(\frac{\boldsymbol{\kappa}_0}{p_0}\cdot\nabla\right)^2 C(\boldsymbol{\rho}_d) \tag{51}$$

Making the change of variables $\boldsymbol{\rho}_s = (\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)/2$, $\boldsymbol{\rho}_d = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$, the ρ_s integral in (37) can easily be performed to yield

$$\frac{p_0^2}{\pi}\int d^2\boldsymbol{\rho}_d \exp(-i\boldsymbol{\gamma}_d\cdot\boldsymbol{\rho}_d)\exp[\sigma^2\Lambda(\boldsymbol{\rho}_d)] \cdot \exp\{-(p_0 + p_s)^2\sigma^2(1 - C(\boldsymbol{\rho}_d)]\} \cdot \exp\left[-\frac{(p_0 + p_s)\sigma^2}{p_0}[\Gamma(0) - \Gamma(\boldsymbol{\rho}_d)]\right] \tag{52}$$

Further assuming that the surface is isotropic yields

$$2p_0^2\int d\rho_d\,\rho_d\,J_0(\gamma_d\rho_d)\exp[\sigma^2\Lambda(\rho_d)] \cdot \exp\{-(p_0 + p_s)^2\sigma^2[1 - C(\rho_d)]\} \cdot \exp\left[-\frac{(p_0 + p_s)\sigma^2}{p_0}[\Gamma(0) - \Gamma(\rho_d)]\right] \tag{53}$$

where J_0 is the zeroth order Bessel function of the first kind.

As with the Kirchhoff approximation, this integral cannot be further reduced for a general surface. However, an often encountered situation in natural surfaces is that the rms surface height is much greater than the wavelength of the field (deep phase modulation). Assuming that for this case $(p_0 + p_s)^2\sigma^2$ is large enough so that the coherent component of the scattered field can be ignored and, if for

$$\delta^2 < 2/[(p_0 + p_s)^2\sigma^2|C^{(2)}|]$$

the correlation coefficient is well approximated by

$$C(\delta) \approx 1 - |C^{(2)}|\,\frac{\delta^2}{2}$$

and, also, the condition

$$\frac{|C^{(4)}|}{|C^{(2)}|(p_0 + p_s)p_0} \ll 1$$

holds, then one can approximate the bistatic cross section by

$$\sigma_0 = 2p_0^2\int d\rho_d\,\rho_d J_0(\gamma_d\rho_d)\exp\left[-(p_0 + p_s)^2\sigma^2|C^{(2)}| \cdot \left(1 - \frac{1/\cos^2\theta_0 + 1/3}{(p_0 + p_s)p_0L^2}\,\frac{C^{(4)}}{|C^{(2)}|}\right)\frac{\rho_d^2}{2L^2}\right] \tag{54}$$

where a term proportional to $(\kappa_0^2/p_0^2)/(p_0 + p_s)^2L^2$ has been neglected. The integral can now be easily evaluated and the result is

$$\sigma_0(\boldsymbol{\kappa}_0, \boldsymbol{\kappa}_s) = 2p_0^2\left(\frac{1}{(p_0 + p_s)^2\langle|\nabla\xi|^2\rangle_{\text{eff}}}\right) \cdot \exp\left[\frac{-\gamma_d^2}{2(p_0 + p_s)^2\langle|\nabla\xi|^2\rangle_{\text{eff}}}\right] \tag{55}$$

where $\langle|\nabla\xi|^2\rangle_{\text{eff}}$, the effective rms slope, is given by

$$\langle|\nabla\xi|^2\rangle_{\text{eff}} = \langle|\nabla\xi|^2\rangle\left(1 - \frac{1/\cos^2\theta_0 + 1/3}{(p_0 + p_s)p_0L^2}\,\frac{C^{(4)}}{|C^{(2)}|}\right) \tag{56}$$

The ratio $|C^{(4)}/C^{(2)}|$ is calculated in the appendix for Gaussian and power law spectra.

