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On the vertical distribution of $\langle\tilde{u}\tilde{w}\rangle$

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Abstract

The aim of this paper is to present an analytical expression for the vertical distribution of the correlation between the horizontal ($\tilde{u}$) and vertical ($\tilde{w}$) wave velocity components. This quantity, $\langle\tilde{u}\tilde{w}\rangle$, which appears explicitly in the time-averaged momentum balance equations, has been shown to play an important role in the vertical distribution of wave-induced currents.

The proposed formulation for $\langle\tilde{u}\tilde{w}\rangle$ is based on an identity that relates the effective (wave) shear stress $\langle\tilde{u}\tilde{w}\rangle$ to the effective (wave) normal stresses ($\langle\tilde{u}\rangle^2$ and $\langle\tilde{w}\rangle^2$) and to the vorticity of the oscillatory flow $\tilde{\omega}$. This general expression has been applied to simplified situations and has been shown to degenerate into other existing formulations with comparable simplifying assumptions, viz. irrotational waves in shallow water over an arbitrary bottom topography and breaking waves over a horizontal bottom.

The model has also been applied to the case of waves interacting with a depth-varying current over a horizontal bottom, in which preliminary results have been obtained for a simplified situation invoking linear (small-amplitude) wave theory.

1. Introduction

The correlation between horizontal ($\tilde{u}$) and vertical ($\tilde{w}$) components of the oscillatory (wave) motion plays an important role in the analysis of the vertical distribution of wave-induced currents, shear stresses, sediment transport, etc. (see e.g. Deigaard and Fredsøe, 1989, De Vriend and Kitou, 1990b and Arcilla et al., 1992, to mention three recent references). Focusing on wave-induced circulation models — which of course need the corresponding shear stress as an input — it is apparent that most models have, until very recently, neglected the $\langle\tilde{u}\tilde{w}\rangle$ contribution ($\langle\ \rangle$ denotes the time-averaging operator) by simply arguing that the two wave velocity components, $\tilde{u}$ and $\tilde{w}$, are 90° out of phase. This result, which corresponds to periodic waves of permanent form, is not valid for real waves propagating over a sloping bottom and with energy dissipation (be it in the bottom boundary layer or in the free surface area for the roller of breaking waves). In conclusion, neglecting

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$\langle\tilde{u}\tilde{w}\rangle$ contributions in the momentum balance equations leads to inconsistencies in the wave-induced stresses. This would also have to result in inconsistencies in any result derived from those momentum balance equations. The fact that some of the obtained results do not appear to show this ill-behaviour is due to the large number of uncertainties and free fit parameters that are included in this kind of models. Furthermore, the smallness of $\langle\tilde{u}\tilde{w}\rangle$ compared to other effective wave stresses, such as $\langle\tilde{u}\rangle^2$ and $\langle\tilde{w}\rangle^2$ (according to what most irrotational wave theories predict), and the lack of reliable experimental data have hindered any definite conclusions. The recent literature on wave-induced circulation in the surf zone clearly illustrates this situation.

This paper focuses on the evaluation of $\langle\tilde{u}\tilde{w}\rangle$ terms for waves in the presence of an ambient current and/or a sloping bottom. For such cases, several sources of non-zero $\langle\tilde{u}\tilde{w}\rangle$ have been described in the literature. They are the following:

(*a*) *Sloping bottom effects.* As already shown in Battjes (1968), when waves propagate over a sloping bottom, wave fronts generally exhibit a vertical curvature in addition to the curvature in the horizontal plane due to refraction. Because of this, the (oscillatory) velocity components $\tilde{u}$ and $\tilde{w}$ are no longer 90° out of phase, and therefore, the correlation $\langle\tilde{u}\tilde{w}\rangle$ is not nil. Considering all *irrotational* effects associated to the sloping bottom, a perturbation solution based on linear wave theory may be obtained for the wave velocity potential. This solution, fulfilling the no-flow bottom boundary condition for the actual sloping bottom has been derived in De Vriend and Kitou (1990a), and turns out to give, in the first-order approximation, a linear distribution of $\langle\tilde{u}\tilde{w}\rangle$ over depth:

$$\langle\tilde{u}\tilde{w}\rangle = G\left(\frac{E}{\rho h}\right)\left[\frac{\partial z_b}{\partial x} + \frac{1}{1+G}\frac{kh}{\tanh(kh)}\frac{\partial h}{\partial x}\left(\frac{z-z_b}{h}\right)\right] - \frac{1}{2}G\left[\frac{\partial}{\partial x}\left(\frac{E}{\rho}\right)\right]\left(\frac{z-z_b}{h}\right) \tag{1}$$

in which x is the horizontal coordinate in the wave propagation direction, z is the vertical coordinate positive upwards, z_b is the bottom vertical coordinate, h is the mean water depth, k is the wavenumber, E is the wave energy density, ρ is the water mass density, and $G = 2kh/\sinh(2kh)$. The corresponding values for the other effective (wave) stresses appearing in the momentum balance equations, namely, $\langle\tilde{u}\rangle^2$ and $\langle\tilde{w}\rangle^2$, coincide with the values obtained from standard linear wave theory.

(*b*) *Wave amplitude gradient effects.* Even in the framework of linear wave theory, wave amplitude gradients are known to generate non-zero $\langle\tilde{u}\tilde{w}\rangle$ (Mei, 1983). This quantity has been usually related to wave energy dissipation (Deigaard and Fredsøe, 1989). When this dissipation takes place in the bottom boundary layer a perturbation solution, similar to the one derived for tidal waves with bottom friction, may be obtained. The resulting $\langle\tilde{u}\tilde{w}\rangle$ distribution is given by Deigaard and Fredsøe (1989):

$$\langle\tilde{u}\tilde{w}\rangle = \left[\frac{\partial}{\partial x}\left(\frac{E}{\rho}\right)\right]\left(1 - \frac{z-z_b}{2h}\right) \tag{2}$$

which is also linear with z. When the dissipation takes place near the surface area — due to the roller associated to breaking waves — it is also easy to derive the wave orbital velocity associated to a linearly decreasing wave height. This is done expressing $\tilde{u}$ and $\tilde{w}$ in terms of $\partial\eta/\partial t$ by using the continuity equation (η is the free-surface elevation with respect to the mean water level). The resulting $\langle\tilde{u}\tilde{w}\rangle$ expression is (Deigaard and Fredsøe, 1989):

$$\langle \tilde{u}\tilde{w} \rangle = -\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{E}{\rho}\right)\right]\left(\frac{z-z_b}{h}\right) \tag{3}$$

