Proceedings of the Eighth (1998) International Offshore and Polar Engineering Conference
Montréal, Canada, May 24-29, 1998
Copyright © 1998 by The International Society of Offshore and Polar Engineers
ISBN 1-880653-34-6 (Set); ISBN 1-880653-37-0 (Vol. III)

Effect of Directional Spreading and Spectral Bandwidth on the Nonlinearity of the Irregular Waves

Marc Prevosto
IFREMER
Brest, France

ABSTRACT

Wherever one stands, deep water, intermediate or shallow water, in extreme conditions the nonlinearity in the wave kinematics is large and has a strong influence on the design parameters. Simple models of the wave kinematics have been studied based on hypotheses of narrowband and unidirectionality.

Obviously a real sea is neither narrowbanded nor unidirectional and the width of the spectral density and the directional spreading influence the nonlinear characteristics of the waves (skewness, asymmetries,...).

A second order directional irregular wave model is used to simulate time series of the free surface elevation. Based on a large simulated data base, a parametric study of the influence of the spectral width and directional spreading is led on several wave characteristics sensitive to nonlinearities (skewness, wave and crest heights distributions, steepness of maximum crest). Three typical situations are analysed which correspond to extreme situations in long and short fetch wind sea. The validity of the simplified assumptions of narrowband and unidirectionality is then discussed.

KEYWORDS

Wave statistics, spectral bandwidth, directional spreading, nonlinearity, second order, crest height, skewness.

INTRODUCTION

Wherever one stands, deep water, intermediate or shallow water, in extreme conditions the nonlinearity in the wave kinematics is large and has a strong influence on the design parameters. This has been showed clearly for the crest height distribution in deep water (Nerzic, 1997). In a very simple model of a real sea wave, let us say an unidirectional harmonic wave, the nonlinearity is driven by a steepness parameter (the amplitude divided by the wavelength) and a dimensionless water depth parameter (the water depth divided by the wavelength). This is also the case for the narrowband models where the amplitude is replaced by H_s the significative height and the wavelength by "a" mean wavelength.

Obviously a real sea is neither narrowbanded nor unidirectional and the width of the spectral density and the directional spreading influence the nonlinear characteristics of the waves (skewness, asymmetry,...).

In this study we consider uniquely in the kinematics of the waves, the elevation of the free surface. A second order directional irregular wave model based on a Stokes expansion and a first order directional Gaussian process is used to simulate time series of the free surface elevation. Based on a large simulated data base, a parametric study of the influence of the spectral width and directional spreading is led on several wave characteristics sensitive to nonlinearities (skewness, wave and crest heights distributions, steepness of maximum crest). Three typical situations are analysed which correspond to extreme situations in long and short fetch wind sea. The validity of the simplified assumptions of narrowband and unidirectionality is then discussed.

EFFECT OF DIRECTIONAL SPREADING AND SPECTRAL BANDWIDTH

At our knowledge, good literature does not exist about the effect of the directional spreading and spectral bandwidth on the nonlinear characteristics of waves, and particularly none can be found for shallow water.

In (Longuet-Higgins, 1963) the skewness of the free surface elevation, in deep water, is bounded with a lower bound corresponding to the superposition of two orthogonal longcrested seas and an upper bound corresponding to a single one:

$$0.44\lambda_{3,\,\mathrm{uni}} \le \lambda_{3,\,\mathrm{dir}} \le 1.01\lambda_{3,\,\mathrm{uni}} \qquad (1)$$

with $\lambda_{3,\,\mathrm{uni}}$ the skewness in the unidirectional case.

Most of the studies about effect of the spectral bandwidth concern the wave height distribution in a Gaussian sea (Longuet-Higgins, 1980), (Naess, 1985), considered as representative of the wave height distribution in nonlinear sea. Tayfun (1983) studied the nonlinear effects on the distribution of crest-to-trough wave heights but without considering the directional spreading.

Effects of the directional spreading and spectral bandwidth on the skewness and kurtosis and cumulative distribution of maximum crest and wave heights have been also studied in (Stansberg, 1995). In that study, wave tank measurements were considered and he concluded that "extreme wave events due to nonlinear modulations are most pronounced in longcrested waves". This remark will be confirmed hereafter in the deep water case.