For backscattering, $\boldsymbol{\gamma}_d = 2\boldsymbol{\kappa}_0$, $p_s = p_0$, so the backscattering cross section can be written as

$$\sigma_0(\boldsymbol{\kappa}_0, -\boldsymbol{\kappa}_0) = \frac{1}{2\langle|\nabla\xi|^2\rangle_{\text{eff}}}\exp[-\tan^2\theta_0/2\langle|\nabla\xi|^2\rangle_{\text{eff}}] \tag{57}$$

Comparing this result with the usual Kirchhoff result, we see that the effect of including wavelength dependent effects is to replace the rms surface slope with an effective, smaller, rms surface slope. This is consistent with the effect inferred from the coherent reflection coefficient: the field scatters from an effective surface which is smoother than the actual surface. A similar result was obtained by *Lynch* [1970] for the one-dimensional case.

5. COMPOSITE SURFACE PERTURBATION EXPANSION

For natural surfaces, it is often the case that the surface is too rough for the momentum transfer perturbation expansion to apply. However, it is frequently possible to split the surface into two components

$$\xi(\boldsymbol{\rho}) = \eta(\boldsymbol{\rho}) + \zeta(\boldsymbol{\rho}) \tag{58}$$

such that $\eta(\boldsymbol{\rho})$ is smooth enough so that the Kirchhoff approximation is valid for it; and $\zeta(\boldsymbol{\rho})$ is such that $(p_s - p_0)\zeta \ll 1$. In this case, one can introduce an additional smallness parameter, $\beta \sim O[(p_s - p_0)\zeta]$, and rewrite (9) as

$$-p_0 \delta_{\gamma,0} = \frac{1}{(2\pi)^2} \int d\boldsymbol{\rho} \exp(-i\boldsymbol{\gamma} \cdot \boldsymbol{\rho}) \exp(ip_0 \xi) \cdot \exp[iq(\alpha)p_0 \eta] \exp[i\beta(p_s - p_0)\zeta] f(\boldsymbol{\rho}) \tag{59}$$

Proceeding in analogy with the previous sections, introduce the expansion for the source function

$$f(\boldsymbol{\rho}) = -p_0 \exp(-ip_0 \xi) \exp\left(\sum_{n=0,\, m=0}^{\infty} \frac{g^{(n,m)}(\boldsymbol{\rho})\alpha^n \beta^m}{n!m!} \right) \tag{60}$$

One can then solve for $g^{(n,m)}$ term by term, as before. Proceeding in this fashion, the first few terms are given by

$$g^{(0,0)} = 0 \tag{61}$$

$$g^{(1,0)} = g^{(1)} \tag{62}$$

$$g^{(2,0)} = g^{(2)} \tag{63}$$

$$g^{(0,1)} = \frac{-i}{(2\pi)^2} \int d^2\kappa_1 \exp(i\boldsymbol{\kappa}_1 \cdot \boldsymbol{\rho})[p'(\boldsymbol{\kappa}_1) - p_0]\bar{\zeta}(\boldsymbol{\kappa}_1) \tag{64}$$

$$g^{(0,2)} = \frac{-1}{(2\pi)^4} \int d^2\kappa_1 \exp(i\boldsymbol{\kappa}_1 \cdot \boldsymbol{\rho}) \int d^2\kappa_2 \bar{\zeta}(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2)\bar{\zeta}(\boldsymbol{\kappa}_2) \cdot \{2[p'(\boldsymbol{\kappa}_2) - p_0][p'(\boldsymbol{\kappa}_1) - p_0] - [p'(\boldsymbol{\kappa}_1) - p_0]^2 - [p'(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2) - p_0][p'(\boldsymbol{\kappa}_2) - p_0]\} \tag{65}$$

$$g^{(1,1)} = \frac{-i}{(2\pi)^2} \int d^2\kappa_1 \exp(i\boldsymbol{\kappa}_1 \cdot \boldsymbol{\rho})[p'(\boldsymbol{\kappa}_1) - p_0] \int d^2\rho_1 \cdot \exp(-i\boldsymbol{\kappa}_1 \cdot \boldsymbol{\rho}_1)\left(\eta(\boldsymbol{\rho}) \frac{\boldsymbol{\kappa}_0 \cdot \boldsymbol{\kappa}_1}{p_0} + \eta(\boldsymbol{\rho}_1) \frac{\boldsymbol{\kappa}_0 \cdot \nabla_1}{p_0} \right)\zeta(\boldsymbol{\rho}_1) \tag{66}$$

where the shorthand notation $p'(\boldsymbol{\kappa}) = p(\boldsymbol{\kappa}_0 + \boldsymbol{\kappa})$ has been used. We have also introduced the symbolic notation

$$\bar{\zeta}(\boldsymbol{\kappa}) = \int d^2\rho \exp(-i\boldsymbol{\kappa} \cdot \boldsymbol{\rho})\zeta(\boldsymbol{\rho}) \tag{67}$$

