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On the transmission of water waves over a shelf

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The first- and second-order problems of wave transmission over a step in an oblique sea are solved using a Green's theorem integral equation with a finite-depth Green's function. The first-order transmission and reflection coefficients are shown to be consistent with previous results obtained by using the method of matching eigenfunction expansions (Newman), the variational formulation (Miles), and Galerkin method (Massel). Comparison of the second-order free wave agrees with Massel. It is shown that the ratio of the second- to first-order maximum amplitude can be over 0.2 for the range where the Stokes theory is valid and that at low frequency the second-order potential is more pronounced than the quadratic interaction of the first-order potentials. © 1997 Published by Elsevier Science Ltd.

1 INTRODUCTION

We consider the reflection and transmission of water waves incident on a depth discontinuity, sketched in Fig. 1, where the $(x–z)$-plane coincides with the undisturbed free surface and the y-axis is vertically upwards. The water depth $H(x)$ is assumed to change at $x = 0$, from one constant $H(x) = h_1$ for $x < 0$ to another $H(x) = h_2$ for $x > 0$. This is an idealized situation of waves propagating over a continental shelf, and has received much attention in the literature.

In the limiting case of long waves, solutions appeared in Lamb[1] and later in Bartholomeusz,[2] who also formulated the integral equation for a more generalized problem of finite wavelength (see also Mei,[3] Section 4.2). More complete solutions were given by Newman,[4] Miles,[5] and Mei and Black.[6] Using Havelock's wavemaker theory Newman formulated an integral equation similar to Bartholomeusz's and presented numerical solutions, which showed good agreement with Miles's solutions obtained by applying a variational technique. Extensions of these approaches can be found in, for example, Kirby and Dalrymple[7] for a trench and Smith[8] for a step with horizontal shear. See Rey *et al.*,[9] and Evans and Linton[10] for reviews of these methods and their subsequent applications.

All these studies are based on the linearized wave theory. Recently there has been the growing recognition of the non-linear effects on wave diffraction, especially, in connection with problems of wave interactions with fixed or floating obstacles. See, for example, Molin,[11] Vada,[12] Eatock Taylor *et al.*,[13] Kim and Yue,[14] McIver and McIver,[15] and Wu.[16] A typical problem is the resonance of a large offshore structure whose natural modes are close to the sum of frequencies of the incident waves.[17] Apart from this engineering interest, it is known that the development of a wave spectrum over such a variable bottom can be understood better by the study of non-linear effects.

The second-order wave consists of several components, which may be separated largely into two parts: a harmonic and a mean-surface variation, i.e. a steady set-up or set-down. Part of the harmonic, as well as the mean-surface variation, comes from interactions of the first-order potentials only, such as the self- and cross-interaction of the incident and reflected waves in the reflected wave field and the self-interaction of the transmitted wave in the transmitted wave field. To account for the non-linear effect at second-order, it is still necessary to calculate the second-order potentials. The computation of the second-order potential is complicated, however, mainly because of the difficulty in dealing with the inhomogeneous free-surface boundary condition and the slowly diminishing behaviour at infinity of its forcing term. Molin[11] (and the subsequent studies) showed that the second-order problem can be solved by further splitting the second-order potential into two parts, an outgoing free wave and a wave locked (in phase) to the the first-order wave system. He was then able to transform the problem to that for the free wave potential, which is similar to a linearized problem.

In the present problem, the free wave may be interpreted as the one generated by the presence of the step and scattered into reflected and transmitted wave fields. Massel[18]

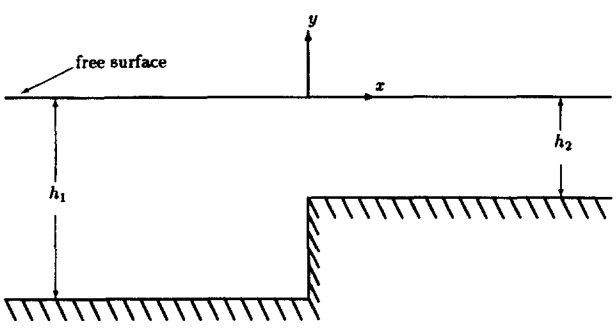

Fig. 1. Sketch of a step shelf.

appears to be the first who solved this problem. By applying Galerkin's method, used previously by Garret[19] for a scattering problem involving a circular dock, he presented numerical results for the first- and second-order solutions. His first-order solution showed a good agreement with the result of Mei and Black[6] obtained by using the closely related variational techniques. In this paper we extend the integral equation to the second-order problem with formulation generalized for an oblique wave. The integral equation is identical to that for the first-order problem, except for the free-surface integral, and has shown success in other nonlinear diffraction problems.[12,14] Thus, particular attention is paid to comparisons with the previous solutions.

2 FORMULATION

2.1 Definitions

The fluid is assumed to be ideal and the flow irrotational, so that the flow is described by a velocity potential $\Phi(x,y,z,t)$ as

$$\Phi(x,y,z,t) = \Phi^{(1)} + \Phi^{(2)} + O(\epsilon^3), \tag{1}$$

where $\epsilon(\ll 1)$ is the wave slope. Here $\Phi^{(1)}$ and $\Phi^{(2)}$ are the first- and second-order potentials, respectively, which are individually governed by Laplace's equation and the appropriate boundary conditions.

