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Short Communication

A note on the diffraction of obliquely incident water waves by a stepwise obstacle

J.P. Rhee

Coastal and Hydraulics Laboratory, USAE Engineer Research and Development Center, CEERD-HC-P, 3909 Halls Ferry Road, Vicksburg, MS 39180-6199, USA

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Abstract

The second-order problem of the propagation of surface gravity waves over an asymmetric rectangular obstacle in an oblique sea is solved numerically using a Green's theorem integral-equation method. Published by Elsevier Science Ltd.

Keywords: Surface gravity waves; Stepwise obstacle; Green's theorem

1. Introduction

Study on the harmonic generation of water waves propagating over a discontinuity in depth continues to receive much attention (for example, Ref. [1]). In addition, in recent years there have been significant advances in nonlinear diffraction theory for problems of the wave-body interactions (see, for example, Refs. [2,3] for computations at third-order). In this short note we extend the Green's theorem integral-equation method for an asymmetrical ridge in an oblique sea, shown in Fig. 1, where, with the (x–z)-plane the undisturbed free-surface and the y-axis positive upwards, the depth $H(x)$ is given by

$$H(x) = \begin{cases} h_1, & x < 0 \\ h_2, & 0 < x < L \\ h_3, & x > L. \end{cases} \tag{1}$$

In particular, we shall compare numerical results with those based on Galerkin's method [4] for a symmetrical ridge ($h_1 = h_3$) in normally incident waves. The present method has previously been used for a shelf (a single vertical step) [5], and the same formulation is employed here.

This potential-flow problem is well known, and in the case of linearized long waves in shallow water an exact solution is available [6]. For similar problems of a ridge linear solutions can be found in Newman [7] and Mei and Black [8], and nonlinear solutions for long waves trapped on a long ridge in Agnon and Mei [9].

2. Definitions

We use largely the same notation as that used in our previous paper [5]. It is assumed that $h_1 \neq h_3$, $h_1 > h_2$, $h_2 < h_3$ with $H(x)$ constant in the z-direction and that waves of first-order amplitude A and frequency ω arrive obliquely at an angle θ with respect to the x axis from $x = -\infty$.

The superscripts (1) and (2) are appended to denote the first- and second-order functions, respectively. The velocity potential Φ is expressed as

$$\Phi(x, y, z, t) = \Phi^{(1)} + \Phi^{(2)} + O(\epsilon^3), \tag{2}$$

where $\epsilon(\ll 1)$ is the wave slope.

We divide the fluid domain into three separate regions by drawing vertical lines at $x = 0$ and $x = L$, and use the subscripts 1, 2, and 3 to denote the regions $x < 0$, $0 < x < L$, and $x > L$, respectively. We then define the wavenumbers k_1, k_2, and k_3, and the corresponding x components k_{1x}, k_{2x}, and k_{3x} as

$$k_1 \tanh k_1 h_1 = k_0, \qquad k_{1x} = (k_1^2 - k_z^2)^{1/2}, \tag{3}$$

$$k_2 \tanh k_2 h_2 = k_0, \qquad k_{2x} = (k_2^2 - k_z^2)^{1/2}, \tag{4}$$

$$k_3 \tanh k_2 h_3 = k_0, \qquad k_{3x} = (k_3^2 - k_z^2)^{1/2}, \tag{5}$$

where $k_0 = \omega^2/g$, g is the gravitational acceleration, and

$$k_z = k_1 \sin \theta. \tag{6}$$

E-mail address: j.rhee@chl.wes.army.mil (J.P. Rhee).

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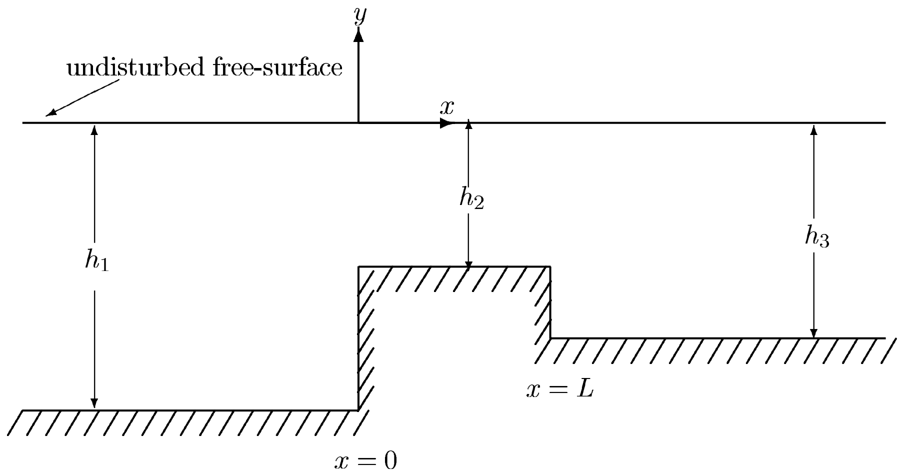

Fig. 1. Sketch of a stepwise obstacle.