Strictly speaking, this integral does not converge for infinite homogeneous surfaces. However, in what follows we shall only be interested in the moments of the scattered field which involve only the moments of $\bar{\zeta}(\boldsymbol{\kappa})$. These moments are well defined. A rigorous spectral representation of the random surface ζ can be obtained in terms of Fourier-Stieltjes integrals [*Yaglom*, 1973]. Since the answers obtained by using this method are the same as those obtained using our symbolic notation, the symbolic notation will be retained for the sake of simplicity.

One can show that in the limit $\eta \to 0$, the results obtained are equivalent (to order β^2) to those obtained by *Winebrenner and Ishimaru* [1985*b*]. They have shown that for $p_s\zeta \ll 1$, one obtains Bragg scattering as the dominant scattering mechanism. On the other hand, it is obvious that in the limit $\eta \to 0$, one obtains the momentum transfer expansion developed in the previous sections. Hence, the expansion method advocated here has the limiting behavior desired of the two-scale approximation. As we shall see in the next section, the behavior of the scattering cross section for random surfaces is also similar to the one derived by more ad hoc methods. It should be noticed that the perturbation expansion developed here also includes coupling terms between the large scale and small scale currents through terms like $g^{(1,1)}$ and higher order terms of the form $g^{(n,m)}$ $(n, m \neq 0)$.

This is in contrast to the usual two-scale theories which neglect this interaction.

6. APPLICATION TO TWO-SCALE ROUGH SURFACES

To apply the expansion developed in the previous section to random rough surfaces, let us assume that statistically the surface ξ can be treated as the sum of two zero mean random variables, η and ζ, such that the first is smooth enough that the momentum transfer expansion is applicable, and the rms amplitude of the second is small compared to the field wavelength. In order to simplify the mathematics, let us further assume that these two variables are not correlated. This implies that the surface correlation function and surface spectrum can be written as

$$\sigma_\xi^2 C_\xi(\boldsymbol{\rho}_d) = \sigma_\eta^2 C_\eta(\boldsymbol{\rho}_d) + \sigma_\zeta^2 C_\zeta(\boldsymbol{\rho}_d) \tag{68}$$

$$\sigma_\xi^2 W_\xi(\boldsymbol{\kappa}) = \sigma_\eta^2 W_\eta(\boldsymbol{\kappa}) + \sigma_\zeta^2 W_\zeta(\boldsymbol{\kappa}) \tag{69}$$

A natural way to accomplish this split for Gaussian surfaces is to split the spectral expansion of the random process ξ at a cutoff wave number κ_c. Then η and ζ are defined as the low and high frequency components of ξ, respectively. For this approach to be viable, there must exist a cutoff frequency such that both of the scattering conditions required by the two-scale expansion are simultaneously satisfied.

The calculation of the scattered field proceeds in complete analogy to the calculations presented in section 4. We will assume that the smallness parameters α and β are of the same order and carry out the calculations to order α^2. In analogy to section 4, we define

$$G_\eta(\boldsymbol{\rho}) = -(p_0 + p_s)\eta(\boldsymbol{\rho}) - i\left(g^{(1,0)}(\boldsymbol{\rho}) + \tfrac{1}{2} g^{(2,0)}(\boldsymbol{\rho})\right) \tag{70}$$

$$G_\zeta(\boldsymbol{\rho}) = -(p_0 + p_s)\zeta(\boldsymbol{\rho}) - i\left(g^{(0,1)}(\boldsymbol{\rho}) + \tfrac{1}{2} g^{(0,2)}(\boldsymbol{\rho})\right) \tag{71}$$

$$G_{\eta\zeta}(\boldsymbol{\rho}) = -ig^{(1,1)}(\boldsymbol{\rho}) \tag{72}$$

so that

$$G_\xi^{(1)} = G_\eta(\boldsymbol{\rho}_1) + G_\zeta(\boldsymbol{\rho}_1) + G_{\eta\zeta}(\boldsymbol{\rho}_1) \equiv G_\eta^{(1)} + G_\zeta^{(1)} + G_{\eta\zeta}^{(1)} \tag{73}$$

$$G_\xi^{(2)} = -[G_\eta(\boldsymbol{\rho}_2) + G_\zeta(\boldsymbol{\rho}_2) + G_{\eta\zeta}(\boldsymbol{\rho}_2)]^*$$
$$\equiv G_\eta^{(2)} + G_\zeta^{(2)} + G_{\eta\zeta}^{(2)} \tag{74}$$