Let the subscripts 1 and 2 denote the regions for $x < 0$ and for $x > 0$, respectively. Let an incident wave train of first-order amplitude A, frequency ω and wavenumber k_1 approach from $x = -\infty$, in water of depth h_1, at an angle θ with respect to the x axis. By assuming that the shelf is a vertical step and that the profile is uniform in the z-direction, we write the first-order potential in the form

$$\Phi^{(1)}(x,y,z,t) = \mathrm{Re}\left[\phi^{(1)}(x,y)e^{i(k_z z - \omega t)}\right], \tag{2}$$

where $k_z = k_1 \sin\theta$. The asymptotic forms of $\phi^{(1)}(x,y)$ are given as

$$\phi^{(1)} \sim -\frac{igA}{\omega}\frac{\cosh k_1(y+h_1)}{\cosh k_1 h_1}(e^{ik_{1x}x} + Re^{-ik_{1x}x}), \quad x \to -\infty \tag{3}$$

and

$$\phi^{(1)} \sim -\frac{igA}{\omega}\frac{\cosh k_2(y+h_2)}{\cosh k_2 h_2}Te^{ik_{2x}x}, \quad x \to +\infty, \tag{4}$$

where R and T are the first-order reflection and transmission coefficients, respectively, and g is the acceleration due to gravity. Here the wavenumbers k_1 and k_2 and their x components k_{1x} and k_{2x} are defined as

$$k_1 \tanh k_1 h_1 = k_0, \quad k_{1x} = (k_1^2 - k_z^2)^{1/2} \tag{5}$$

$$k_2 \tanh k_2 h_2 = k_0, \quad k_{2x} = (k_2^2 - k_z^2)^{1/2}, \tag{6}$$

where $k_0 = \omega^2/g$.

The free-surface boundary condition can be found, for instance, in Mei[3] (Section 12.9) as

$$\frac{\partial^2\Phi^{(2)}}{\partial t^2} + g\frac{\partial\Phi^{(2)}}{\partial y} = -\frac{\partial}{\partial t}(\nabla\Phi^{(1)})^2 + \frac{1}{g}\frac{\partial\Phi^{(1)}}{\partial t}\frac{\partial}{\partial y}\left(\frac{\partial^2\Phi^{(1)}}{\partial t^2} + g\frac{\partial\Phi^{(1)}}{\partial y}\right) \quad \text{on } y = 0. \tag{7}$$

From this and eqn (2), $\Phi^{(2)}$ may be written in the form

$$\Phi^{(2)}(x,y,z,t) = \mathrm{Re}\left[\phi^{(2)}(x,y)e^{2i(k_z z - \omega t)}\right] + \Phi_0^{(2)}, \tag{8}$$

where $\Phi_0^{(2)}$ gives at most the third-order contribution to the free surface elevation and will be neglected here.

The free-surface elevation $y = \zeta(x,z,t)$ is given by

$$\zeta(x,z,t) = -\frac{1}{g}\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2\right]_{y=\zeta}, \tag{9}$$

which can be expanded in the form

$$\zeta(x,z,t) = \zeta^{(1)} + \zeta^{(2)} + \bar{\zeta} + O(\epsilon^3), \tag{10}$$

where $\zeta^{(1)}$ is the first-order free-surface elevation

$$\zeta^{(1)} = \left[\frac{i\omega}{g}\phi^{(1)}\right]_{y=0} e^{i(k_z z - \omega t)} \tag{11}$$

and $\bar{\zeta}$ is the time-independent part of the second-order ζ

$$\bar{\zeta} = -\frac{1}{4g}\left[\left|\frac{\partial\phi^{(1)}}{\partial x}\right|^2 - (k_0^2 - k_z^2)\left|\phi^{(1)}\right|^2\right]_{y=0}. \tag{12}$$

$\zeta^{(2)}$, the time-dependent part of the second-order ζ, consists of several components. Denoting the contributions from $\phi^{(1)}$ and $\phi^{(2)}$ by $\zeta_{1st}^{(2)}$ and $\zeta_{2nd}^{(2)}$, respectively,

$$\zeta^{(2)} = \zeta_{1st}^{(2)} + \zeta_{2nd}^{(2)}, \tag{13}$$

where

$$\zeta_{1st}^{(2)} = -\frac{1}{4g}\left[\left(\frac{\partial\phi^{(1)}}{\partial x}\right)^2 + (3k_0^2 - k_z^2)(\phi^{(1)})^2\right]_{y=0} e^{2i(k_z z - \omega t)}, \tag{14}$$

$$\zeta_{2nd}^{(2)} = \left[2\frac{i\omega}{g}\phi^{(2)}\right]_{y=0} e^{2i(k_z z - \omega t)} = \zeta_f^{(2)} + \zeta_l^{(2)}. \tag{15}$$

Note that $\zeta_{2nd}^{(2)}$ is further split into the contributions from the free-wave ($\zeta_f^{(2)}$) and locked-wave ($\zeta_l^{(2)}$) potentials.

2.2 Review of first-order problem

The problem for $\phi^{(1)}$ is defined as

$$\nabla^2\phi^{(1)} - k_z^2\phi^{(1)} = 0 \text{ in the fluid,} \tag{16}$$

$$\partial\phi^{(1)}/\partial y - k_0\phi^{(1)} = 0 \text{ on } y = 0, \tag{17}$$

$$\partial\phi^{(1)}/\partial n = 0 \text{ on } y = H(x), \tag{18}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

An integral equation can be obtained for the unknown $\phi^{(1)}$ at $x = 0$, $-h_1 < y < y_2$ by applying Green's theorem to $\phi^{(1)}$ and a suitable Green's function defined for the entire fluid domain. Evans[20] has derived such a Green's function which can be used for more general problems of a irregular step shape, but a numerical solution using this function has yet to be seen because of the cumbersome procedures for evaluating the complicated Green's function.[10]

In the present case of a vertical step an integral equation is obtained for the unknown $\partial\phi^{(1)}/\partial x$ at $x = 0$, $-h_2 < y < 0$ by simply using a finite-depth Green's function for wave source.