SECOND ORDER MODEL

The nonlinear model of the free surface elevation of a directional sea which we have used is a classical second order model based on a Stokes expansion and a first order directional Gaussian process (see e.g. (Ding, 1994)). The nonlinear model of the elevation process is the superposition of two processes:

$$\eta(t) = \eta_1(t) + \eta_2(t) \qquad (2)$$

The first order part η_1 of this model is a directional Gaussian process (superposition of Airy waves with random phases and amplitudes):

$$\eta_1(t) = \int_K dB(\vec{\mathbf{k}}) \exp(-i\omega t) \qquad (3)$$

with the integration domain defined by:

$$K = \{((\omega, \vec{\mathbf{k}}), (-\omega, -\vec{\mathbf{k}})) \text{ with } k \in [0, \infty], \beta \in [-\pi, \pi]\} \qquad (4)$$

the vectorial wavenumber $\vec{\mathbf{k}}$ and the angular frequency ω defined by:

$$\omega^2 = gk\tanh kh \ , \ \vec{\mathbf{k}} = k\begin{pmatrix}\cos\beta \\ \sin\beta\end{pmatrix} \ , \ k = |\vec{\mathbf{k}}| \qquad (5)$$

and with the wavenumber directional spectral density $(\vec{\mathbf{k}})$ related to $dB(\vec{\mathbf{k}})$ a Brownian, independent increment process, by:

$$S(\vec{\mathbf{k}})d\vec{\mathbf{k}} = E(dB(\omega, \vec{\mathbf{k}})dB^*(\omega, \vec{\mathbf{k}})) \qquad (6)$$

where E denotes the expectation, and with:

$$dB(-\omega, -\vec{\mathbf{k}}) = dB^*(\omega,, \vec{\mathbf{k}}) \ , \ E(dB(\omega, \vec{\mathbf{k}})) = 0 \qquad (7)$$

The second order part can be written as:

$$\eta_2(t) = \iint_{KK} H_2(\overrightarrow{\mathbf{k_1}}, \overrightarrow{\mathbf{k_2}})dB(\overrightarrow{\mathbf{k_1}})dB(\overrightarrow{\mathbf{k_2}})\exp(-i(\omega_1 + \omega_2)t) + c_{\tau_2} \qquad (8)$$

More detailed expressions of first and second order parts are given in appendix.

SIMULATED DATA

Model of directional spectrum

The directional spectrum of the simulated sea-states has been based on a Jonswap spectrum as point spectrum S(f) and the Mitsuyasu (Mitsuyasu, 1975) directional distribution $H(f,\theta)$ for the directional part. The frequency-angular spectrum is then calculated by:

$$S(f, \theta) = S(f)H(f, \theta) \qquad (9)$$

$$\text{with } H(f, \theta) = A(s)\cos^{2s}\left(\frac{\theta - \bar{\theta}}{2}\right) \qquad (10)$$

where A(s) is a normalization factor to ensure that $\int_{-\pi}^{\pi} H(f, \theta)d\theta = 1$. $H(f,\theta)$ is considered to equal 0 outside $[-\pi/2,\pi/2]$.

The coefficient s is frequency-dependent and given by:

$$s = \begin{cases} s_p\left(\dfrac{f}{f_p}\right)^5 & f < f_p \\ s_p\left(\dfrac{f}{f_p}\right)^{-\frac{5}{2}} & f \geq f_p \end{cases} \qquad (11)$$

The spectrum is then introduced in (3) and (8) in considering

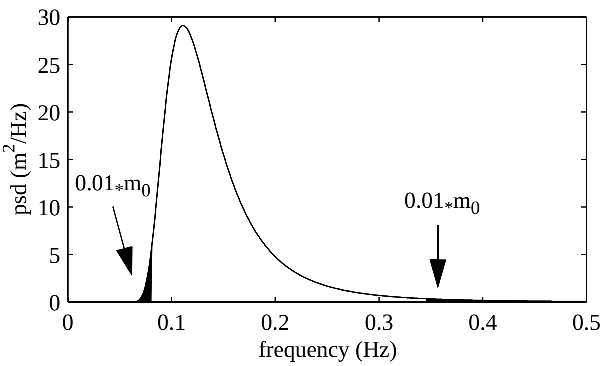
$$S(\vec{\mathbf{k}})d\vec{\mathbf{k}} = S(f, \theta)dfd\theta \qquad (12)$$

Discretization

The Fourier transforms of equations (3) and (8) have been discretized in the frequency and angular domains. The frequency domain has been discretized in 1024 points with a sampling frequency of 1Hz. The angular domain $[-\pi/2,\pi/2]$ has been divided in 120 equal sectors.