In addition, we define

$$m_1 \tanh m_1 h_1 = 4k_0, \qquad m_{1x} = (m_1^2 - (2k_z)^2)^{1/2}, \tag{7}$$

$$m_2 \tanh m_2 h_2 = 4k_0, \qquad m_{2x} = (m_2^2 - (2k_z)^2)^{1/2}, \tag{8}$$

$$m_3 \tanh m_3 h_3 = 4k_0, \qquad m_{3x} = (m_3^2 - (2k_z)^2)^{1/2}. \tag{9}$$

We rewrite the potentials $\Phi^{(1)}$ and $\Phi^{(2)}$ as

$$\Phi^{(1)}(x,y,z,t) = \mathrm{Re}[\phi^{(1)}(x,y)\mathrm{e}^{\mathrm{i}(k_z z - \omega t)}], \tag{10}$$

$$\Phi^{(2)}(x,y,z,t) = \mathrm{Re}[\phi^{(2)}(x,y)\mathrm{e}^{2\mathrm{i}(k_z z - \omega t)}] + \Phi_0^{(2)}, \tag{11}$$

where $\Phi_0^{(2)}$ is time-independent.

With $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the problem for $\phi^{(1)}$ is defined as

$$\nabla^2\phi^{(1)} - k_z^2\phi^{(1)} = 0 \qquad \text{in the fluid}, \tag{12}$$

$$\partial\phi^{(1)}/\partial y - k_0\phi^{(1)} = 0 \qquad \text{on } y = 0, \tag{13}$$

$$\partial\phi^{(1)}/\partial n = 0 \text{ on } y = H(x) \tag{14}$$

and the problem for $\phi^{(2)}$ is defined as

$$\nabla^2\phi^{(2)} - 4k_z\phi^{(2)} = 0 \qquad \text{in the fluid}, \tag{15}$$

$$\partial\phi^{(2)}/\partial y - 4k_0\phi^{(2)} = q(x) \qquad \text{on } y = 0, \tag{16}$$

$$\partial\phi^{(2)}/\partial n = 0 \qquad \text{on } y = H(x), \tag{17}$$

where $q(x)$ can be found in the form

$$q(x) = \frac{\mathrm{i}\omega}{g}\left[\left(\frac{\partial\phi^{(1)}}{\partial x}\right)^2 + \frac{3}{2}\left(k_0^2 - k_z^2\right)(\phi^{(1)})^2 + \frac{1}{2}\phi^{(1)}\frac{\partial^2\phi^{(1)}}{\partial x^2}\right]_{y=0}. \tag{18}$$

The free-surface elevation $y = \zeta(x,z,t)$ is given by

$$\zeta(x,z,t) = -\frac{1}{g}\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{g}\frac{\partial\Phi}{\partial t}\frac{\partial\Phi^2}{\partial y\partial t}\right]_{y=0} + O(\epsilon^3). \tag{19}$$

The second-order free-surface elevation, $\zeta^{(2)}$, consists of several components. Denoting the contributions from $\phi^{(1)}$ and $\phi^{(2)}$ by $\zeta_{1\mathrm{st}}^{(2)}$ and $\zeta_{2\mathrm{nd}}^{(2)}$, respectively, and the time-independent part by $\bar\zeta^{(2)}$, we write

$$\zeta^{(2)} = \zeta_{1\mathrm{st}}^{(2)} + \zeta_{2\mathrm{nd}}^{(2)} + \bar\zeta^{(2)}, \tag{20}$$

where

$$\zeta_{1\mathrm{st}}^{(2)} = -\frac{1}{4g}\left[\left(\frac{\partial\phi^{(1)}}{\partial x}\right)^2 + \left(3k_0^2 - k_z^2\right)(\phi^{(1)})^2\right]_{y=0}\mathrm{e}^{2\mathrm{i}(k_z z - \omega t)}, \tag{21}$$