For the depth h, the Green's function G may be written in the form[21]

$$G(x, y; \xi, \eta; k) = \frac{i2k\cosh\{k(\eta+h)\}\cosh\{k(y+h)\}}{k_x\{2kh+\sinh(2kh)\}}e^{i|x-\xi|k_x} \tag{19}$$

$$+ \sum_{n=1}^{\infty} \frac{2\alpha_n\cos\{\alpha_n(\eta+h)\}\cos\{\alpha_n(y+h)\}}{\beta_n\{2\alpha_n h+\sin(2\alpha_n h)\}}e^{-|x-\xi|\beta_n},$$

where (ξ,η) denotes a field point in the fluid region, $k\tanh(kh) = k_0$, $k_x = +(k^2 - k_z^2)^{1/2}$, $\beta_n = +(\alpha_n^2 + k_z^2)^{1/2}$, and α_n ($n = 1,2,3,\ldots$) are the positive and real roots of

$$\alpha_n\tan(\alpha_n h) = -k_0. \tag{20}$$

By using the identity

$$\lim_{\xi\to\pm 0} \partial G(0, y; \xi, \eta)/\partial x = \pm\frac{1}{2}\delta(y-\eta) \tag{21}$$

and by assuming the continuity of $\phi^{(1)}$ and $\partial\phi^{(1)}/\partial x$ at $x = 0$, $-h_2 < y < 0$, the integral equation for $\partial\phi^{(1)}/\partial x$ at $x = 0$, $-h_2 < y < 0$ may be written as[2]

$$\int_{-h_2}^{0} \frac{\partial\phi^{(1)}}{\partial x}(G_1 + G_2)\,dy = -A_0(0, \eta), \tag{22}$$

where G_1 and G_2 denote G for h_1 and G for h_2, respectively, i.e.

$$G_1 = G(x, y; \xi, \eta; k_1) \text{ and } G_2 = G(x, y; \xi, \eta; k_2), \tag{23}$$

and $A_0(\xi,\eta)$ is a contribution from the incident wave potential to the line integral at $x = -\infty$, i.e.

$$A_0(\xi, \eta) = -\frac{igA}{\omega}\frac{\cosh k_1(\eta+h_1)}{\cosh k_1 h_1}e^{ik_{1x}\xi}. \tag{24}$$

After eqn (22) yields $\partial\phi^{(1)}/\partial x$ on $x = 0$, $\phi^{(1)}$ is known everywhere in the fluid. From Green's theorem, we have separate solutions for $x \le 0$ and $x \ge 0$,

$$\phi^{(1)}(\xi, \eta) = \int_{-h_1}^{0}\left(G_1\frac{\partial\phi^{(1)}}{\partial x} - \phi^{(1)}\frac{\partial G_1}{\partial x}\right)dy + A_0(\xi, \eta) \text{ for } \xi \le 0, \tag{25}$$

$$\phi^{(1)}(\xi, \eta) = \int_{-h_2}^{0}\left(-G_2\frac{\partial\phi^{(1)}}{\partial x} + \phi^{(1)}\frac{\partial G_2}{\partial x}\right)dy \text{ for } \xi \ge 0. \tag{26}$$

To find the transmission and reflection coefficients R and T defined by eqn (3) and eqn (4) requires $\phi^{(1)}$ at $x = \pm\infty$. These asymptotic potentials are obtained from eqn (25) and eqn (26), and then the magnitudes of R and T are

$$|R| = \left|\frac{i\omega}{gA}\phi^{(1)}(-\infty, 0)e^{-ik_{1x}x} - 1\right|, \tag{27}$$

$$|T| = \left|\frac{i\omega}{gA}\phi^{(1)}(+\infty, 0)e^{-ik_{2x}x}\right|. \tag{28}$$

2.3 Second-order transmission

The problem for $\phi^{(2)}$ is defined as

$$\nabla^2\phi^{(2)} - 4k_z^2\phi^{(2)} = 0 \text{ in the fluid,} \tag{29}$$

$$\partial\phi^{(2)}/\partial y - 4k_0\phi^{(2)} = q(x) \text{ on } y = 0, \tag{30}$$

$$\partial\phi^{(2)}/\partial n = 0 \text{ on } y = H(x), \tag{31}$$

where $q(x)$ can be found in the form

$$q(x) = \frac{i\omega}{g}\left[\left(\frac{\partial\phi^{(1)}}{\partial x}\right)^2 + \frac{3}{2}(k_0^2 - k_z^2)(\phi^{(1)})^2 + \frac{1}{2}\phi^{(1)}\frac{\partial^2\phi^{(1)}}{\partial x^2}\right]_{y=0}. \tag{32}$$

The far-field behaviour of the forcing function $q(x)$ is found by using eqn (3) and eqn (4),

$$q(x) = -\frac{igA^2}{2\omega}\begin{cases} C_1 e^{2ik_{1x}x} + C_2 e^{-2ik_{1x}x} + C_3, & x \to -\infty \\ D_1 e^{2ik_{2x}x}, & x \to +\infty \end{cases}, \tag{33}$$

where

$$C_1 = 3(k_0^2 - k_1^2),\ C_2 = C_1 R^2,\ C_3 = \left(6(k_0^2 - k_z^2) + 2k_{1x}^2\right)R,$$

$$D_1 = 3(k_0^2 - k_2^2)T^2.$$

When $x \to -\infty$, the first term in eqn (33) results from the self-interaction of the first-order incident waves, the second term from the self-interaction of the first-order reflected waves, and the constant term (C_3) from the interaction of the first-order incident and reflected waves. Note that, for $k_1h_1 = \infty$, the wave field in $x < 0$ is represented by the constant term because there is no contribution to the incident and reflected waves at second order. When $x \to +\infty$, $q(x)$ represents the self-interaction of the first-order transmitted wave.

To specify the radiation condition, $\phi^{(2)}$ is separated into two parts: a free wave which corresponds to a solution to the homogeneous free-surface condition and a locked wave which corresponds to a solution to the non-homogeneous free-surface condition. The far-field form of $\phi^{(2)}$ is then

$$\phi^{(2)} \sim -\frac{igA^2}{2\omega}\left[R^{(2)}\frac{\cosh m_1(y+h_1)}{\cosh m_1 h_1}e^{-im_{1x}x} - \frac{\cosh 2k_1(y+h_1)}{4k_0\sinh^2 k_1 h_1}(C_1 e^{2ik_{1x}x} + C_2 e^{-2ik_{1x}x}) - \frac{1}{4k_0}C_3\right],\quad x \to -\infty, \tag{34}$$

$$\phi^{(2)} \sim -\frac{igA^2}{2\omega}\left[T^{(2)}\frac{\cosh m_2(y+h_2)}{\cosh m_2 h_2}e^{im_{2x}x} - \frac{\cosh 2k_2(y+h_2)}{4k_0\sinh^2 k_2 h_2}D_1 e^{2ik_{2x}x}\right],\quad x \to +\infty, \tag{35}$$

where $m_1\tanh m_1 h_1 = 4k_0$, $m_2\tanh m_2 h_2 = 4k_0$, $m_{1x} = (m_1^2 - (2k_z)^2)^{1/2}$, $m_{2x} = (m_2^2 - (2k_z)^2)^{1/2}$. In both eqn (34) and eqn (35), the first term is the potential representing an outgoing free wave, and $R^{(2)}$ and $T^{(2)}$ are related to their magnitudes. The remainder is the response to quadratic forcing of the first-order incident, reflected and transmitted waves.