To order α^2 of approximation, the coupling between the surfaces η and ζ appears only through the averages $\langle G_{\eta\zeta}\rangle$, $\langle \eta G_{\eta\zeta}\rangle$, and $\langle \zeta G_{\eta\zeta}\rangle$, or their complex conjugates. However, since we have assumed that η and ζ are uncorrelated zero mean variables, these averages can easily be seen to vanish. This implies that all averages can be split into products of η and ζ averages, which will be written as $\langle\ \rangle_\eta$ and $\langle\ \rangle_\zeta$ respectively. Thus the expression for the mean scattered field becomes

$$\langle \psi_s \rangle = \frac{-ip_0 e^{ikr_s}}{2\pi r_s} \int d^2\rho_1 \exp[-i\boldsymbol{\gamma}_d \cdot \boldsymbol{\rho}_1] \cdot \langle \exp(iG_\eta^{(1)})\rangle_\eta \langle \exp(iG_\zeta^{(1)})\rangle_\zeta \tag{75}$$

Similarly, the expression for the mean scattered intensity becomes

$$\langle \psi_s \psi_s^* \rangle = \frac{p_0^2}{4\pi^2 r_s^2} \int d^2\rho_1 \int d^2\rho_2 \exp[-i\boldsymbol{\gamma}_d \cdot \boldsymbol{\rho}_d] \cdot \langle \exp[i(G_\eta^{(1)} + G_\eta^{(2)})]\rangle_\eta \langle \exp[i(G_\zeta^{(1)} + G_\zeta^{(2)})]\rangle_\zeta \tag{76}$$

The averages involving η have already been calculated in section 4. A straightforward calculation shows that

$$\langle \exp(iG_\zeta^{(1)})\rangle_\zeta = \exp\left[-\frac{1}{2}(p_0 + p_s)^2\sigma_\zeta^2 - \frac{\sigma_\zeta^2(p_0 + p_s)}{(2\pi)^2} \cdot \int d^2\kappa_1 (p_1' - p_0) W(\kappa_1)\right] \tag{77}$$

and

$$\langle \exp[i(G_\zeta^{(1)} + G_\zeta^{(2)})]\rangle_\zeta = L(p_0, p_s) \exp[-(p_0 + p_s)^2 \sigma_\zeta^2 C_\zeta(\boldsymbol{\rho}_d)] \cdot \exp\left\{\frac{\sigma_\zeta^2}{(2\pi)^2} \int d^2\kappa_1 \exp(i\boldsymbol{\kappa}_1 \cdot \boldsymbol{\rho}_d) W(\kappa_1)(p_1' - p_0) \cdot [(p_1' - p_0)^* + 2(p_0 + p_s)]\right\} \tag{78}$$

$$\langle \exp [i(G_\zeta^{(1)} + G_\zeta^{(2)})] \rangle_\zeta = L(p_0, p_s) \cdot \exp \left[\frac{\sigma_\zeta^2}{(2\pi)^2} \int d^2\kappa_1 \exp (i\boldsymbol{\kappa}_1 \cdot \boldsymbol{\rho}_d) W(\kappa_1) |(p_1' + p_s)|^2 \right] \tag{79}$$

where $p_1' = p'(\boldsymbol{\kappa}_1)$, and

$$L(p_0, p_s) = \exp \left[-(p_0 + p_s)^2 \sigma_\zeta^2 - 2 \operatorname{Re} \left(\frac{\sigma_\zeta^2 (p_0 + p_s)}{(2\pi)^2} \int d^2\kappa_1 W(\kappa_1)(p_1' - p_0) \right) \right] \tag{80}$$

is an attenuation factor which is independent of position.