$$\zeta_{2\mathrm{nd}}^{(2)} = \left[2\frac{i\omega}{g}\phi^{(2)}\right]_{y=0}\mathrm{e}^{2\mathrm{i}(k_z z - \omega t)} = \zeta_f^{(2)} + \zeta_l^{(2)}, \tag{22}$$

$$\bar\zeta^{(2)} = -\frac{1}{4g}[|\frac{\partial\phi^{(1)}}{\partial x}|^2 - \left(k_0^2 - k_z^2\right)|\phi^{(1)}|^2]_{y=0}. \tag{23}$$

Note that $\zeta_{2\mathrm{nd}}^{(2)}$ is further split into the contributions from the free-wave ($\zeta_f^{(2)}$) and locked-wave ($\zeta_l^{(2)}$) potentials.

3. Integral equation

For constant depth h, a Green's function G satisfying $\nabla^2 G = -\delta(x-\xi)(y-\eta)$, Eqs. (13) and (14), and suitable radiation conditions at $x = \pm\infty$ can be found as (see

Ref. [10])

$$G(x,y;\xi,\eta;k) = \frac{\mathrm{i}2k\cosh\{k(\eta+h)\}\cosh\{k(y+h)\}}{k_x\{2kh+\sinh(2kh)\}}\mathrm{e}^{\mathrm{i}|x-\xi|k_x} + \sum_{n=1}^{\infty}\frac{2\alpha_n\cos\{\alpha_n(\eta+h)\}\cos\{\alpha_n(y+h)\}}{\beta_n\{2\alpha_n h+\sin(2\alpha_n h)\}}\mathrm{e}^{-|x-\xi|\beta_n}, \quad (24)$$

where $k\tanh(kh) = k_0$, $k_x = +(k^2-k_z^2)^{1/2}$, and $\beta_n = +(\alpha_n^2+k_z^2)^{1/2}$ with α_n $(n = 1,2,3,\cdots)$ being the positive and real roots of

$$\alpha_n\tan(\alpha_n h) = -k_0. \quad (25)$$

We denote the Green's functions for the three regions of h_1, h_2, and h_3 by G_1, G_2, and G_3, respectively, i.e.

$$G_1 = G(x,y;\xi,\eta;k_1), \qquad G_2 = G(x,y;\xi,\eta;k_2), \qquad G_3 = G(x,y;\xi,\eta;k_3). \quad (26)$$

For the first-order problem, applying Green's theorem to $\phi^{(1)}$ and the Green's function for each region, we find

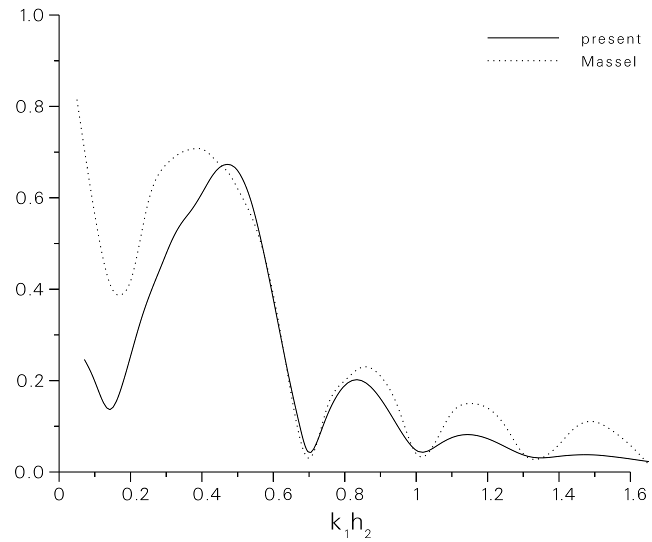

Fig. 2. A comparison of the numerical results with Massel [4]. $|\zeta_f^{(2)}|/A$ at $x = +\infty$ is plotted against k_1h_2, for $\theta = 0$ degrees (normal incidence), $h_1/h_2 = 2$, $h_3/h_2 = 2$, $L/h_2 = 8$, and $A/h_2 = 0.134$.