An integral equation can be obtained by closely following the procedure described in the previous subsection. Let the Green's functions G_1 and G_2 be redefined as

$$G_1 = G(x, y; \xi, \eta; m_1) \text{ and } G_2 = G(x, y; \xi, \eta; m_2). \tag{36}$$

Applying Green's theorem to the regions of h_1 and h_2,

$$\phi^{(2)}(\xi,\eta) = \int_{x=-\infty}^{x=0} G_1 q(x)\,dx + \int_{y=-h_2}^{y=0} G_1\frac{\partial\phi^{(2)}}{\partial x}dy - \int_{y=-h_1}^{y=0}\phi^{(2)}\frac{\partial G_1}{\partial x}dy + A_{-\infty}(\xi,\eta),\quad \text{for } \xi \le 0, \tag{37}$$

$$\phi^{(2)}(\xi,\eta) = \int_{x=0}^{x=+\infty} G_2 q(x)\,dx - \int_{y=-h_2}^{y=0} G_2\frac{\partial\phi^{(2)}}{\partial x}dy + \int_{y=-h_2}^{y=0}\phi^{(2)}\frac{\partial G_2}{\partial x}dy + A_{+\infty}(\xi,\eta),\quad \text{for } \xi \ge 0, \tag{38}$$

where $A_{\pm\infty}$ are the contributions from the line integrals at $x = \pm\infty$, i.e.

$$A_{-\infty}(\xi,\eta) = \int_{-h_1}^{0}\left(\phi^{(2)}\frac{\partial G_1}{\partial x} - G_1\frac{\partial\phi^{(2)}}{\partial x}\right)dy, \tag{39}$$

$$A_{+\infty}(\xi,\eta) = \int_{-h_2}^{0}\left(G_2\frac{\partial\phi^{(2)}}{\partial x} - \phi^{(2)}\frac{\partial G_2}{\partial x}\right)dy. \tag{40}$$

Upon substituting the asymptotic potentials eqn (34) and eqn (35) into eqn (39) and eqn (40), respectively, and observing that the integrands proportional to $R^{(2)}$ and $T^{(2)}$ vanish, we obtain $A_{\pm\infty}$ in the forms

$$A_{-\infty}(\xi,\eta) = \frac{igA^2}{2\omega}\int_{-h_1}^{0}\left(\left[\frac{\cosh 2k_1(y+h_1)}{4k_0\sinh^2 k_1 h_1}(C_1 e^{2ik_{1x}x} + C_2 e^{-2ik_{1x}x}) + \frac{1}{4k_0}C_3\right]\frac{\partial G_1}{\partial x} + G_1\left[2ik_{1x}\frac{\cosh 2k_1(y+h_1)}{4k_0\sinh^2 k_1 h_1}(C_1 e^{2ik_{1x}x} - C_2 e^{-2ik_{1x}x})\right]\right)dy, \tag{41}$$

$$A_{+\infty}(\xi,\eta) = \frac{igA^2}{2\omega}\int_{-h_2}^{0}\left(G_2\left[\frac{\cosh 2k_2(y+h_2)}{4k_0\sinh^2 k_2 h_2}D_1 e^{2ik_{2x}x}\right] + \left[2ik_{2x}\frac{\cosh 2k_2(y+h_2)}{4k_0\sinh^2 k_2 h_2}D_1 e^{2ik_{2x}x}\right]\frac{\partial G_2}{\partial x}\right)dy. \tag{42}$$

Assuming the continuity of $\phi^{(2)}$ and $\partial\phi^{(2)}/\partial x$ at $x = 0$, the integral equation for $\partial\phi^{(2)}/\partial x$ at $x = 0$ can be written as

$$\int_{y=-h_2}^{y=0}(G_1 + G_2)\frac{\partial\phi^{(2)}}{\partial x}dy = -\int_{x=-\infty}^{x=0} G_1 q(x)\,dx + \int_{x=0}^{x=+\infty} G_2 q(x)\,dx - A_{-\infty}(0_-,\eta) + A_{+\infty}(0_+,\eta). \tag{43}$$

When $\partial\phi^{(2)}/\partial x$ has been obtained on $x = 0$, $\phi^{(2)}$ is found everywhere in the fluid through the use of the expressions eqn (37) and eqn (38). The constants $R^{(2)}$ and $T^{(2)}$ in eqn (34) and eqn (35) are then determined with $\phi^{(2)}$ computed at large $|x|$.

Now, the free-surface elevation $\zeta^{(2)}_{(2nd)}$ defined by eqn (15) can be obtained everywhere. In particular, we find the elevation of the free wave $\zeta_f^{(2)}$ at $|x| = \infty$ in the forms

$$\zeta_f^{(2)} = A^2 R^{(2)} e^{-im_{1x}x} e^{2i(k_z z - \omega t)},\quad x \to -\infty, \tag{44}$$

$$\zeta_f^{(2)} = A^2 T^{(2)} e^{im_{2x}x} e^{2i(k_z z - \omega t)},\quad x \to +\infty. \tag{45}$$

3 REMARKS ON NUMERICAL PROCEDURES

A system of N linear algebraic equations was solved by discretizing eqn (43) (and eqn (22)). We chose to divide the line, $x = 0$ and $h_2 < y < 0$, into N equal segments and approximated the velocity by a piecewise step function. The solutions were then sought for the mid points of the segments. To deal with the problem of the singularity in the velocity in the vicinity of the sharp corner of the step, we simply increased the total number of segments N until sufficient accuracy was obtained. The present results were computed with $32 \le N \le 100$. In evaluating the Green's function, the logarithmic singularity was separated explicitly, and the integrals were calculated analytically. The numerical techniques used are standard. What follows is a brief description for the procedures handling the integrals in the right-hand side of eqn (43) (and eqn (37) and eqn (38)).