As it is explained in (Chen, 1994), the non-linear interactions between long and short waves are very badly calculated with a short Stokes expansion and to be taken into account accurately it is better to introduced a modulated wave-mode approach. This approach furnishes a so-called hybrid model (Zhang, 1996). If this approach seems necessarily in the computation of the kinematics in the crest, in the case of free-surface elevation, a more simple method could be used for the calculation of the non-linear interactions. It consists merely in a truncation of the spectral density, so removing very long and short waves. The waves in the frequency bands corresponding to 1% of the total variance m0 ($H_S = 4\sqrt{m_0}$) were removed as illustrated in fig. 1.

Fig. 1. Truncated spectrum (Hs=6m, Tp=9s, γ=1)



Sea-state characteristics

Three types of (H_s, T_p, h (water depth)) configurations have been considered. A first one (H_s=12m,T_p=14s, h=100m) corresponds to an offshore long fetch situation (typically the North Sea), a second one (H_s=6m,T_p=9s, h=100m) corresponds to an offshore medium fetch situation (typically the Aegean sea) and the last one to the previous one propagated on shallow water (H_s=4m,T_p=9s, h=10m). For each of these situations the 16 combinations of the parameters γ of the Jonswap model (γ = 1,3,10,100) and s of the directional distribution (s = 1000,16,7,2) were considered. For each of these couples (γ,s), 100 time series have been generated from the discretized Fourier transforms. We observe that some of the (γ,s) couples are not really realistic, e.g. (γ = 100,s = 2) which would correspond to a spreaded swell. In fact, all the couples have been considered, the aim of this study being to proceed to a systematic parametric analysis in order to better understand the effect of both sea-state parameters on nonlinear wave characteristics.

The different step of the simulations are as follows. First, simulation of the first-order part $\eta_1(t)$ of the elevation for a given (γ,s) couple (eq. (3)). Secondly, calculation of the corresponding second-order part $\eta_{2,dir}(t)$, (eq. (8)). Finally, in order to compare to a second-order unidirectional sea-state elevation with the same first-order elevation, calculation of the dB(k) corresponding to a unidirectional $\eta_1(t)$ first-order part by inverse Fourier transform from:

$$\eta_1(t) = \int_{-\infty}^{\infty} dB(k)\exp(-i\omega t) \qquad (13)$$

and secondly, with these dB(k), calculation of the second-order part:

$$\eta_{2,\,uni}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_2(k_1, k_2)dB(k_1)dB(k_2)\exp(-i(\omega_1 + \omega_2)t) \qquad (14)$$

So we will be able to compare the statistical parameters of these three elevation processes:

- the linear process $\eta_{lin} = \eta_1(t)$,
- the non-linear unidirectional process $\eta_{uni} = \eta_1(t)+\eta_{2,uni}(t)$,
- the non-linear directional process $\eta_{dir} = \eta_1(t)+\eta_{2,dir}(t)$.

Two examples of a wave obtained from these three elevation processes are given in figures 2 & 3. We see here clearly the effect of nonlinearity and of directional spreading, effect that we will find again in the statistical parameters analysis. In deep water depth as in shallow water depth the nonlinearity, as well known, increases crest elevation. But the directional spreading associated to a second-order nonlinear model, decreases the crest elevation in deep water depth (fig. 2) even though it increases the crest elevation in shallow water depth (fig. 3).

These two different behaviours come from the different nonlinearities concerned in each case. In deep water depth only a free surface nonlinearity is concerned, but in shallow water nonlinearity is predominantly a bottom effect one.