$$\phi^{(1)}(\xi,\eta) = \begin{cases} \int_{-h_1}^{0}\left[G_1\frac{\partial\phi^{(1)}}{\partial x} - \phi^{(1)}\frac{\partial G_1}{\partial x}\right]_{x=0}\mathrm{d}y + A_0(\xi,\eta), & \xi < 0, \\ \int_{-h_2}^{0}\left[-G_2\frac{\partial\phi^{(1)}}{\partial x} + \phi^{(1)}\frac{\partial G_2}{\partial x}\right]_{x=0}\mathrm{d}y + \int_{-h_2}^{0}\left[G_2\frac{\partial\phi^{(1)}}{\partial x} - \phi^{(1)}\frac{\partial G_2}{\partial x}\right]_{x=L}\mathrm{d}y, & 0 < \xi < L, \\ \int_{-h_3}^{0}\left[-G_3\frac{\partial\phi^{(1)}}{\partial x} + \phi^{(1)}\frac{\partial G_3}{\partial x}\right]_{x=L}\mathrm{d}y, & \xi > L, \end{cases} \quad (27)$$

where $A_0(\xi,\eta)$ is a contribution from the incident wave potential to the line integral at $x = -\infty$, i.e.

$$A_0(\xi,\eta) = -\frac{\mathrm{i}gA}{\omega}\frac{\cosh k_1(\eta+h_1)}{\cosh k_1h_1}\mathrm{e}^{\mathrm{i}k_{1x}\xi}. \quad (28)$$

We assume the continuity of $\phi^{(1)}$ and $\partial\phi^{(1)}/\partial x$ at the discontinuity in depth (at $x = 0$ and $x = L$). Since $\partial G/\partial x$ becomes a delta function as $\xi \to 0$ and $\xi \to L$, then we obtain a set of integral equations for $\partial\phi^{(1)}/\partial x$ over the vertical boundaries ($x = 0$, $-h_2 < y < 0$ and $x = L$, $-h_2 < y < 0$) as

$$\int_{-h_2}^{0}[G_1(0,y;0,\eta) + G_2(0,y;0,\eta)]\frac{\partial\phi^{(1)}}{\partial x}(0,y)\mathrm{d}y - \int_{-h_2}^{0}\left[G_2(L,y;0,\eta) + 2\int_{-h_2}^{0}G_3(L,y;L,u)\frac{\partial G_2}{\partial x}(L,u;0,\eta)\mathrm{d}u\right]\frac{\partial\phi^{(1)}}{\partial x}(L,y)\mathrm{d}y = -A_0(0,\eta), \quad (29)$$

$$\int_{-h_2}^{0}\left[G_2(0,y;L,\eta) - 2\int_{-h_2}^{0}G_1(0,y;L,u)\frac{\partial G_2}{\partial x}(0,u;L,\eta)\mathrm{d}u\right]\frac{\partial\phi^{(1)}}{\partial x}(0,y)\mathrm{d}y - \int_{-h_2}^{0}[G_2(L,y;L,\eta) + G_3(L,y;L,\eta)]\frac{\partial\phi^{(1)}}{\partial x}(L,y)\mathrm{d}y = 2\int_{h_2}^{0}A_0(0,y)\frac{\partial G_2}{\partial x}(0,y;L,\eta)\mathrm{d}y. \quad (30)$$

For the second-order problem, we redefine the Green's functions as

$$G_1 = G(x, y; \xi, \eta; m_1), \qquad G_2 = G(x, y; \xi, \eta; m_2), \qquad G_3 = G(x, y; \xi, \eta; m_3) \tag{31}$$

for the three regions of h_1, h_2, and h_3, respectively. Then applying Green's theorem to the regions of h_1, h_2, and h_3,