These infinite integrals are not convergent in the usual sense, since the integrands G_1q and G_2q oscillate with everlasting amplitude as $|x| \rightarrow \infty$. For numerical solutions, we apply Green's theorem in the finite fluid domain, replacing the intervals $(-\infty, 0)$ and $(0, +\infty)$ with $(-x_l, 0)$ and $(0, x_r)$, respectively, where x_l and x_r are finite. For x_l, x_r large enough, we can find the limits so that the solution is virtually unaffected no matter how much we increase x_l, x_r beyond these limits. We found that in general the solutions are qualitatively independent of the values of x_l and x_r, if x_l, $x_r > 3L$, where L is the wavelength. All results presented in this paper were obtained with x_l, $x_r \geq 10L$.

For each k_1h_1 and k_2h_2, values of $T^{(2)}$ and $R^{(2)}$ were evaluated accurate to two decimal places (four decimal places for T and R) by repeated computations with increasing N, x_l, x_r, and the number of subintervals for x_l and x_r.

4 RESULTS

To compare these computations with the previous solutions for an infinite step, we use $h_2/h_1 = 1/10$ and $h_2/h_1 = 10/1$ with normally incident waves ($\theta = 0$). Following Newman,[4] we express $T = |T|\, e^{i\delta T}$ and $R = |R| e^{i\delta R}$ and append subscripts 1 and 2 to T and R to denote the case ofwaves incident from the deep side ($h_1 > h_2$) and the case of waves incident from the shallow side ($h_1 < h_2$), respectively.

Fig. 2 shows $|T_1|$, $|T_2|$, and $|R|$. Note that $|R_1| = |R_2| = |R|$.[22,23] For large values of k_2h_2 (> 1), the present reflection and transmission coefficients agree well with the two previous solutions—there is little difference within graphical depiction. For small values of k_2h_2 (< 1), however, we observe the effect of a finite step. For instance, $|T_1|$ for a finite step is actually higher than, but within $\approx 6\%$ of, the results for an infinite step for $0.2 < k_2h_2 < 0.7$. Note that the present as well as previous results can be checked with the energy conservation law, $|R|^2 + |T|^2\, C_{g_2}/C_{g_1} = 1$, where C_{g_1} and C_{g_2} denote the group velocities for regions of h_1 and h_2, respectively, and with the expression given by Newman[23] $|T_1T_2| = 1 - |R|^2$. Also, the results are

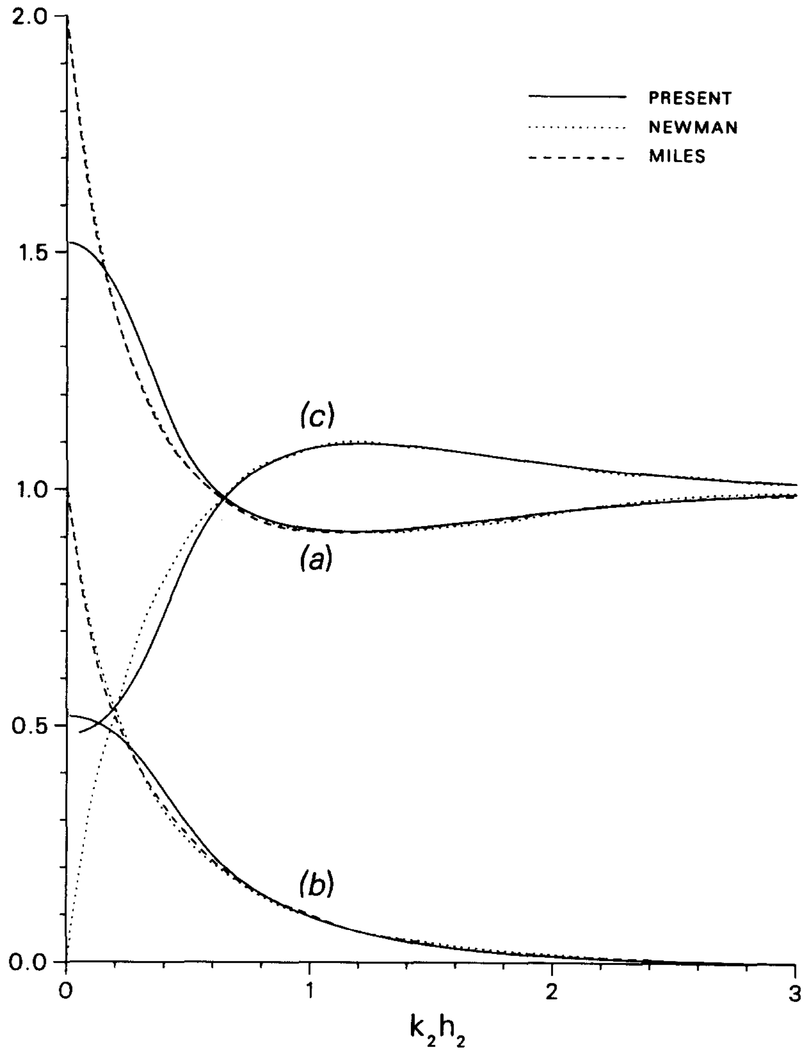

Fig. 2. First-order transmission and reflection coefficients for $h_2/h_1 = 1/10$ (——), compared with Newman [4] (···) and Miles [5] (- - -): (a) $|T_1|$ and (b) $|R_1|$ ($= |R_2|$). (c) is $|T_2|$ for the case where an incident wave arrives from the shallow side with $h_2/h_1 = 10/1$.