Fig. 2. Comparison of the shape of a wave, water depth = 100m

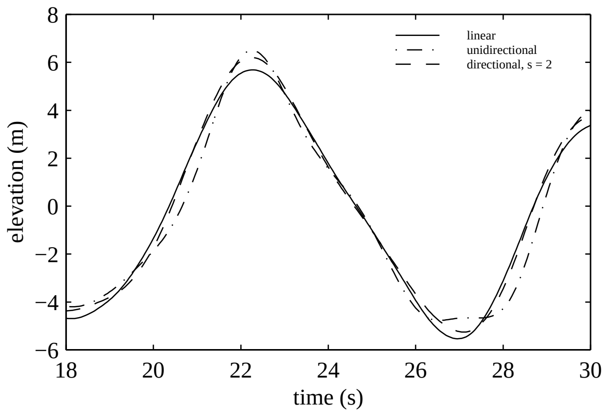
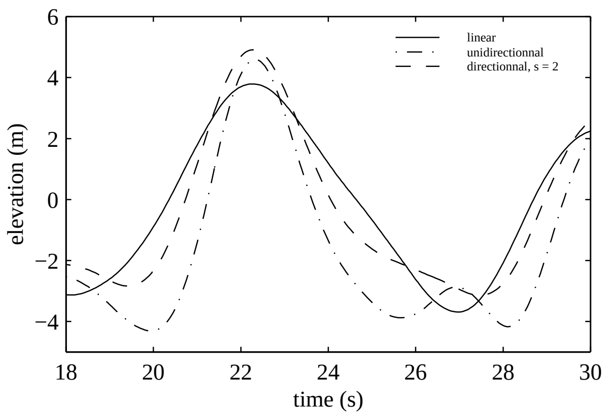


Fig. 3. Comparison of the shape of a wave, water depth = 10m



PROCESSING OF THE DATA

On each set of 100 time series, skewness, maximum crest height, maximum wave height and maximum-crest front steepness have been computed. Here crest front steepness is defined as:

$$s_{cf} = \frac{2C}{\lambda(4T)} \tag{15}$$

with, for the maximum crest height, C and T as defined in figure 4, and $\lambda(t)$ the dispersion relation function between wave period and wavelength (see (Myrhaug, 1984)).

In a second step the mean skewness and crest front steepness have been computed and a Gumbel law has been fitted on the two other set of maximum wave characteristics:

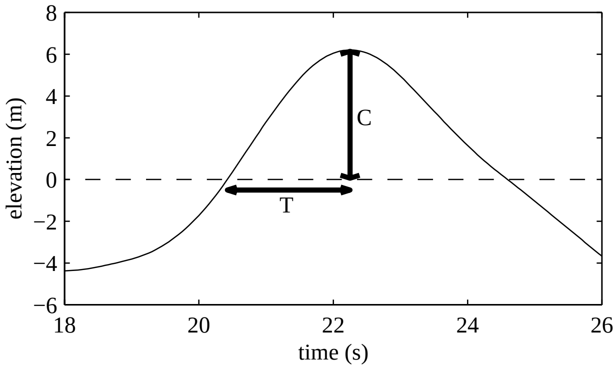
$$G(y) = \mathrm{Prob}(X_N \le y) = \exp\left(-\exp\left(-\frac{y - a_N}{b_N}\right)\right) \tag{16}$$

with N the number of waves of the time history, considered as constant for a particular T_p.

The two parameters of the Gumbel law, the mode a_N and the scale b_N, have been estimated in using a Maximum Likelihood Method. All the fits are very good, as well on the maximum crest as on the maximum wave heights.

These results are summarized in tables 1-5 and commented hereafter.

Fig. 4. Definition of front crest steepness



ANALYSIS OF STATISTICAL PARAMETERS

As indicated previously, the skewness has been calculated empirically on the set of the time series to be consistent with the analyses of the other parameters. But it could have been calculated directly from the directional spectra as proposed in (Longuet-Higgins, 1963) or (Ding, 1994). The formulas are given at the end of the appendix.

As it has been often demonstrated, (Stansberg, 1995), (Vinje, 1989), (Vinje, 1994), (Nerzic, 1997), the mean steepness, e.g. defined as $s = H_S k_p/(2\pi)$, is the dominant factor of the nonlinearity in deep water. The three types of situations considered here correspond to different mean steepness (H_s=12m,T_p=14s, h=100m $\Rightarrow$ s = 0.041), (H_s=6m,T_p=9s, h=100m $\Rightarrow$ s = 0.047) and (H_s=4m,T_p=9s, h=10m $\Rightarrow$ s = 0.048). These differences between the mean steepnesses are sufficiently low to compare the sensitivities to the directional spreading or to the bandwidth.