From these results, it follows that the coherent reflection coefficient, $R_{\eta\zeta}$, for the two-scale surface is given by

$$R_{\eta\zeta} = \exp [-2p_0^2(\sigma_\eta^2 + \sigma_\zeta^2)] \exp [-\sigma_\eta^2 \Gamma_\eta(0)] \cdot \exp \left[-\frac{2p_0\sigma_\zeta^2}{(2\pi)^2} \int d^2\kappa_1 (p_1' - p_0) W(\kappa_1) \right] \tag{81}$$

One recognizes the first exponential as the coherent reflection coefficient in the Kirchhoff approximation for scattering from the surface ξ. The second exponential is due to curvature effects of the large scale surface. The last exponential is the additional contribution due to the small scale surface. It can be seen that this expression reduces to the one which includes curvature corrections derived in section 4 in the high frequency limit by expanding the difference $(p_1' - p_0)$ about p_0 and retaining terms to order α^2. One concludes that the effect of the small scale surface on the coherent reflection coefficient is to induce an additional attenuation which is, to first order, equal to that predicted by the Kirchhoff approximation. A numerical evaluation of the coherent reflection coefficient in the limit of vanishing η is presented by *Winebrenner and Ishimaru* [1985*b*]. They conclude that, for smooth surfaces, the reflection coefficient is nearly identical to that predicted by the Kirchhoff approximation. For rougher surfaces, on the other hand, significant differences may be introduced by keeping the higher order terms.

As in section 4, the scattering cross section cannot be computed analytically without making further assumptions about the surface. To compare with the standard results on two-scale surfaces, we will assume, as in section 4, that the deep phase modulation approximation applies; i.e., $(p_0 + p_s)^2\sigma_\eta^2 \gg 1$. In this approximation, we may neglect the coherent component of the scattered field in the calculation of the scattering cross section. In addition, we will assume that the small scale surface is much smaller than the field wavelength so that the exponential in (79) may be expanded. Keeping terms of order β^2 and lower, interchanging the orders of integration and proceeding as in section 4, one obtains

$$\sigma_0(\boldsymbol{\kappa}_0, \boldsymbol{\kappa}_s) = \sigma_0^{(1)}(\boldsymbol{\kappa}_0, \boldsymbol{\kappa}_s) + \sigma_0^{(2)}(\boldsymbol{\kappa}_0, \boldsymbol{\kappa}_s) \tag{82}$$

where

$$\sigma_0^{(1)}(\boldsymbol{\kappa}_0, \boldsymbol{\kappa}_s) = 2p_0^2 \left(\frac{L(p_0, p_s)}{(p_0 + p_s)^2 \langle |\nabla\eta|^2 \rangle_{\text{eff}}} \right) \cdot \exp \left[\frac{-\gamma_d^2}{2(p_0 + p_s)^2 \langle |\nabla\eta|^2 \rangle_{\text{eff}}} \right] \tag{83}$$

$$\sigma_0^{(1)}(\boldsymbol{\kappa}_0, \boldsymbol{\kappa}_s) = \frac{2p_0^2}{(2\pi)^2} \left(\frac{L(p_0, p_s)}{(p_0 + p_s)^2 \langle |\nabla\eta|^2 \rangle_{\text{eff}}} \right) \cdot \int d^2\kappa_1 |p_s + p_1'|^2 W(\kappa_1) \cdot \exp \left[\frac{-(\boldsymbol{\gamma}_d - \boldsymbol{\kappa}_1)^2}{2(p_0 + p_s)^2 \langle |\nabla\eta|^2 \rangle_{\text{eff}}} \right] \tag{84}$$

We recognize $\sigma^{(1)}$ as the specular cross section for the surface η attenuated by the factor $L(p_0, p_s)$ due to the small scale roughness. The second term is similar in structure (aside from the curvature corrections due to the large scale surface) to the two-scale cross sections derived by previous authors [*McDaniel and Gorman*, 1983; *McDaniel*, 1986; *Brown*, 1978]. However, these authors start with the equivalent of the Kirchhoff approximation and obtain a factor of $(p_0 + p_s)^2$ inside the integral instead of $|p_s + p_1'|^2$ (in effect, the last integral inside the exponential in (78) is ignored). In the limit of vanishing large scale surface ($\langle |\nabla\eta|^2 \rangle_{\text{eff}} = 0$), both approximations give the correct Bragg backscatter cross section, as can be easily seen. However, for general bistatic scattering, our approach gives the correct scattering cross section

$$\sigma_0 = \frac{4k_0^4 \cos^2\theta_0 \cos^2\theta_s\, \sigma_\zeta^2 W(\gamma_d)}{\pi} \tag{85}$$

The approach which starts from the Kirchhoff approximation, on the other hand, results in the bistatic cross section given by

$$\sigma_0 = \frac{k_0^4(\cos\theta_0 + \cos\theta_s)^2\sigma_\zeta^2 W(\boldsymbol{\gamma}_d)}{\pi} \tag{86}$$

which does not contain the appropriate angular factors.