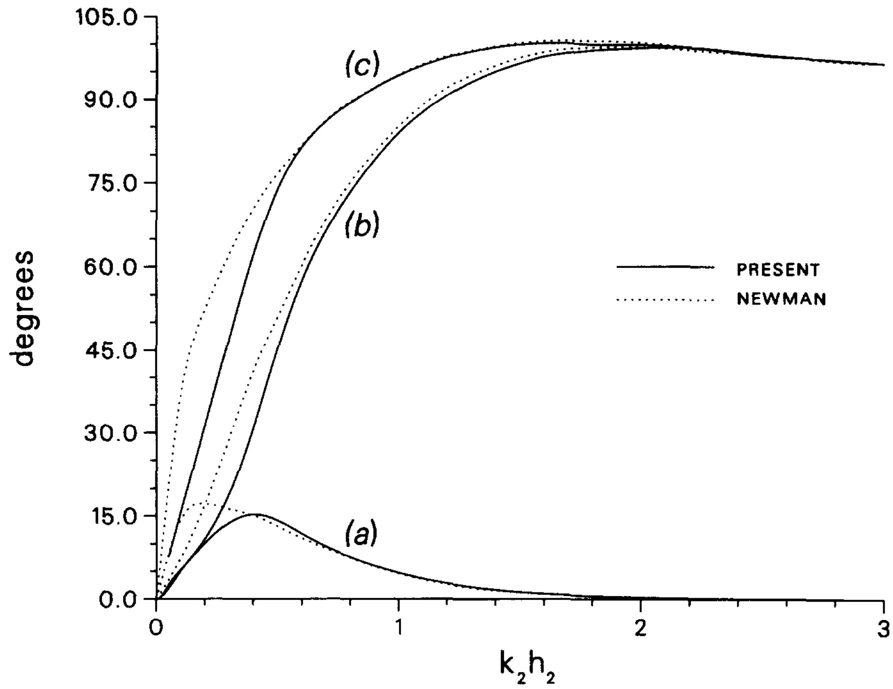

Fig. 3. Phase shifts of the first-order transmission and reflection coefficients for $h_2/h_1 = 1/10$ (——), compared with Newman[4] (···): (a) δT_1 ($= \delta T_2$), (b) $-\delta R_1$, and (c) $(\pi + \delta R_2)$ for the case where an incident wave arrives from the shallow side with $h_2/h_1 = 10/1$.

consistent with the shallow water solutions[1,2]

$$|T| = \frac{2}{1+(h_2/h_1)^{1/2}}, \quad |R| = \frac{|1-(h_2/h_1)^{1/2}|}{1+(h_2/h_1)^{1/2}}. \tag{46}$$

Note that $|T_1| \to 1.52$, $|T_2| \to 0.48$, and $|R| \to 0.52$ as $k_2h_2 \to 0$.

Fig. 3 shows the comparisons of the phase shifts δT, δR_1, and δR_2, where $|\delta T_1| = |\delta T_2| = |\delta T|$. Note that these results confirm the expression,[23] $\delta R_1 + \delta R_2 = \pi + 2\delta T$.

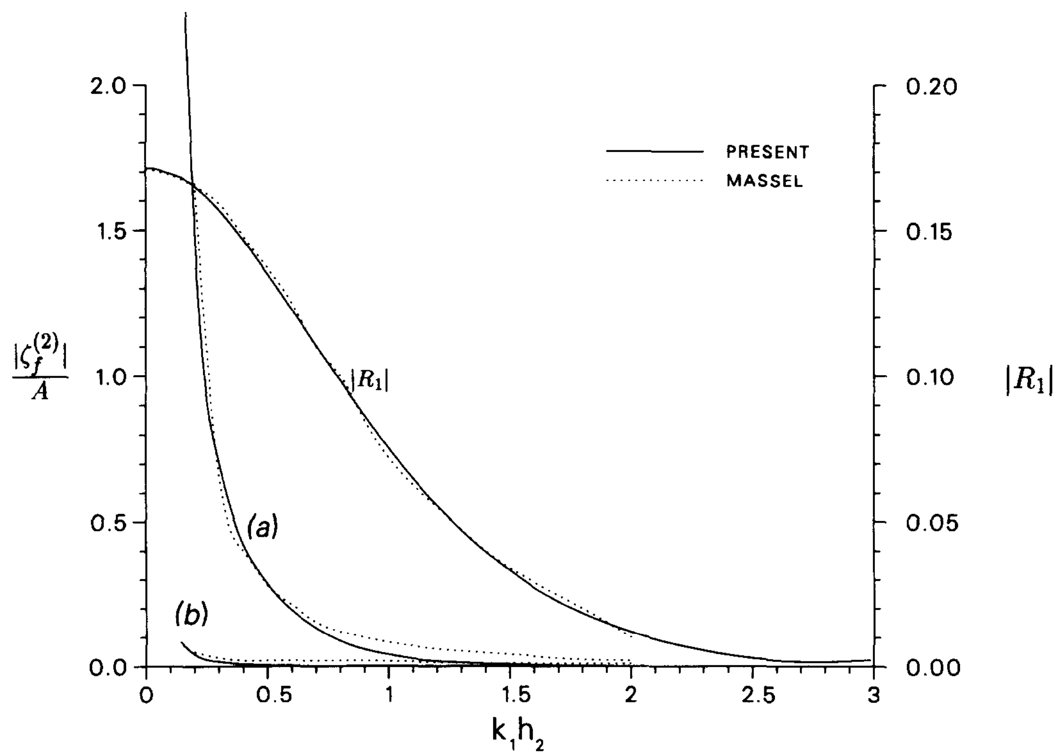

Fig. 4. Second-order free-wave amplitudes for $h_2/h_1 = 1/2$, $A/h_2 = 0.134$ and $A = 0.134$ (——), compared with Massel[18] (···) on normal incidence ($\theta = 0$): (a) at $x = +\infty$ defined by eqn (45) and (b) at $x = -\infty$ defined by eqn (44). Also shown is the comparison for first-order reflection coefficients $|R_1|$. Note that the parameter k_1h_2 is the same as kh_1 in Massel's Fig. 6.