Skewness

If we observe, for the deep water case, the values of the mean skewness ratio (directional/unidirectional), given in table 1, two tendencies appear clearly. The skewness decreases when the bandwidth increases and when the directional spreading increases. But what seems very strange is that for a very weak spreading (s = 1000) the ratio is higher than unity.

Table 1. Skewness ratio, directional/unidirectional

	H_s = 12m , T_p = 14s , depth = 100m			
	γ = 1	**γ = 3**	**γ = 10**	**γ = 100**
s = 2	0.71	0.74	0.78	0.84
s = 7	0.87	0.91	0.94	1.00
s = 16	0.95	1.01	1.05	1.12
s = 1000	1.03	1.04	1.05	1.10

As explanation we recall here the results given in (Longuet-Higgins, 1963). The transfer function H_2 which enters in the calculation of the skewness (eq. 35) can be written as:

$$H_2(\vec{k_1}, \omega_1, \vec{k_2}, \omega_2) = (k_1 k_2)^{3/2} f(\eta, \gamma) \tag{17}$$

where γ is the angle between $\vec{k_1}$ and $\vec{k_2}$ and η, which indicates the closeness of k_1 and k_2, is given by:

$$\eta = \frac{(k_1 + k_2)}{2(k_1 k_2)^{1/2}} \tag{18}$$

The function $f(\eta, \gamma)$ is plotted in figure 5. We observe on this graph that for very close wavenumber ($\eta = 1.001$, $\eta = 1.01$) the value of the function is higher in the directional case than in the unidirectional case, and finally Longuet-Higgins demonstrated that $f(\eta, \gamma)$ is bounded by the unidirectional case by:

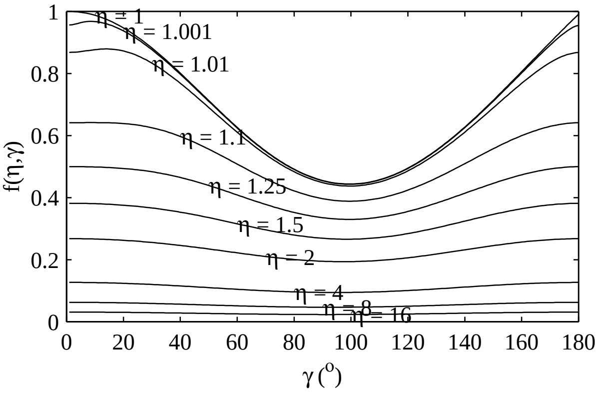
$$0.44 f(\eta, 0) \le f(\eta, \gamma) \le 1.01 f(\eta, 0) \tag{19}$$

This relation led him to a similar relation for the skewness:

$$0.44 \lambda_{3,\,\mathrm{uni}} \le \lambda_{3,\,\mathrm{dir}} \le 1.01 \lambda_{3,\,\mathrm{uni}} \tag{20}$$

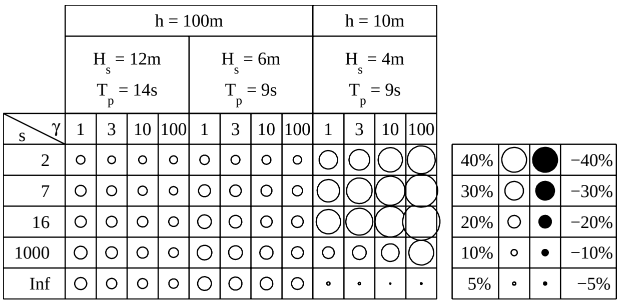
In our simulations the ratio is up to 1.12 and never below 0.71. The ratio 0.44 of eq. 20 corresponds to a very particular condition, two swells with perpendicular directions but the ratio 1.01 to a more common one, a simple swell (e.g. s = 1000). The higher ratios that we found, corresponding to the upper bound in eq. 20, could be explained by the fact that this bound was calculated for infinite water depth and that we consider here wavelength of 300m for water depth of 100m. The finite water depth situation increases strongly this contrast between directional and unidirectional case and explain our higher values.

Fig. 5. Graphs of $f(\eta,\gamma)$ for various values of η



In table 2, the values of the skewness for the nonlinear directional case are plotted. These values are all around 0.2 for the 100m water depth situations but very sensitive to the directional spreading in shallow water.