It should finally be remarked that, whenever there are gradients in the wave amplitude field, even if they are not due to energy dissipation, there will be a non-zero $\langle \tilde{u}\tilde{w} \rangle$ contribution. This case may be illustrated by gradients in wave amplitude due to diffraction, e.g. behind a detached breakwater. Expressions to evaluate $\langle \tilde{u}\tilde{w} \rangle$ for such a case have been proposed in Arcilla et al. (1992) in which, starting from a wave velocity potential ϕ:

$$\phi = -\frac{ig}{\sigma}AZ(z-z_b)\cdot e^{iS} \tag{4}$$

the $\langle \tilde{u}\tilde{w} \rangle$ term is derived in a straightforward manner by assuming a spatially varying wave amplitude field $A(x,y)$:

$$\langle \tilde{u}\tilde{w} \rangle = \frac{1}{2}\frac{g^2}{\sigma^2}\frac{\partial Z}{\partial z}ZA\frac{\partial A}{\partial x} \tag{5}$$

where $Z(z-z_b)$ is the vertical shape function of the velocity potential, A the wave amplitude, σ the angular wave frequency, and S the phase function.

(*c*) *Vorticity effects induced by viscosity near solid boundaries.* The oscillatory boundary-layer streaming (Longuet-Higgins, 1953) is a consequence of the $\langle \tilde{u}\tilde{w} \rangle$ contribution, which is in turn produced by the vorticity generation and diffusion in the bottom boundary layer. This vorticity-based explanation is an alternative to the dissipation-based explanation and should lead to equivalent expressions. However, since the bottom (oscillatory) boundary layer is not considered in detail in this paper, this point will not be further analyzed.

(*d*) *Vorticity effects induced by depth-varying currents.* In the presence of ambient depth-varying currents, a nonlinear vorticity transfer from the current to the wave motion takes place (Peregrine, 1976), which induces effective stresses $\langle \tilde{u}\tilde{w} \rangle$ directly related, as will be shown later, to the vorticity transfer (Rivero and Arcilla, 1993).

In what follows an attempt is made to develop a theory to evaluate $\langle \tilde{u}\tilde{w} \rangle$. This theory should encompass the various sources of non-zero wave correlations just described. It will be shown that the quantity $\langle \tilde{u}\tilde{w} \rangle$ is strongly related to the vorticity of the oscillatory flow.

2. Governing equation

The physical problem is here confined, for simplicity, to a 2DV situation, in which waves propagate along the x-direction. The z-axis is directed vertically upwards with origin fixed at a given reference level. See Fig. 1.

The velocity field (u,w) is decomposed into a mean component (U,W) and an oscillatory component $(\tilde{u},\tilde{w})$:

$$u = U + \tilde{u} \tag{6a}$$

$$w = W + \tilde{w} \tag{6b}$$

By definition, the time-averaged value of the oscillatory component is zero:

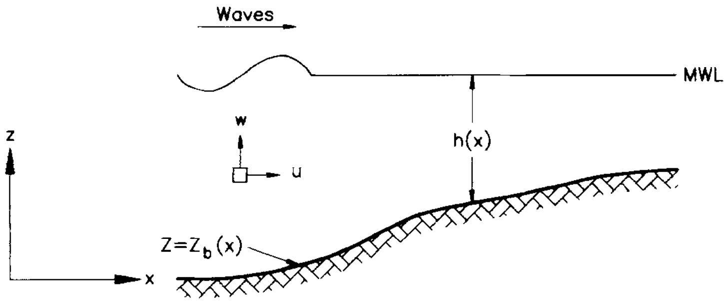

Fig. 1. Domain definition sketch (2DV situation).

$$\langle \tilde{u} \rangle = 0 \tag{7a}$$

$$\langle \tilde{w} \rangle = 0 \tag{7b}$$

where ⟨ ⟩ denotes the time-averaging operator over a wave period.

In a similar way, the scalar vorticity ω, defined as

$$\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \tag{8}$$

is split into a mean component (Ω) and an oscillatory component ($\tilde{\omega}$):

$$\omega = \Omega + \tilde{\omega} \tag{9}$$

where

$$\Omega = \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x} \tag{10a}$$

$$\tilde{\omega} = \frac{\partial \tilde{u}}{\partial z} - \frac{\partial \tilde{w}}{\partial x} \tag{10b}$$

Notice that, according to definition (7), the time-averaged value of the oscillatory vorticity $\tilde{\omega}$ is also nil:

$$\langle \tilde{\omega} \rangle = 0 \tag{11}$$

As will be shown here, the quantity $\langle \tilde{w}\tilde{\omega} \rangle$ is directly related to the vertical distribution of $\langle \tilde{u}\tilde{w} \rangle$. Multiplying $\tilde{w}$ by $\tilde{\omega}$ (10b), it is obtained:

$$\tilde{w}\tilde{\omega} = \tilde{w}\frac{\partial \tilde{u}}{\partial z} - \tilde{w}\frac{\partial \tilde{w}}{\partial x} \tag{12}$$

which may be rearranged to yield

$$\tilde{w}\tilde{\omega} = \frac{\partial}{\partial z}(\tilde{u}\tilde{w}) - \tilde{u}\frac{\partial \tilde{w}}{\partial z} - \tilde{w}\frac{\partial \tilde{w}}{\partial x} \tag{13}$$