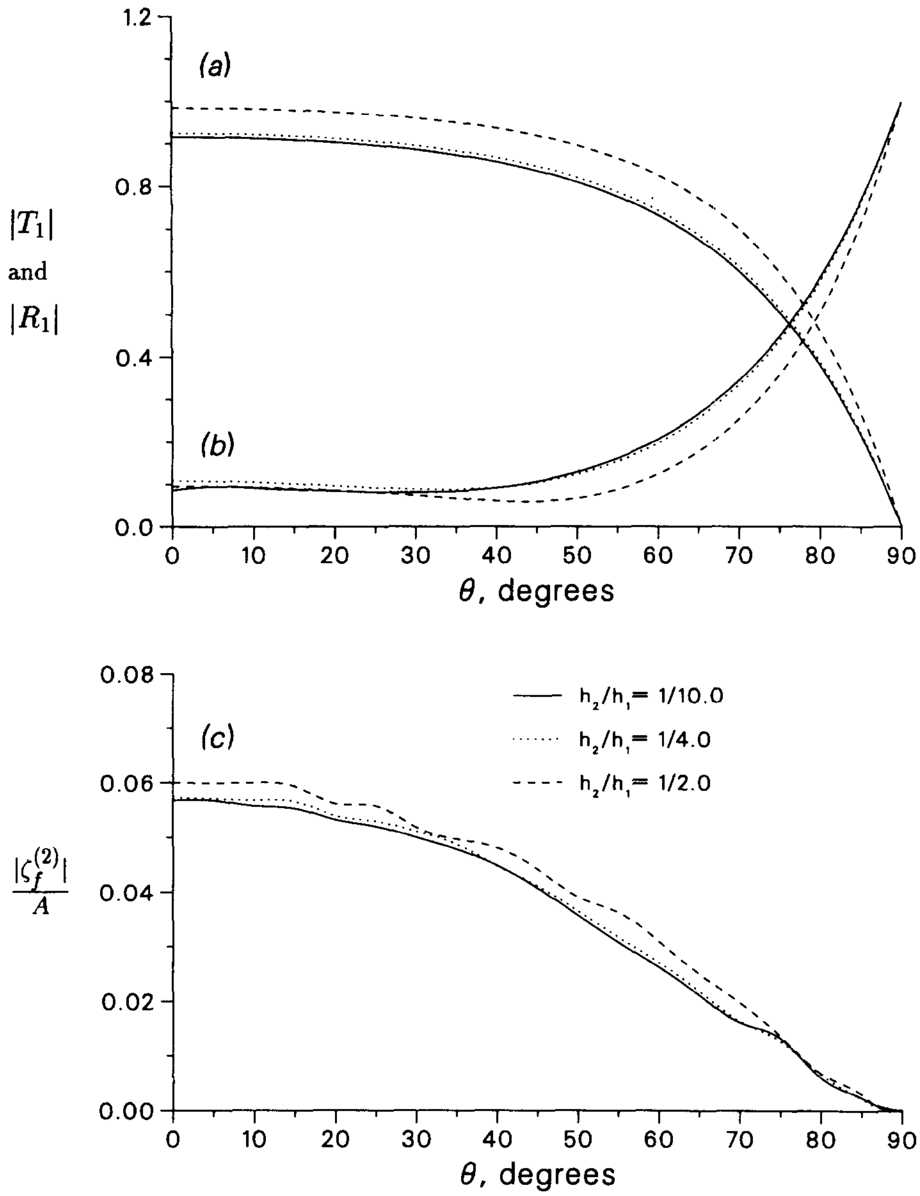

Fig. 5. Result for an oblique wave incident from $x = -\infty$ ($\theta = 0$ for normal incidence) with $h_2/h_1 = 1/10$ (——), $h_2/h_1 = 1/4$ (···) and $h_2/h_1 = 1/2$ (- - -) ($k_2h_2 = 1.0$, $A/h_2 = 0.1$): (a) at the first-order transmission coefficient $|T_1|$, (b) the first-order reflection coefficient $|R_1|$, and (c) the second-order free wave at $x = +\infty$ defined by eqn (45).

Fig. 4 shows the second-order free wave amplitude compared with the result of Massel[18] for normal incidence ($\theta = 0$). With the scale provided by the graphs in Massel, there seems little difference between the two results forsmall k_2h_2 (< 0.7). Some disparity shown at large k_2h_2 (> 1.0) appears well within the range of the numerical error. The two results appear to agree well for the magnitude of the first-order reflection coefficient $|R_1|$.

Fig. 5 shows an example demonstrating the effect of the incidence angle θ on the first- and second-order solutions, with k_2h_2 and A/h_2 fixed at 1.0 and 0.1, respectively, for depth ratios $h_2/h_1 = 10$, 4 and 2. The amplitude of the second-order free wave ($|\zeta_f^{(2)}|$) at $x = -\infty$ is not shown because its magnitude $|\zeta_f^{(2)}/A|$ is small (in the order < 0.001) and comparable to the numerical error. Results appear to be sensitive to values of k_2h_2 in a quantitative sense, but the trends seem similar in other cases.

For an obliquely incident wave, we recall that the conditions eqn (3) and eqn (4) require that $k_z < k_1$, $k_z < k_2$. Thus, when $h_1 < h_2$, a wave incident from the shallow side no longer satisfies these conditions if $\sin\theta > k_2/k_1$.[5],[20] In this case (not shown), the wave undergoes total reflection at first order with the amplitude decreasing exponentially for $x > 0$. However, if $\sin\theta < m_2/2k_1$, the second-order free wave can propagate to the deep side ($\phi^{(2)} \propto e^{im_{2r}x}$ as $x \to \infty$).