Table 2. Skewness, directional

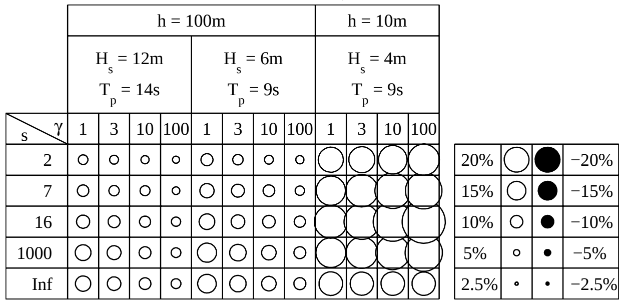


We observe particularly that the skewness is very high (up to 0.6) with the spreading sea-states and close to zero in the unidirectional case. In fact it indicates a change in the wave shapes, but, as we will see hereafter, does not demonstrate the absence of asymmetry between crest and trough heights.

Mode crest

The analysis of the ratios of the most probable value of the crest heights between directional and linear case (table 3) shows a slight decrease of the crest with spreading, in the deep water case. Globally, for the 100m water depth, the crest ratio varies from 6% (g = 100, s = 2) to 15% (g = 1, s = 1000). The values of the ratios are higher for (H_s = 6m, T_p = 9s), a sea-state which corresponds to a higher mean steepness. This confirms, in deep water, the conclusion of (Stansberg, 1995) about the severity of longcrested sea. For the shallow water depth situation the increase is higher and can reach 35% (g = 100, s = 16). The effect of the bandwidth is opposite in deep water and shallow water. The nonlinear effect is increased in deep water and decreased in shallow water.

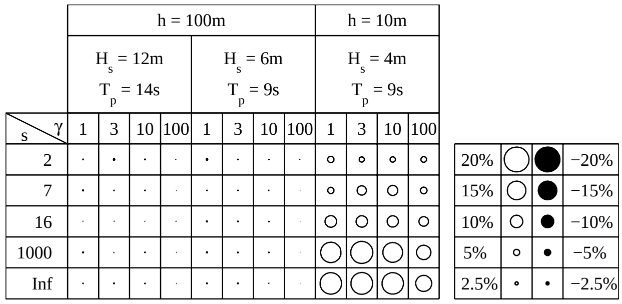
Table 3. Mode crest ratio, directional/linear



Mode wave

It has been often mentioned in previous studies that, in deep water, wave heights are weakly affected by the nonlinearities. We see here (table 4) that neither spreading nor bandwidth change this fact. On the contrary, in finite depth (water depth = 10m) the effect of the nonlinearities is completely different with an increase of up to 18%. In that case, a high spreading diminishes the wave height ratio below 5%.

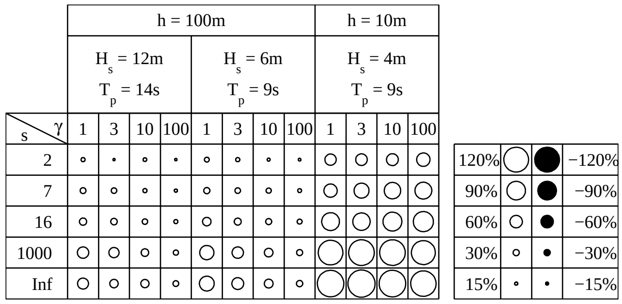
Table 4. Mode wave ratio, directional/linear



Crest front steepness

The mean maximum-crest front steepness follows, in deep water, the same pattern (table 5) as the crest height with an increase of up to 75% in the case of a unidirectional and Pierson-Moskowitz spectrum model.

Table 5. Mean maximum-crest front ratio, directional/linear



In deep and shallow water the effect decreases with the spreading. In deep water, the spreading can almost completely cancel the effect of the nonlinearity. In shallow water, the bandwidth has no effect on the front steepness, but the nonlinearity has a very strong effect on the maximum-crest front steepness for longcrested seas (increasing up to 130%). Of course, we have not considered in this study the different breaking phenomena.

CONCLUSIONS

Taking into account the directional spreading and the spectral bandwidth in the nonlinear models of prediction of crest heights has significative effects. If in deep water depth the hypothesis of unidirectionality is conservative (the longcrested sea is the most severe), it is not the case when the depth is sufficiently shallow to affect the nonlinear behaviour of the kinematics. For the wave height and maximum-crest front steepness, the hypothesis of unidirectionality is always conservative whatever the water depth.