Invoking now the continuity equation for the oscillatory motion:

$$\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{w}}{\partial z}=0 \tag{14}$$

it follows

$$\tilde{w}\tilde{\omega}=\frac{\partial}{\partial z}(\tilde{u}\tilde{w})+\tilde{u}\frac{\partial \tilde{u}}{\partial x}-\tilde{w}\frac{\partial \tilde{w}}{\partial x}=\frac{\partial}{\partial z}(\tilde{u}\tilde{w})+\frac{1}{2}\left[\frac{\partial}{\partial x}(\tilde{u}^2-\tilde{w}^2)\right] \tag{15}$$

from which, after time-averaging, it may be written

$$\frac{\partial}{\partial z}\langle \tilde{u}\tilde{w}\rangle=\langle \tilde{w}\tilde{\omega}\rangle-\frac{1}{2}\left[\frac{\partial}{\partial x}(\langle \tilde{u}^2\rangle-\langle \tilde{w}^2\rangle)\right] \tag{16}$$

This identity relates the vertical variation of the effective (wave) shear stress $\langle \tilde{u}\tilde{w}\rangle$ to the oscillatory vorticity $\tilde{\omega}$, still unknown. The two effective (wave) normal stresses $\langle \tilde{u}^2\rangle$ and $\langle \tilde{w}^2\rangle$ are assumed to be given by any wave theory, e.g. linear sinusoidal theory, therefore allowing to evaluate $\langle \tilde{u}\tilde{w}\rangle$ once the oscillatory vorticity is known.

Eq. (16) is, in this sense, a *circular expression* relating wave correlations between themselves and to the oscillatory vorticity. It is not, therefore, an alternative or more complete wave model to calculate wave correlations, such as the perturbation solution proposed in Bijker et al. (1974). This solution, based on linear wave theory plus a perturbation term due to the oscillatory bottom boundary layer, provides values for $\langle \tilde{u}^2\rangle$, $\langle \tilde{w}^2\rangle$ and $\langle \tilde{u}\tilde{w}\rangle$. The $\langle \tilde{u}^2\rangle$ and $\langle \tilde{w}^2\rangle$ values so obtained could also be used to feed Eq. (16), from which a $\langle \tilde{u}\tilde{w}\rangle$ value could be, in turn, obtained. The added value of Eq. (16) is, thus, the possibility to obtain $\langle \tilde{u}\tilde{w}\rangle$ in terms of $\langle \tilde{u}^2\rangle$ and $\langle \tilde{w}^2\rangle$ (assuming $\tilde{\omega}$ known), much easier to calculate using any wave theory.

3. Irrotational wave motion

When no ambient currents are present, the oscillatory velocity field (far from solid boundaries) may be assumed essentially irrotational (i.e. $\tilde{\omega}=0$) as long as the free surface remains simply connected and wave breaking does not occur. In such a case, the governing equation for $\langle \tilde{u}\tilde{w}\rangle$ (16) will read

$$\frac{\partial}{\partial z}\langle \tilde{u}\tilde{w}\rangle=-\frac{1}{2}\left[\frac{\partial}{\partial x}(\langle \tilde{u}^2\rangle-\langle \tilde{w}^2\rangle)\right] \tag{17}$$

Invoking linear (small-amplitude) wave theory, the term inside brackets on the right-hand side of Eq. (17) is seen to be independent of the vertical coordinate (Longuet-Higgins and Stewart, 1962):

$$\langle \tilde{u}^2\rangle-\langle \tilde{w}^2\rangle=G\left(\frac{E}{\rho h}\right) \tag{18}$$

where $G=2kh/\sinh(2kh)$, k is the wavenumber, h is the mean water depth, $E=\rho gH^2/8$ is the wave energy density, H is the wave height, ρ is the fluid density, and g is the gravity acceleration.

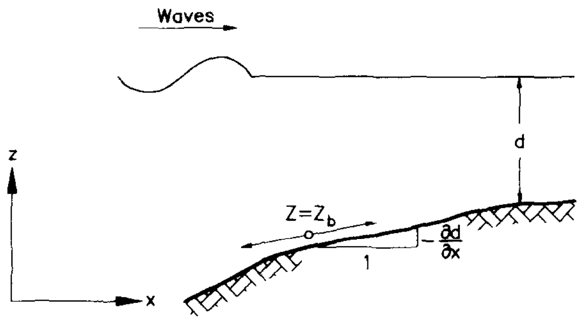

Fig. 2. Simplified model for the oscillatory wave motion near the bed.

Upon vertical integration of Eq. (17) from the top of the bottom boundary layer ($z=z_b$) to any level z, an explicit expression for $\langle\tilde{u}\tilde{w}\rangle$ may be obtained:

$$\langle\tilde{u}\tilde{w}\rangle=\langle\tilde{u}\tilde{w}\rangle_{(z=z_b)}-\frac{1}{2}\left[\frac{\partial}{\partial x}\left(G\frac{E}{\rho h}\right)\right](z-z_b) \tag{19}$$

This expression states that the vertical distribution of $\langle\tilde{u}\tilde{w}\rangle$ over depth is, in this case, linear. Notice that the term inside brackets on the right-hand side of (19), basically varying as $\partial/\partial x(H^2/h)$, is positive for shoaling waves (i.e. $\langle\tilde{u}\tilde{w}\rangle$ decreases with z), and negative for dissipative waves (i.e.$\langle\tilde{u}\tilde{w}\rangle$ increases with z).

An approximate value of $\langle\tilde{u}\tilde{w}\rangle$ at the bed ($z=z_b$) may be found from the kinematic boundary condition at the bottom ($\tilde{w}_b=-\tilde{u}_b\partial d/\partial x$, where the subscript b indicates near-bed values and d is the still-water depth), as suggested in (Putrevu and Svendsen, 1993), even though these authors do not actually evaluate the $\langle\tilde{u}\tilde{w}\rangle$ term. The procedure, illustrated in Fig. 2, consists in calculating $\langle\tilde{u}\tilde{w}\rangle_{(z=z_b)}$ from the corresponding $\tilde{u}_b$ and $\tilde{w}_b$ values:

$$\tilde{u}_b=V_{orb}(t) \tag{20a}$$

$$\tilde{w}_b=-V_{orb}(t)\frac{\partial d}{\partial x} \tag{20b}$$

where $V_{orb}(t)$ is the horizontal component of the near-bed orbital velocity at the bottom, given by linear theory:

$$V_{orb}(t)=\frac{\sigma H}{2\sinh(kh)}\cos(\sigma t) \tag{21}$$

and $\sigma=\sqrt{gk\tanh(kh)}$ is the angular wave frequency.