This disagreement is a reflection of the well known fact that the source function (or surface current) for the Kirchhoff approximation does not coincide with the source function for the SPM [*Marvin*, 1980]. However, as can be easily seen, these two approximations do coincide in the appropriate shared region of validity (small surface height, small momentum transfer) if the second-order corrections obtained in section 4 are added to the Kirchhoff approximation. This is a further confirmation of the basic soundness of the approach put forward in this paper.

The difference for the formula derived here and that obtained from the Kirchhoff approximation is not readily apparent for small large-scale slopes, the case for which most empirical data are available. In this limit, the exponential term in (84) restricts the contributions to the integral to values of the wave vector such that $\boldsymbol{\kappa}_1 \approx \boldsymbol{\gamma}_d$, or $p_1' \approx p_s$. Then, to first order of approximation, both formulas coincide. This will not be the case, however, when the large scale surface has large rms slope.

To summarize the results in this section: we have derived an expression for the scattering cross section from two-scale surfaces from our perturbation series. This expression is similar to previous expressions in the literature for small large-scale surface slopes and curvatures. However, the expressions differ when the large scale surface curvature or slopes are large. To our knowledge, there are no experimental data which allow for the discrimination between these two expressions. The advantage of the expressions derived here is that the cross section was derived without the need of making a priori assumptions (for example, validity of the Kirchhoff approximation to first order). In addition, the procedure clearly defines the regions of validity of the approximation and presents a systematic way of including higher order effects.

7. CONCLUSIONS

We have presented a new perturbation expansion for calculating the scattered field using the extinction theorem. This method, which is valid for smooth surfaces or near-nadir incidence, allows for the rigorous derivation of the Kirchhoff approximation and allows for the inclusion of higher order corrections (for example, curvature corrections) in a systematic manner. All the terms of the perturbation expansion for the source function can be obtained as closed form analytic expressions involving the derivatives of the scattering surface. An application of this perturbation expansion to random surfaces showed how the inclusion of curvature corrections to the scattered field effectively smoothed the scattering surface. This result is in accordance with the heuristic discussions of *Hagfors* [1966] and *Tyler* [1976] and with the quantitative results by *Lynch* [1970] for one-dimensional surfaces.

It was then shown how this perturbation method could be united with the SPM phase perturbation method [*Winebrenner and Ishimaru*, 1985*b*] to calculate the field scattered from two-scale surfaces in a systematic way. This two-scale perturbation expansion was then applied to random rough surfaces to obtain their scattering cross section. The results obtained are similar to the ones previously derived in the literature [*McDaniel and Gorman*, 1983; *McDaniel*, 1986; *Brown*, 1978] in the case of small large-scale surface curvatures and slopes. When these parameters are large, however, significant differences may be introduced. It was further shown that the effect of the small scale surface on the specular point scattering contribution was to introduce an additional attenuation factor. This, again, is in accordance with the heuristic discussion of *Tyler* [1976].

At present, there are no available data of which we are aware that will allow for the testing of this approximation. The comparison of the results against exact numerical calculations is currently under way and will be reported on in a future publication. An extension of the method presented here to vector electromagnetic scattering is also currently under way.

APPENDIX

In the text we consider a zero mean, homogeneous random rough surface $\xi(\boldsymbol{\rho})$ with standard deviation σ. It is also assumed that the correlation between the surface height and slope can be neglected. The correlation coefficient is defined by

$$C(\boldsymbol{\rho}_d) = C(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) = \frac{1}{\sigma^2}\langle \xi(\boldsymbol{\rho}_1)\xi(\boldsymbol{\rho}_2)\rangle \tag{87}$$

The surface power spectrum, $W(\boldsymbol{\kappa})$, is defined by the relation

$$\sigma^2 C(\boldsymbol{\rho}_d) = \frac{\sigma^2}{(2\pi)^2}\int d^2\kappa \exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}_d W(\boldsymbol{\kappa}) \tag{88}$$

normalized such that

$$1 = \frac{1}{(2\pi)^2}\int d^2\kappa\, W(\boldsymbol{\kappa}) \tag{89}$$

If the random rough surface is isotropic in addition to being homogeneous, the correlation coefficient is a function of ρ_d only, while the power spectrum is a function of κ only. They are related by the equation

$$\sigma^2 C(\rho_d) = \frac{\sigma^2}{2\pi}\int d\kappa\, \kappa J_0(\kappa\rho_d)W(\kappa) \tag{90}$$