Fig. 6 shows the ratio of the second- to first-order maximum wave amplitude at $x = \pm\infty$, for $\theta = 0$ and $h_2/h_1 = 1/10$ with $A/h_2 = 0.1$. The small ratio of A/h_2 is chosen for the expansion eqn (1) to be valid. Recalling Ursell[24] we restrict the solution to the range $k_2A/(k_2h_2)^3 \ll 1$ as k_2h_2

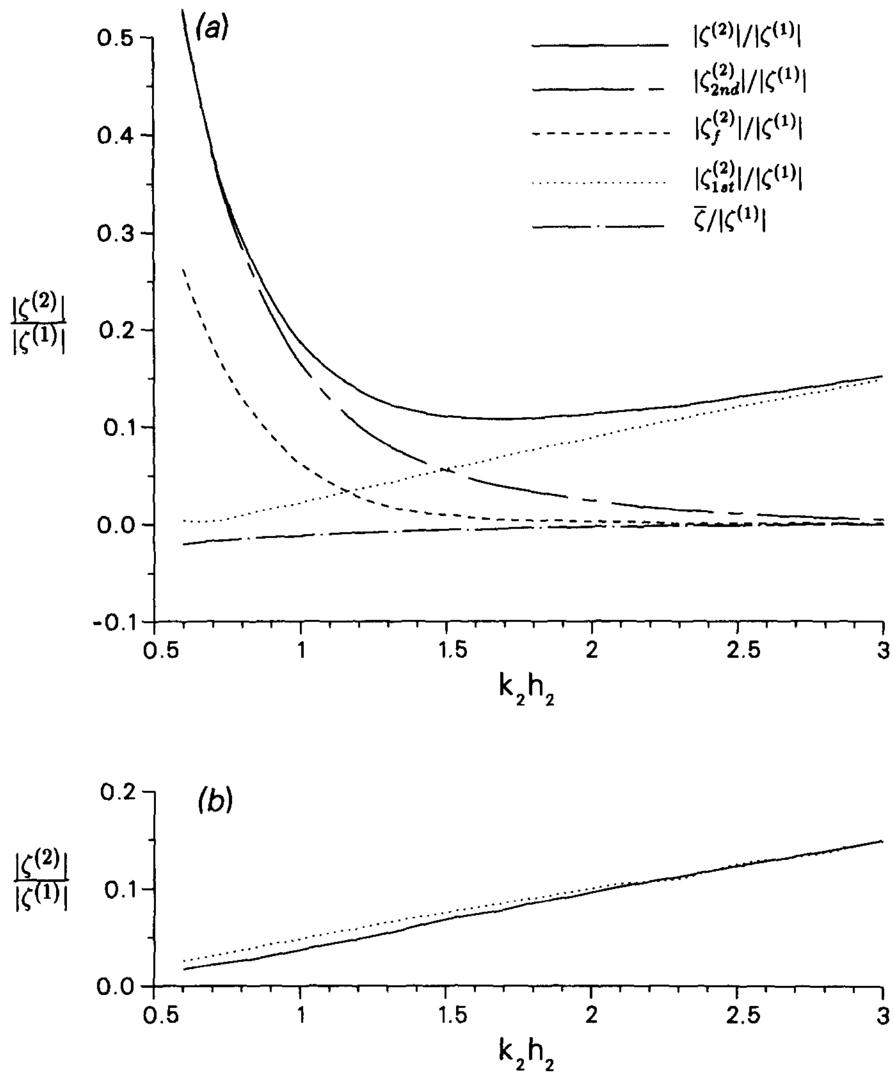

Fig. 6. Ratio of second- to first-order wave amplitude for normal incidence ($\theta = 0$) with $h_2/h_1 = 1/10$ and $A/h_2 = 0.1$, at (a) $x = +\infty$ and (b) $x = -\infty$. (——) $|\zeta^{(2)}|/|\zeta^{(1)}|$; (— - —) $|\zeta^{(2)}_{2\mathrm{nd}}|/|\zeta^{(1)}|$; (- - -) $|\zeta^{(2)}_{1\mathrm{st}}|/|\zeta^{(1)}|$; (— · —) $\bar{\zeta}/|\zeta^{(1)}|$. Results are calculated from eqns (11)–(15) using the maximum amplitudes.

becomes small. In addition to the ratio of maximum $|\zeta^{(2)}|$ to maximum $|\zeta^{(1)}|$, the figure shows the components of $|\zeta^{(2)}|$ and $\bar{\zeta}$; $\bar{\zeta}$ is always negative corresponding to set-down.

At $x = +\infty$, we find that $|\zeta^{(2)}|/|\zeta^{(1)}|$ reaches the value greater than 0.3 when k_2h_2 becomes small (< 1.0) but within the range where the theory is valid. The ratio remains above 0.1 over the entire frequency range. The curves also demonstrate that the second-order effects are determined by the contributions mainly from the second-order solutions for small k_2h_2 and from the quadratic contributions of the first-order solution for large k_2h_2.

At $x = -\infty$, $\zeta^{(2)}$ is largely composed of the contributions from $\zeta^{(2)}_{1\mathrm{st}}$. Our computations find $\zeta^{(2)}_{2\mathrm{nd}}$ and $\bar{\zeta}$ negligible. Also,the ratio of the reflected to transmitted free wave amplitude ($|\zeta^{(2)}_{\mathrm{f}}|$) is of the order $O(0.01)$ for the various values of h_2/h_1.

5 CONCLUDING REMARKS

We have obtained the first- and second-order solutions to the problem of wave transmission over a step using an integral equation with a finite-depth Green's function in oblique waves.

The present first-order results for a finite step are consistent with previous solutions for an infinite step obtained by using the method of matching eigenfunction expansions[4] the variational formulation,[5] and Galerkin's method.[18] In the limit of shallow water, the results also confirm the theory of Lamb[1] and Bartholomeusz.[2] Within the graphical accuracy given by Massel, the second-order free wave appears to agree with that of Massel.

From the example shown, we find that the ratio of the second to first-order maximum wave amplitude can be over

0.2 for the range where the Stokes theory is valid. In particular, at low frequency the contributions from the second-order solutions are more pronounced than the quadratic contributions from the first-order solutions. The study thus demonstrates that, though subtle, the contributions from the second-order harmonics to the sea spectrum development can be significant, especially for transmitted waves.

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