In deep water, the higher the bandwidth is the more severely the nonlinearity changes the wave characteristics. In shallow water the bandwidth does not have very significative effect.

The nonlinearities brought into the kinematics in shallow water are different from the deep water situation, where only the free surface nonlinearity is concerned. This changes completely the effect of the directional spreading and of the bandwidth with sometimes opposite effects compared to the deep water situation.

REFERENCES

Chen, L., Zhang, J. (1994) "On interaction between intermediate-depth long waves and deep-water short waves", *Offshore Technol. Res. Center Report*, no. A5575, p. 37.

Ding, P.-X., Sun, F., Yu, Z.-W. (1994). "Study of the second-order nonlinear characteristics of ocean waves (1) -- theoretical derivation", *Sci. China Ser. B*, vol. 37, no. 5, pp. 625-633.

Longuet-Higgins, M.S. (1980). "On the distribution of the heights of sea waves: some effects of nonlinearity and finite band width", *J. Geophys. Res.*, vol. 85, no. C3, pp. 1519-1523.

Longuet-Higgins, M.S. (1963). "The effect of non-linearities on statistical distributions in the theory of sea waves", *J. Fluid Mech.*, vol. 17, pp. 459-480.

Mitsuyasu, H., Tasai, F., Suhara, T., Mizuno, S., Ohkuso, M., Honda, T., Rikiishi, K. (1975). "Observations of the directional spectrum of ocean waves using a cloverleaf buoy", *Journ. Physic. Oceanogr.*, vol. 5, pp. 750-760.

Myrhaug, D., Kjeldsen, S.P. (1984). "Parametric modelling of joint probability density distributions for steepness and asymmetry in deep water waves", *Appl. Ocean Res.*, vol. 6, no. 4, pp. 207-220.

Naess, A. (1985). "On the distribution of crest to trough wave heights", *Ocean Eng.*, vol. 12, no. 3, pp. 221-234.

Nerzic, R., Prevosto, M. (1997). "A Weibull-Stokes Model for the Distribution of Maximum Wave and Crest Heights", *Proc. of the 7th ISOPE Conf.*, vol. III, pp. 367-377.

Stansberg, C.T. (1995). "Effects from directionality and spectral bandwidth on nonlinear spatial modulations of deep-water surface gravity wave trains", *Proc. 24th ICCE*, vol. 2, pp. 579-593.

Tayfun, M.A. (1983). "Nonlinear effects on the distribution of crest-to-trough wave heights", *Ocean Eng.*, vol. 10, no. 2, pp. 97-106.

Vinje, T. (1989). "The statistical distribution of wave heights in a random seaway", *Applied Ocean Res.*, vol. 11, no. 3.

Vinje, T., Haver, S. (1994). "On the non-Gaussian structure of ocean waves", *Proc. 7th Intl Conf on the Behaviour of Offshore Structures*, vol. 2, pp. 453-480.

Zhang, J., Chen, L., Ye, M., Randall, R.E. (1996) "Hybrid wave model for unidirectional irregular waves, Part I. Theory and numerical scheme", *Appl. Ocean Res.*, vol. 18, nos 2-3, pp. 77-92.

APPENDIX: SECOND ORDER MODEL

First-order part:

$$\eta_1(\vec{\mathbf{r}}, t) = \int_K dB(\vec{\mathbf{k}}) \exp(i(\vec{\mathbf{k}} \bullet \vec{\mathbf{r}} - \omega t)) \tag{21}$$

with the wavenumber spectral density $S(\vec{\mathbf{k}})$ related to $B(\vec{\mathbf{k}})$ a Brownian, independent increment process, by:

$$S(\vec{\mathbf{k}})d\vec{\mathbf{k}} = E(dB(\omega, \vec{\mathbf{k}})dB^*(\omega, \vec{\mathbf{k}})) \\ \text{and} \quad dB(-\omega, -\vec{\mathbf{k}}) = dB^*(\omega, \vec{\mathbf{k}}) \;, \quad E(dB(\omega, \vec{\mathbf{k}})) = 0 \tag{22}$$

where E denotes the expectation.