$$\phi^{(2)}(\xi,\eta) = \begin{cases} \int_{-h_2}^{0}\left[G_1\frac{\partial\phi^{(2)}}{\partial x}\right]_{x=0}\mathrm{d}y - \int_{-h_1}^{0}\left[\phi^{(2)}\frac{\partial G_1}{\partial x}\right]_{x=0}\mathrm{d}y + \int_{x=-\infty}^{x=0} G_1 q(x)\mathrm{d}x + \int_{-h_1}^{0}\left[\phi^{(2)}\frac{\partial G_1}{\partial x} - G_1\frac{\partial\phi^{(2)}}{\partial x}\right]_{x=-\infty}\mathrm{d}y, & \xi < 0, \\ \int_{-h_2}^{0}\left[-G_2\frac{\partial\phi^{(2)}}{\partial x} + \phi^{(2)}\frac{\partial G_2}{\partial x}\right]_{x=0}\mathrm{d}y + \int_{-h_2}^{0}\left[G_2\frac{\partial\phi^{(2)}}{\partial x} - \phi^{(2)}\frac{\partial G_2}{\partial x}\right]_{x=L}\mathrm{d}y + \int_{x=0}^{x=L} G_2 q(x)\mathrm{d}x, & 0 < \xi < L, \\ -\int_{-h_2}^{0}\left[G_3\frac{\partial\phi^{(2)}}{\partial x}\right]_{x=L}\mathrm{d}y + \int_{-h_3}^{0}\left[\phi^{(2)}\frac{\partial G_3}{\partial x}\right]_{x=L}\mathrm{d}y + \int_{x=L}^{x=+\infty} G_3 q(x)\mathrm{d}x + \int_{-h_3}^{0}\left[G_3\frac{\partial\phi^{(2)}}{\partial x} - \phi^{(2)}\frac{\partial G_3}{\partial x}\right]_{x=+\infty}\mathrm{d}y, & \xi > L, \end{cases} \tag{32}$$

where $q(x)$ is defined by Eq. (18).

In a manner analogous to establishing the integral equations for the first-order problem, assuming the continuity of $\phi^{(2)}$ and $\partial\phi^{(2)}/\partial x$ at $x = 0$ and $x = L$ we obtain a set of integral equations for $\partial\phi^{(2)}/\partial x$ over the vertical boundaries ($x = 0$, $-h_2 < y < 0$ and $x = L$, $-h_2 < y < 0$) as

$$\begin{aligned} &\int_{-h_2}^{0}[G_2(0,y;0,\eta) + G_1(0,y;0,\eta)]\frac{\partial\phi^{(2)}}{\partial x}(0,y)\mathrm{d}y \\ &\quad - \int_{-h_2}^{0}\left[G_2(L,y;0,\eta) - 2\int_{-h_2}^{0} G_3(L,y;L,u)\frac{\partial G_2}{\partial x}(L,u;0,\eta)\mathrm{d}u\right]\frac{\partial\phi^{(2)}}{\partial x}(L,y)\mathrm{d}y \\ &\quad = P_0(0,\eta), \end{aligned} \tag{33}$$

$$\begin{aligned} &\int_{-h_2}^{0}\left[G_2(0,y;L,\eta) - 2\int_{-h_2}^{0} G_1(0,y;L,u)\frac{\partial G_2}{\partial x}(0,u;L,\eta)\mathrm{d}u\right]\frac{\partial\phi^{(2)}}{\partial x}(0,y)\mathrm{d}y - \int_{-h_2}^{0}[G_2(L,y;L,\eta) \\ &\quad + G_3(L,y;L,\eta)]\frac{\partial\phi^{(2)}}{\partial x}(L,y)\mathrm{d}y \\ &\quad = Q_0(L,\eta), \end{aligned} \tag{34}$$

where

$$\begin{aligned} P_0(0,\eta) = &\int_{x=0}^{x=L} G_2(x,0;0,\eta)q(x)\mathrm{d}x - \int_{x=-\infty}^{x=0} G_1(x,0;0,\eta)q(x)\mathrm{d}x - \int_{-h_1}^{0}\Bigg(\phi^{(2)}(-\infty,y)\frac{\partial G_1}{\partial x}(-\infty,y;0,\eta) \\ &- G_1(-\infty,y;0,\eta)\frac{\partial\phi^{(2)}}{\partial x}(-\infty,y)\Bigg)\mathrm{d}y - 2\int_{-h_2}^{0}\Bigg[\int_{-h_3}^{0}\Bigg(G_3(\infty,u;L,y)\frac{\partial\phi^{(2)}}{\partial x}(\infty,u) - \phi^{(2)}(\infty,u)\frac{\partial G_3}{\partial x}(\infty,u;L,y)\Bigg)\mathrm{d}u \\ &+ \int_{x=L}^{x=\infty} G_3(x,0;L,y)q(x)\mathrm{d}x\Bigg]\frac{\partial G_2}{\partial x}(L,y;0,\eta)\mathrm{d}y, \end{aligned} \tag{35}$$