According to this model, it is easily found that

$$\langle\tilde{u}\tilde{w}\rangle_{(z=z_b)}=-\langle V_{orb}^2(t)\rangle\frac{\partial d}{\partial x}=-G\left(\frac{E}{\rho h}\right)\frac{\partial d}{\partial x} \tag{22}$$

which is seen to coincide with the near-bed value for $\langle\tilde{u}\tilde{w}\rangle$ given by (De Vriend and Kitou, 1990a).

An alternative expression for $\langle\tilde{u}\tilde{w}\rangle$ at the top of the bottom boundary layer ($z=z_b$) may be obtained from the boundary layer streaming solution over a horizontal bottom (see e.g. Phillips, 1977, p. 55):

$$\langle \tilde{u}\tilde{w} \rangle_{(z=z_b)} = -\frac{1}{2} k\delta G\left(\frac{E}{\rho h}\right) \tag{23}$$

where k is the wavenumber, $\delta = (2\nu/\sigma)^{1/2}$ is a measure of the boundary layer thickness, and ν is the molecular viscosity. This $\langle \tilde{u}\tilde{w} \rangle_{(z=z_b)}$ value has been used e.g. in (Putrevu and Svendsen, 1993) using an eddy viscosity coefficient ν_t instead of the ν value for laminar flow. Expression (22) has been preferred in the context of this paper because it explicitly includes bottom slope effects and does not need to define and calculate ν_t (or ν).

Upon substitution of (22) into (19), an approximate expression describing the vertical distribution of $\langle \tilde{u}\tilde{w} \rangle$ in the case of irrotational wave motion is obtained:

$$\langle \tilde{u}\tilde{w} \rangle = -G\left(\frac{E}{\rho h}\right)\frac{\partial d}{\partial x} - \frac{1}{2}\left[\frac{\partial}{\partial x}\left(G\frac{E}{\rho h}\right)\right](z - z_b) \tag{24}$$

This equation is, though similar, essentially different from the one presented in De Vriend and Kitou (1990a), given by Eq. (1) in this paper. The reason for this discrepancy may be ascribed to the different methodologies applied in the derivation of $\langle \tilde{u}\tilde{w} \rangle$: Eq. (24) has been derived from identity (16), after assuming irrotational wave motion and invoking linear theory — to evaluate $\langle \tilde{u}^2 \rangle$, $\langle \tilde{w}^2 \rangle$, and the near-bed value of $\langle \tilde{u}\tilde{w} \rangle$ — whereas Eq. (1) was obtained by calculating the wave velocity components $\tilde{u}$ and $\tilde{w}$ from a velocity potential ϕ, which was found from a first-order perturbation for mild-sloping bottom of the horizontal-bottom linear-wave solution. These two equations, (24) and (1), however, may easily be seen to coincide in the shallow-water approximation. In this case ($kh \to 0$), the resulting distribution of $\langle \tilde{u}\tilde{w} \rangle$ may be written as follows:

$$\langle \tilde{u}\tilde{w} \rangle = -\left(\frac{E}{\rho h}\right)\frac{\partial d}{\partial x} - \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{E}{\rho}\right) - \left(\frac{E}{\rho h}\right)\frac{\partial h}{\partial x}\right]\left(\frac{z - z_b}{h}\right) \tag{25}$$

The expression proposed in Deigaard and Fredsøe (1989) to evaluate $\langle \tilde{u}\tilde{w} \rangle$ for dissipative breaking waves — given by Eq. (3) in this paper — may be in turn considered a particular case of Eq. (25) if the following two assumptions are made:

- horizontal bottom ($\partial d/\partial x = 0$)
- negligible horizontal variations of mean water level ($\partial h/\partial x = 0$)

In the following an attempt is made to give approximate estimates of the vertical distribution of $\langle \tilde{u}\tilde{w} \rangle$ for several simplified situations:

3.1. Non-dissipative water waves

3.1.1. *Horizontal bottom.* According to (22) $\langle \tilde{u}\tilde{w} \rangle$ is, in this case, zero at the bottom. If it is further assumed that averaged wave properties do not vary in the x-direction (i.e. wave energy dissipation is neglected), Eq. (24) gives directly $\langle \tilde{u}\tilde{w} \rangle = 0$ at any z-level, which is the well-known outcome from irrotational wave theories over a horizontal bed.

3.1.2. *Sloping bottom.* From the kinematic boundary conditions at the free surface and at the bottom, Battjes (1968) showed that wave fronts generally exhibit a vertical curvature whenever waves propagate over a sloping bottom (see Fig. 3). The vertical variation of the

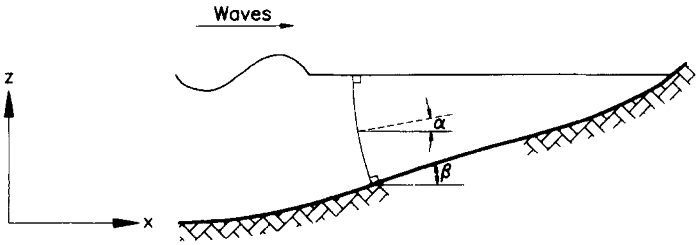

Fig. 3. Vertical curvature of wave fronts over a sloping bottom (after Battjes, 1968).

wave front can be approximated, in this case, by a circle with a linear variation with depth of the corresponding angle α (from 0 at the mean free surface to β at the bottom). This kinematic description already allows an evaluation of the $\langle\tilde{u}\tilde{w}\rangle$-correlation, as suggested by (Svendsen and Lorenz, 1989; appendix), which, though different from zero, is not the complete one. The reason is that other effects, such as shoaling, are not fully included in this simple kinematic description. The general expression (24) can be further elaborated for this case applying the energy flux conservation law.