$$\omega^2 = gk\tanh kh \;, \quad \vec{\mathbf{k}} = k\begin{pmatrix}\cos\beta \\ \sin\beta\end{pmatrix} = k\vec{\mathbf{d}} \;, \quad k = |\vec{\mathbf{k}}| \tag{23}$$

$$K = \{((\omega, \vec{\mathbf{k}}), (-\omega, -\vec{\mathbf{k}})) \text{ with } k \in [0, \infty], \beta \in [-\pi, \pi]\} \tag{24}$$

Second-order part:

$$\eta_2(\vec{\mathbf{r}}, t) = \\ \iint_{KK} H_2(\vec{\mathbf{k_1}}, \omega_1, \vec{\mathbf{k_2}}, \omega_2)dB(\vec{\mathbf{k_1}})dB(\vec{\mathbf{k_2}})\exp(i(\vec{\varphi_1} + \vec{\varphi_2})) + c_{\eta_2} \tag{25}$$

with:

$$\vec{\varphi_1} = (\vec{\mathbf{k_1}} \bullet \vec{\mathbf{r}}) - \omega_1 t \qquad \vec{\varphi_2} = (\vec{\mathbf{k_2}} \bullet \vec{\mathbf{r}}) - \omega_2 t \tag{26}$$

$$\vec{\mathbf{k_1}} = \begin{pmatrix}k_1\cos\beta_1 \\ k_1\sin\beta_1\end{pmatrix} \qquad \vec{\mathbf{k_2}} = \begin{pmatrix}k_2\cos\beta_2 \\ k_2\sin\beta_2\end{pmatrix} \tag{27}$$

$$H_2(\vec{\mathbf{k_1}}, \omega_1, \vec{\mathbf{k_2}}, \omega_2) = \frac{1}{2g}\Big(2(\omega_1 + \omega_2)D(\vec{\mathbf{k_1}}, \omega_1, \vec{\mathbf{k_2}}, \omega_2) + \\ \omega_1\omega_2 + \omega_1^2 + \omega_2^2 - \frac{g^2\vec{\mathbf{k_1}} \bullet \vec{\mathbf{k_2}}}{\omega_1\omega_2}\Big) \tag{28}$$

and c_{η_2} a constant to ensure that $E(\eta_2) = 0$:

$$c_{\eta_2} = \int_K \frac{k}{\sinh 2kh} S(\vec{\mathbf{k}})d\vec{\mathbf{k}} \tag{29}$$

D is given by:

$$D(\vec{\mathbf{k_1}}, \omega_1, \vec{\mathbf{k_2}}, \omega_2) = \frac{2(\omega_1 + \omega_2)(g^2\vec{\mathbf{k_1}} \bullet \vec{\mathbf{k_2}} - \omega_1^2\omega_2^2)}{N(\vec{\mathbf{k_1}}, \omega_1, \vec{\mathbf{k_2}}, \omega_2)} + \\ \frac{g^2(k_1^2\omega_2 + k_2^2\omega_1) - \omega_1\omega_2(\omega_1^3 + \omega_2^3)}{N(\vec{\mathbf{k_1}}, \omega_1, \vec{\mathbf{k_2}}, \omega_2)} \tag{30}$$

$$D(\vec{\mathbf{k_1}}, \omega_1, -\vec{\mathbf{k_1}}, -\omega_1) = 0 \tag{31}$$

with:

$$N(\vec{\mathbf{k_1}}, \omega_1, \vec{\mathbf{k_2}}, \omega_2) = 2\omega_1\omega_2({\omega_+}^2 - g|\vec{\mathbf{k_+}}|\tanh|\vec{\mathbf{k_+}}|h) \tag{32}$$

$$\vec{\mathbf{k_+}} = \vec{\mathbf{k_1}} + \vec{\mathbf{k_2}} \qquad \omega_+ = \omega_1 + \omega_2 \tag{33}$$

Skewness

The cumulants are:

$$\lambda_1 = 0 \,, \; \lambda_2 = \int_K S(\vec{\mathbf{k}})d\vec{\mathbf{k}} \tag{34}$$

$$\lambda_3 = 6\iint_{KK} H_2(\vec{\mathbf{k_1}}, \omega_1, \vec{\mathbf{k_2}}, \omega_2)S(\vec{\mathbf{k_1}})S(\vec{\mathbf{k_2}})d\vec{\mathbf{k_1}}d\vec{\mathbf{k_2}} \tag{35}$$

The skewness:

$$\gamma_3 = \frac{\lambda_3}{\lambda_2^{3/2}} \tag{36}$$