$$\begin{aligned} Q_0(L,\eta) = &\int_{x=0}^{x=L} G_2(x,0;L,\eta)q(x)\mathrm{d}x - \int_{x=L}^{x=\infty} G_3(x,0;L,\eta)q(x)\mathrm{d}x - \int_{-h_3}^{0}\Bigg(G_3(\infty,y;L,\eta)\frac{\partial\phi^{(2)}}{\partial x}(\infty,y) \\ &- \phi^{(2)}(\infty,y)\frac{G_3}{\partial x}(\infty,y;L,\eta)\Bigg)\mathrm{d}y - 2\int_{-h_2}^{0}\Bigg[\int_{-h_3}^{0}\Bigg(G_1(-\infty,u;0,y)\frac{\partial\phi^{(2)}}{\partial x}(-\infty,u) - \phi^{(2)}(-\infty,u)\frac{\partial G_1}{\partial x}(-\infty,u;0,y)\Bigg) \\ &\times\mathrm{d}u - \int_{x=-\infty}^{x=0} G_1(x,0;0,y)q(x)\mathrm{d}x\Bigg]\frac{\partial G_2}{\partial x}(0,y;L,\eta)\mathrm{d}y. \end{aligned} \tag{36}$$

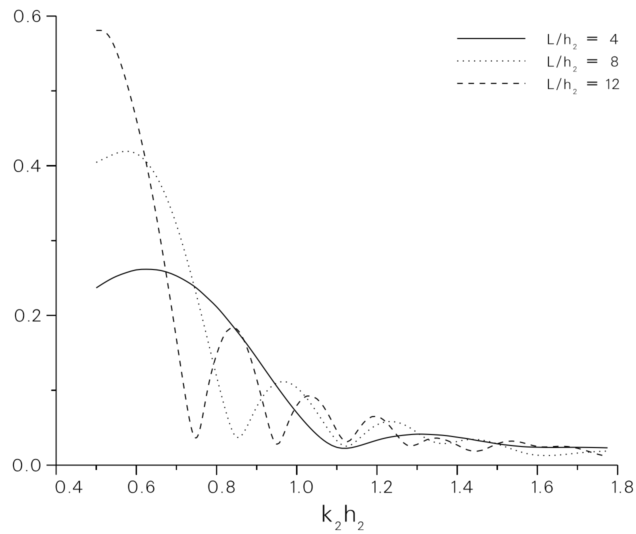

Fig. 3. $|\zeta_f^{(2)}|/A$ at $x = +\infty$ is plotted against k_2h_2, for $\theta = 45$ degrees, $h_1/h_2 = 4$, $h_3/h_2 = 2$, $L/h_2 = 4, 8, 12$ and $A/h_2 = 0.1$.

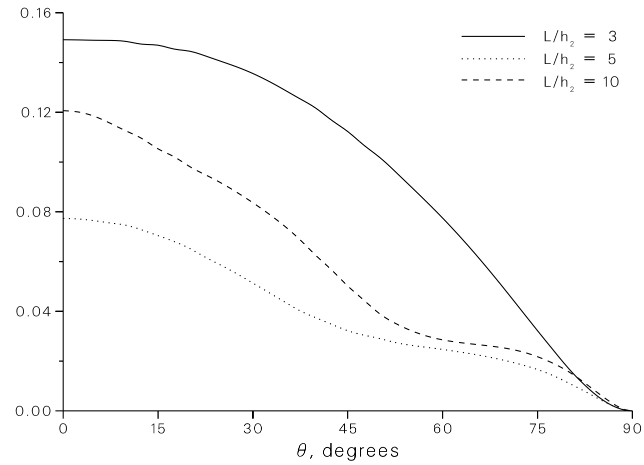

Fig. 4. Results for oblique wave incidence. $|\zeta_f^{(2)}|/A$ at $x = +\infty$ is plotted against the incident-wave angle θ, with $k_2h_2 = 1.0$, $A/h_2 = 0.1$, $h_1/h_2 = 3$, $h_3/h_2 = 2$, and $L/h_2 = 3, 5, 10$.

4. Numerical examples

The numerical procedures for solving Eqs. (29) and (30) (and Eqs. (33) and (34)) are standard. A simultaneous set of $2N$ algebraic equations was solved by dividing each of the two vertical boundaries at $x = 0$ and $x = L$ into N equal segments of length of h_2/N and by approximating $\partial\phi^{(1)}/\partial x$ (and $\partial\phi^{(2)}/\partial x$) by a piecewise step function, which is constant over each segment and evaluated at the midpoints of the segments. The present results were computed with $50 \le 2N \le 150$.