For simplicity, shallow water conditions will be assumed:

$$G=1 \tag{26a}$$

$$C_g=\sqrt{gh} \tag{26b}$$

in which C_g is the group velocity of the wave train and G has already been defined. The conservation of the oscillatory energy flux can then be written as:

$$\frac{\partial}{\partial x}(EC_g)=\frac{\partial}{\partial x}(E\sqrt{gh})=0 \tag{27}$$

from which it is easy to obtain

$$\frac{\partial E}{\partial x}=-\frac{1}{2}\frac{E}{h}\frac{\partial h}{\partial x} \tag{28}$$

Substituting (28) into the expression for the vertical distribution of $\langle\tilde{u}\tilde{w}\rangle$ in the shallow-water approximation (24), it yields:

$$\langle\tilde{u}\tilde{w}\rangle=\frac{E}{\rho h}\left[-\frac{\partial d}{\partial x}+\frac{3}{4}\left(\frac{z-z_b}{h}\right)\frac{\partial h}{\partial x}\right] \tag{29}$$

The resulting vertical distribution of $\langle\tilde{u}\tilde{w}\rangle$, assuming $\partial d/\partial x=\partial h/\partial x$ (which seems quite reasonable since no large gradients in mean water level $\langle\eta\rangle$ are expected for non-dissipative waves), is

$$\langle\tilde{u}\tilde{w}\rangle=-\left(\frac{E}{\rho h}\right)\frac{\partial h}{\partial x}\left[1-\frac{3}{4}\left(\frac{z-z_b}{h}\right)\right] \tag{30}$$

and is shown in Fig. 4. This result could also be obtained from (1) as shallow water conditions have been assumed.

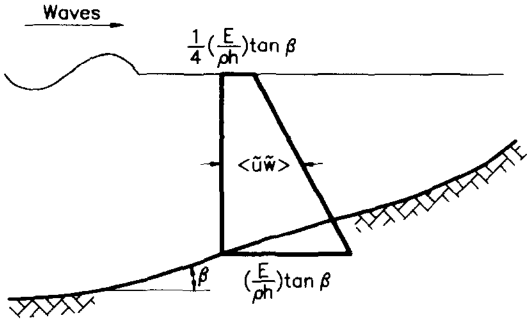

Fig. 4. Vertical distribution of $\langle \tilde{u}\tilde{w} \rangle$ in the case of non-dissipative waves propagating over a sloping bottom.

3.2. Dissipative water waves

In this case, where wave amplitude gradients become important, the vertical distribution of $\langle \tilde{u}\tilde{w} \rangle$ is given again by the already derived Eq. (24):

$$\langle \tilde{u}\tilde{w} \rangle = -G\left(\frac{E}{\rho h}\right)\frac{\partial d}{\partial x} - \left[\frac{\partial}{\partial x}\left(\frac{1}{2}G\frac{E}{\rho h}\right)\right](z - z_b),$$

The only other expression presently available to evaluate $\langle \tilde{u}\tilde{w} \rangle$ for the case of dissipative breaking waves was proposed in Deigaard and Fredsøe (1989), and is given by Eq. (3) in this paper:

$$\langle \tilde{u}\tilde{w} \rangle = -\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{E}{\rho}\right)\right]\left(\frac{z - z_b}{h}\right)$$

This equation, derived for shallow-water waves ($G = 1$) propagating over a horizontal bottom ($\partial d/\partial x = 0$) and neglecting mean water level variations ($\partial h/\partial x = 0$), may be also considered a particular case of the general expression (24). The resulting vertical distribution of $\langle \tilde{u}\tilde{w} \rangle$ for this case is shown in Fig. 5.

However, sloping bottom effects and mean water level variations can be important in the calculation of $\langle \tilde{u}\tilde{w} \rangle$, and a more general expression like (24) should be preferred when evaluating $\langle \tilde{u}\tilde{w} \rangle$. In the general case of uneven bottom topography, the present formulation (24) differs from that presented by Deigaard and Fredsøe, given by Eq. (3), in three main aspects (see also Fig. 6 for a comparison):

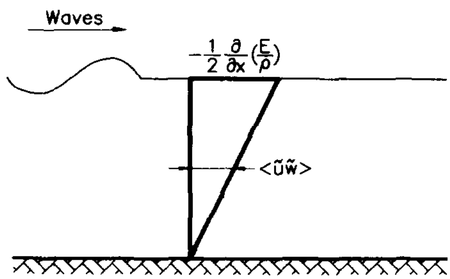

Fig. 5. Vertical distribution of $\langle \tilde{u}\tilde{w} \rangle$ in the case of dissipative waves propagating over a horizontal bottom.

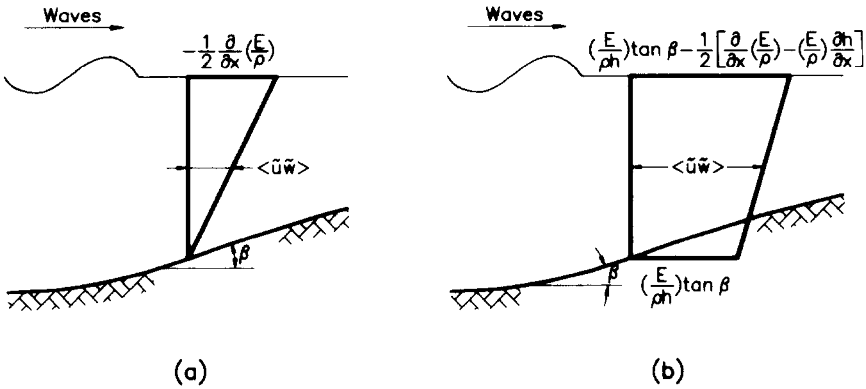

Fig. 6. Comparison of vertical distributions of $\langle \tilde{u}\tilde{w} \rangle$ in the case of dissipative waves propagating over a sloping bottom: (a) Deigaard and Fredsøe (1989) model; (b) present model.

(i) Eq. (3) has been derived for long waves in shallow water, thereby neglecting vertical variations of the orbital (wave) velocity field; Eq. (24) accounts for these variations through the parameter G.