As typical examples of natural surface power spectra, consider the Gaussian and the band-limited power law spectra defined respectively by

$$W_g(\kappa) = \pi L^2 \exp\left[-\frac{\kappa^2L^2}{4}\right] \tag{91}$$

$$W_p(\kappa) = 2\pi\frac{p-2}{1-\mu^{p-2}}\frac{\kappa_l^{p-2}}{\kappa^p} \tag{92}$$

where L is the correlation length of the Gaussian surface; $\mu = \kappa_l/\kappa_u$ is the fractional bandwidth of the power spectrum; κ_l is the low wave number cutoff; and κ_u is the high wave number cutoff. Band-limited power law spectra are good approximations to many natural surfaces, for example, the ocean surface [*Kitaigorodskii*, 1987] where $\frac{10}{3} \le p \le 4$, or planetary surfaces, where it is estimated that $p \approx 3$ [*Tyler*, 1976].

In section 4, one needs to obtain expressions for the short distance behavior of the correlation coefficient. For the Gaussian case, the correlation coefficient is easily calculated to be given by

$$C(\delta) = e^{-\delta^2} \tag{93}$$

$$C(\delta) = \sum_{k=0}^{\infty}\frac{\delta^{2k}}{(2k)!}C^{(2k)}(0) \tag{94}$$

where $\delta = \rho_d/L$ and

$$C^{(2k)}(0) = \frac{(2k)!(-1)^k}{k!} = \frac{d^{2k}}{d\delta^{2k}}C(\delta)\bigg|_{\delta=0} \tag{95}$$

To obtain a similar expression for the power law spectrum, replace W_p into (90), make the change of variables $\delta = \kappa_u\rho_d$ and $x = \kappa/\kappa_u$, and expand the Bessel function about zero. Interchanging summation and integration, one obtains

$$C(\delta) = \sum_{k=0}^{\infty}\frac{\delta^{2k}}{(2k)!}C^{(2k)}(0) \tag{96}$$

$$C^{(2k)}(0) = \frac{(2k)!}{k!}\frac{p-2}{2k+2-p}\frac{1-\mu^{2k+2-p}}{1-\mu^{p-2}}(-1/4)^k\mu^{p-2} \tag{97}$$

The special case of $k = 1$, $p = 4$, can be obtained from this formula by using the limit

$$\lim_{x\to 0}\left(\frac{1-\mu^x}{x}\right) = -\ln\mu \tag{98}$$

Note that in this case, the "typical" length L is $1/\kappa_u$ which does not coincide with the correlation length, which is of order $1/\kappa_l$. This implies that, for power law surfaces, the moments of the derivatives are determined by the high frequency cutoff, while the shape of the surface itself is dependent on the low frequency cutoff.

Using these expressions, one can derive the ratio $|C^{(4)}/C^{(2)}|$ used in section 4:

Gaussian

$$|C^{(4)}/C^{(2)}| = 12 \tag{99}$$

Power law

$$|C^{(4)}/C^{(2)}| = \frac{3}{4}\frac{4-p}{1-\mu^{4-p}}\frac{1-\mu^{6-p}}{6-p} \tag{100}$$

An additional result needed in section 4 is the short range behavior for the quantity $(\nabla^2 + ((\boldsymbol{\kappa}_0/p_0)\cdot\nabla)^2)C(\delta)$. Making the change of variables $\rho_d = L\delta$, this can be written as

$$\frac{1}{L^2}\left(\frac{1}{\cos^2\theta_0}\frac{\partial^2}{\partial\delta^2} + \frac{1}{\delta}\frac{\partial}{\partial\delta}\right)C(\delta)$$

Using the short range behavior obtained for the correlation function, the short range behavior for the quantity can be expanded as

$$\frac{1}{L^2}\sum_{k=1}^{\infty}\frac{\delta^{2k-2}}{(2k-2)!}C^{(2k)}(0)\alpha^{(2k)}$$

where

$$\alpha^{(2k)}=\frac{1}{\cos^2\theta_0}+\frac{1}{2k-1}$$

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