The integrands $G_1q(x)$ and $G_3q(x)$ oscillate with slowly-varying amplitude as $|x| \to \infty$. For numerical computation, we applied Green's theorem in the finite fluid domain. The free-surface integrals in Eqs. (35) and (36) were evaluated by replacing the infinite intervals $[-\infty, 0]$ and $[L, \infty]$ with the finite intervals $[-x_l, 0]$ and $[L, x_r]$, respectively. The values of $\phi^{(2)}(\pm\infty, y)$ and their derivatives (in Eqs. (35) and (36)) were evaluated by use of the asymptotic forms of $\phi^{(2)}(\pm\infty, y)$ (see Ref. [5]) with large values of x_l and x_r. The computations were repeated by increasing the values x_l and x_r until the numerical results were qualitatively independent from x_l and x_r. The second-order results presented here were computed accurate to two decimal places.

We recall that $h_2 < h_3$ and that the present method fails when a linear caustic is present. In addition, we note that the validity of the second-order solutions are limited to the range $k_2A/(k_2h_2)^3 \ll 1$ as $k_2h_2 \to 0$ [11].

The presentation of the first-order results is omitted here, but we remark that we found the present results virtually identical to the results of Mei and Black [8], for $\theta = 0$ with $h_1 = h_3$ and $L/h_2 = 4, 8, 12$.

Fig. 2 shows the second-order free wave amplitude $|\zeta_f^{(2)}|$ (normalized by A) at $x = +\infty$ compared with Massel [4] for normally incident waves ($\theta = 0$) with $h_1 = h_3$. With the scales [4] an accurate comparison is difficult, but we notice some apparent disparity between the two solutions at smaller k_1h_2 (<0.4) and at large k_1h_2 (>0.9).

Fig. 3 shows the second-order free wave amplitude $|\zeta_f^{(2)}|$ (normalized by A) at $x = +\infty$ for obliquely incident waves ($\theta = 45°$) with an asymmetric obstacle ($h_1 \neq h_2$).

Fig. 4 demonstrates the effect of the incidence angle θ on the second-order free wave amplitude $|\zeta_f^{(2)}|$ (normalized by A) at $x = +\infty$. The results were obtained with k_2h_2 and A/h_2 fixed at 1.0 and 0.1, respectively.

The computations were performed on the Cray T3E, and the computing time required to generate the present results was 2000–3000 CPU seconds for each calculation.

In conclusion, the diffraction of obliquely incident water waves by a stepwise obstacle was investigated using the integral-equation method based upon Green's theorem. With the Green's function, the method facilitates dealing with problems of oblique waves. The results are in good agreement with the previous first-order results [8] but in some qualitative disagreement with the previous second-order results [4].

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References

[1] Brossard J, Chagdali M. Experimental investigation of the harmonic generation by waves over a submerged plate. Coast Engng 2001;42:277–90.

[2] Faltinsen OM, Newman JN, Vinje T. Nonlinear wave loads on a slender vertical cylinder. J Fluid Mech 1995;289:179–98.
[3] Liu Y, Xue M, Yue DKP. Computations of fully nonlinear three-dimensional wave–wave and wave–body interactions. Part 2. Nonlinear waves and forces on a body. J Fluid Mech 2001;438:41–66.
[4] Massel SR. Harmonic generation by waves propagating over a submerged step. Coast Engng 1983;7:357–80.
[5] Rhee JP. On the transmission of water waves over a shelf. Appl Ocean Res 1997;19:161–9.
[6] Mei CC. The applied dynamics of ocean surface waves. Singapore: World Scientific, 1989.
[7] Newman JN. Propagation of water waves past long two-dimensional obstacles. J Fluid Mech 1965;23:23–9.
[8] Mei CC, Black JL. Scattering of surface waves by rectangular obstacles in water of finite depth. J Fluid Mech 1969;38:499–511.
[9] Agnon Y, Mei CC. Trapping and resonance of long shelf waves due to groups of short waves. J Fluid Mech 1988;195:201–21.
[10] Heins AE. Water waves over a channel of finite depth with a dock. Am J Math 1948;70:730–48.
[11] Ursell F. The long wave paradox in the theory of gravity waves. Proc Cambridge Philos Soc 1953;49:685–94.