(ii) Eq. (3) assumes a zero value of $\langle \tilde{u}\tilde{w} \rangle$ at the bed — as it was derived for a horizontal bottom — while the present model starts from non-zero values of $\langle \tilde{u}\tilde{w} \rangle$ due to bed slope effects — see Eq. (22) and the non-zero values of $\langle \tilde{u}\tilde{w} \rangle$ at the bed suggested by e.g. Svendsen and Lorenz (1989), De Vriend and Kitou (1990a) and Putrevu and Svendsen (1993).

(iii) The vertical variation of $\langle \tilde{u}\tilde{w} \rangle$ is, according to Eq. (3),

$$\frac{\partial \langle \tilde{u}\tilde{w} \rangle}{\partial z} = -\frac{1}{2h}\left[\frac{\partial}{\partial x}\left(\frac{E}{\rho}\right)\right] \tag{31}$$

and according to Eq. (24), assuming shallow-water waves ($G=1$) — see also Eq. (25),

$$\frac{\partial \langle \tilde{u}\tilde{w} \rangle}{\partial z} = -\frac{1}{2h}\left[\frac{\partial}{\partial x}\left(\frac{E}{\rho}\right)-\left(\frac{E}{\rho h}\right)\frac{\partial h}{\partial x}\right] \tag{32}$$

Although identical in the case of horizontal bottom and negligible mean water level variations ($\partial h/\partial x=0$), these two expressions are different otherwise. Notice that the effects of sloping bottom and mean water level are explicitly included in Eq. (32). For decreasing water depth ($\partial h/\partial x<0$), for instance, the present model, given by Eq. (32), predicts a smaller vertical variation of $\langle \tilde{u}\tilde{w} \rangle$ than Eq. (31), which, although derived for a horizontal bed, is also applied for sloping bottom cases.

The $\langle \tilde{u}\tilde{w} \rangle$ distribution proposed in Deigaard and Fredsøe (1989) for bottom boundary layer dissipation — given by Eq. (2) in this paper — cannot be compared to Eq. (24). The reason is that all the argumentation behind Eq. (24) makes extensive use of linear wave theory disregarding the bottom boundary layer and its effects.

It should be finally remarked that all these expressions have been derived assuming irrotational wave motion, which is not fully consistent with the dissipative character of the waves.

4. Rotational wave motion

In the presence of depth-varying currents, wave motion is no longer irrotational, since, as will be shown below, there exists in general a vorticity transfer from the mean (current) motion to the oscillatory (wave) motion. In this case, the vertical distribution of $\langle \tilde{u}\tilde{w} \rangle$ is governed by Eq. (16):

$$\frac{\partial}{\partial z}\langle \tilde{u}\tilde{w} \rangle = \langle \tilde{w}\tilde{\omega} \rangle - \frac{1}{2}\left[\frac{\partial}{\partial x}(\langle \tilde{u}^2 \rangle - \langle \tilde{w}^2 \rangle)\right]$$

in which vorticity-induced effects appear explicitly in the $\langle \tilde{w}\tilde{\omega} \rangle$ term.

The (scalar) oscillatory vorticity $\tilde{\omega}$ may be obtained from the vorticity transport equation, which reads, ignoring turbulent interactions,

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + w\frac{\partial \omega}{\partial z} = 0 \tag{33}$$

or, upon substitution of definitions (6) and (9),

$$\frac{\partial \Omega}{\partial t} + \frac{\partial \tilde{\omega}}{\partial t} + U\frac{\partial \Omega}{\partial x} + U\frac{\partial \tilde{\omega}}{\partial x} + \tilde{u}\frac{\partial \Omega}{\partial x} + \tilde{u}\frac{\partial \tilde{\omega}}{\partial x} + W\frac{\partial \Omega}{\partial z} + W\frac{\partial \tilde{\omega}}{\partial z} + \tilde{w}\frac{\partial \Omega}{\partial z} + \tilde{w}\frac{\partial \tilde{\omega}}{\partial z} = 0 \tag{34}$$

In the following an attempt is made to estimate the $\tilde{\omega}$ distribution, and thus, $\langle \tilde{w}\tilde{\omega} \rangle$ and $\langle \tilde{u}\tilde{w} \rangle$, for a simplified situation. The following assumptions, corresponding to the case of a horizontal wave–current flume, are made:

- Horizontal bottom
- Uniform mean flow in the x-direction (i.e. $\partial U/\partial x = 0$, $\partial \Omega/\partial x = 0$)
- Stationary current field (i.e. $\partial U/\partial t = 0$, $\partial \Omega/\partial t = 0$)

Notice that these assumptions also imply that the mean vertical velocity W is zero everywhere. In this situation, Eq. (34) yields

$$\frac{\partial \tilde{\omega}}{\partial t} + U\frac{\partial \tilde{\omega}}{\partial x} + \tilde{u}\frac{\partial \tilde{\omega}}{\partial x} + \tilde{\omega}\frac{\partial \Omega}{\partial z} + \tilde{w}\frac{\partial \tilde{\omega}}{\partial z} = 0 \tag{35}$$

and, after some rearrangements,

$$\frac{\partial \tilde{\omega}}{\partial t} + (U + \tilde{u})\frac{\partial \tilde{\omega}}{\partial x} + \tilde{w}\frac{\partial \tilde{\omega}}{\partial z} = -\tilde{w}\frac{\partial^2 U}{\partial z^2} \tag{36}$$

In a frame of reference $\{x',z\}$ moving with the current velocity $U(x' = x - Ut)$, this equation will read

$$\frac{\partial \tilde{\omega}}{\partial t} + \tilde{u}\frac{\partial \tilde{\omega}}{\partial x'} + \tilde{w}\frac{\partial \tilde{\omega}}{\partial z} = -\tilde{w}\frac{\partial^2 U}{\partial z^2} \tag{37}$$

or, in terms of the total time-derivative, $D/Dt = \partial/\partial t + \tilde{u}(\partial/\partial x') + \tilde{w}(\partial/\partial z)$,

$$\frac{D\tilde{\omega}}{Dt} = -\tilde{w}\frac{\partial^2 U}{\partial z^2} = -\tilde{w}U''(z) \tag{38}$$

This equation indicates that in the presence of an ambient current field $U(z)$, the wave (oscillatory) motion will remain irrotational as long as $U''(z)=0$, i.e. when the vertical current profile is either uniform or linear. Otherwise ($U''(z)\neq 0$), a vorticity transfer will take place from the current motion to the wave motion, so that the wave field will no longer be irrotational (Peregrine, 1976).

Numerical solutions of the non-linear Eq. (37) for finite-amplitude waves with an arbitrary depth-varying current field are given in e.g. Dalrymple (1977) and Thomas (1990). For simplicity, this paper will deal only with solutions for small-amplitude waves, for which the non-linear terms in Eq. (37) may be disregarded to give

$$\frac{\partial \tilde{\omega}}{\partial t} = -\tilde{w}\frac{\partial^2 U}{\partial z^2} = -\tilde{w}U''(z) \tag{39}$$

It can easily be shown (Peregrine, 1976) that this equation, together with the continuity equation (14) and the definition of oscillatory vorticity $\tilde{\omega}$ (10b), applied to simple harmonic wave motion, leads to the Rayleigh equation of classical inviscid stability theory for the vertical (wave) oscillatory component $\tilde{w}=\tilde{w}_o\cos(kx-\sigma t)$:

$$\frac{\partial^2 \tilde{w}_o}{\partial z^2} - \left(k^2 + \frac{kU''}{kU-\sigma}\right)\tilde{w}_o = 0 \tag{40}$$

for which analytical solutions only exist when $U''(z)=0$. Numerical solutions for an otherwise arbitrary current velocity profile $U(z)$ may be found in e.g. Peregrine (1976) and Thomas (1981).

Eq. (39) indicates that the rate of variation of $\tilde{\omega}$ is proportional to the vertical velocity $\tilde{w}$, and its sign depends on the sign of U'', i.e. waves following or opposing current (see Fig. 7). According to this simple interpretation of Eq. (39), and assuming quasi-elliptic orbital paths for the fluid particles in the new frame of reference, for the waves following current case (Fig. 7a) $\tilde{\omega}$ turns out to be positive in the upper half of the cycle (with positive horizontal orbital velocities) and negative in the lower half (with negative horizontal orbital velocities). For the waves opposing current case (Fig. 7b) $\tilde{\omega}$ has opposite signs.

The quantity $\langle \tilde{w}\tilde{\omega} \rangle$, needed to estimate the $\langle \tilde{u}\tilde{w} \rangle$ distribution, given by Eq. (16), may easily be shown to be zero in this case since $\tilde{w}$ and $\tilde{\omega}$ appear to be 90° out of phase to the leading order of approximation. The $\langle \tilde{w}\tilde{\omega} \rangle$ term can also be obtained from Eq. (39) multiplied by $\tilde{\omega}$:

$$\tilde{\omega}\frac{\partial \tilde{\omega}}{\partial t} = -\tilde{w}\tilde{\omega}\frac{\partial^2 U}{\partial z^2} \tag{41}$$

from which, after some rearrangements,

$$\tilde{w}\tilde{\omega}\frac{\partial^2 U}{\partial z^2} = -\frac{1}{2}\frac{\partial \tilde{\omega}^2}{\partial t} \tag{42}$$

Assuming stationary conditions for combined wave–current motion, it follows immediately, after time-averaging:

$$\langle \tilde{w}\tilde{\omega} \rangle \frac{\partial^2 U}{\partial z^2} = -\frac{1}{2}\left\langle \frac{\partial \tilde{\omega}^2}{\partial t} \right\rangle = 0 \tag{43}$$

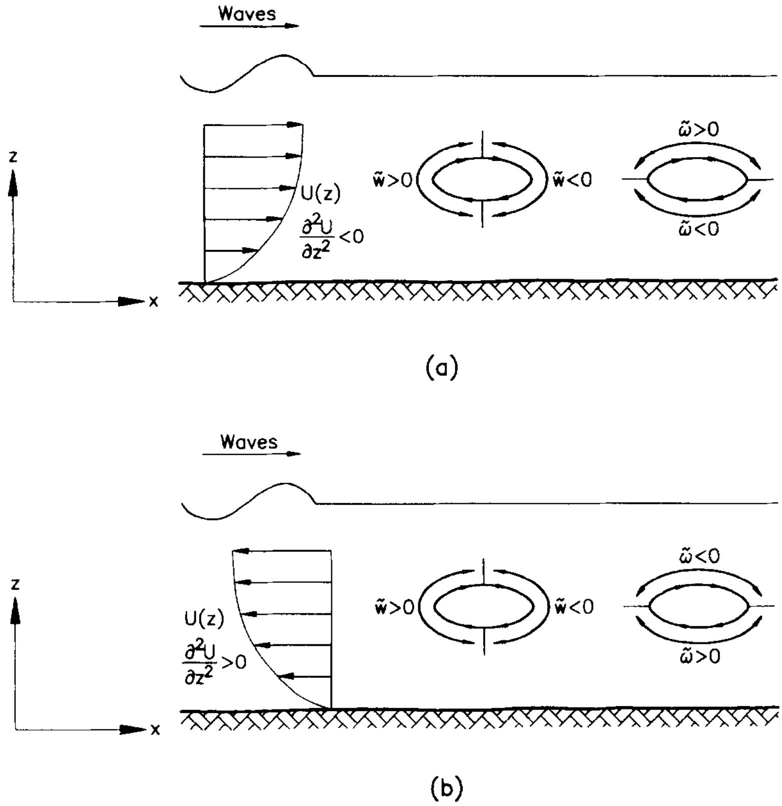

Fig. 7. Variation of $\tilde{w}$ and $\tilde{\omega}$ during one wave cycle for small-amplitude waves: (a) Waves following current, (b) Waves opposing current.

The vertical distribution of $\langle \tilde{u}\tilde{w} \rangle$, given by Eq. (16), will thus read for this case:

$$\frac{\partial}{\partial z}\langle \tilde{u}\tilde{w} \rangle = -\frac{1}{2}\left[\frac{\partial}{\partial x}(\langle \tilde{u}^2 \rangle - \langle \tilde{w}^2 \rangle)\right] \tag{44}$$

from which, assuming $\langle \tilde{u}\tilde{w} \rangle$ to be zero at the (horizontal) bed, and invoking linear wave theory (see Jonsson, 1990 for a discussion), it follows, after vertical integration,

$$\langle \tilde{u}\tilde{w} \rangle = -\frac{1}{2}G\left[\frac{\partial}{\partial x}\left(\frac{E}{\rho}\right)\right]\left(\frac{z-z_b}{h}\right) \tag{45}$$

This expression is consistent with Eq. (24) obtained for irrotational waves. This is not surprising considering that wave properties, viz. $\langle \tilde{u}^2 \rangle$ and $\langle \tilde{w}^2 \rangle$, have been evaluated using linear wave theory, which is the same irrotational model used in the previous section.

5. Preliminary consequences of the presented $\langle \tilde{u}\tilde{w} \rangle$ distribution

The vertical distribution of mean shear stresses $\langle \tau \rangle$ (associated with viscous and/or turbulent effects) can be found from the time-averaged horizontal momentum equations after assuming a given mean pressure distribution (usually, $\langle p\rangle = \langle p_{\rm h}\rangle - \rho\langle \tilde{w}^2\rangle$, where $\langle p_{\rm h}\rangle$ is the mean hydrostatic pressure). For simplicity, only a simplified cross-shore momentum balance equation will be here considered (see e.g. Svendsen, 1984 for the assumptions and motivation of this equation):

$$\frac{\partial}{\partial z}\left(\frac{\langle \tau_{xz}\rangle}{\rho}\right) = g\frac{\partial\langle\eta\rangle}{\partial x} + \frac{\partial}{\partial x}\left(\langle \tilde{u}^2\rangle - \langle \tilde{w}^2\rangle\right) + \frac{\partial}{\partial z}\langle \tilde{u}\tilde{w}\rangle \tag{46}$$

which shows the vertical variation of the mean shear stress $\langle \tau_{xz}\rangle$ in the $\{x,z\}$ vertical plane.

As already mentioned before, the $\langle \tilde{u}\tilde{w}\rangle$ contribution has been, until very recently, neglected throughout the water column except in the bottom boundary layer, in which it led to the streaming solution (Longuet-Higgins, 1953). The relevance of the $\langle \tilde{u}\tilde{w}\rangle$ contribution in the vertical distribution of $\langle \tau_{xz}\rangle$ can be easily assessed after substitution of identity (16) into Eq. (46):

$$\frac{\partial}{\partial z}\left(\frac{\langle \tau_{xz}\rangle}{\rho}\right) = g\frac{\partial\langle\eta\rangle}{\partial x} + \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\langle \tilde{u}^2\rangle - \langle \tilde{w}^2\rangle\right)\right] + \langle \tilde{w}\tilde{\omega}\rangle \tag{47}$$

Since the $\langle \tilde{w}\tilde{\omega}\rangle$ term will be, in general, unknown — it depends on the vertical distribution of the current velocity — it may be set to zero as a first approximation, in which case Eq. (47) would read

$$\frac{\partial}{\partial z}\left(\frac{\langle \tau_{xz}\rangle}{\rho}\right) = g\frac{\partial\langle\eta\rangle}{\partial x} + \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\langle \tilde{u}^2\rangle - \langle \tilde{w}^2\rangle\right)\right] \tag{48}$$

It may thus be seen that the $\langle \tilde{u}\tilde{w}\rangle$-term first effect is to halve the normal wave stress contribution to $\langle \tau_{xz}\rangle$. These stresses are usually small compared to $g(\partial\langle\eta\rangle/\partial x)$ inside the surf zone (Svendsen and Lorenz, 1989) but that is not necessarily the case outside the breaker region (Putrevu and Svendsen, 1993). This means that the $\langle \tilde{u}\tilde{w}\rangle$ contribution, and therefore the need to calculate it, is expected to be more significant in areas in which mean water level gradients do not dominate the momentum balance equation.

6. Summary and conclusions

An alternative equation to evaluate $\langle \tilde{u}\tilde{w}\rangle$ has been derived from the definition of oscillatory vorticity. This equation has been further elaborated for the case of irrotational wave motion making use of linear wave theory. The obtained results coincide or degenerate into previously presented expressions for irrotational wave motion (shallow-water approximation) over a sloping bottom (De Vriend and Kitou, 1990a) and surf-zone breaking waves over a horizontal bottom (Deigaard and Fredsøe, 1989).

The near-bed value of $\langle \tilde{u}\tilde{w}\rangle$, starting point to calculate the vertical distribution of $\langle \tilde{u}\tilde{w}\rangle$, has been obtained by a very simplistic kinematic argument. Further elaborations of the proposed equations should include a more accurate treatment of the bottom boundary layer and its effects.

For the case of rotational wave motion the general equation has been particularized using a set of simplifying assumptions that correspond to a horizontal wave–current flume. The reason is that it is expected that data accurate and detailed enough to test this kind of equations can only be obtained with this type of flume (see e.g. the LIP-13G data set described in Luth et al. (1994).

The proposed expressions, though not an alternative wave theory, allow to calculate the effective (wave) shear stress $\langle \tilde{u}\tilde{w} \rangle$ in terms of the effective (wave) normal stresses, $\langle \tilde{u}^2 \rangle$ and $\langle \tilde{w}^2 \rangle$, and the oscillatory vorticity $\tilde{\omega}$. This kind of expression highlights the relative importance of the various physical mechanisms contributing to $\langle \tilde{u}\tilde{w} \rangle$. The obtained expressions can also be used with ease for standard validation test cases. They can also be employed in the time-averaged momentum balance equations in which they play a fundamental role at least from a qualitative standpoint (i.e. to achieve consistency). Ongoing research therefore includes applying the new $\langle \tilde{u}\tilde{w} \rangle$ equation to wave–current interaction and wave-induced circulation problems.

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