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Nonlinearity and nonequilibrium together in Nature: wind waves in the open Ocean

wind excited waves

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Abstract. We derive scaling laws for the steady spectrum of wind excited waves, neglecting surface tension and taking air and water as inviscid, an approximation valid at large wind speed. Independently of the wind speed, there exists an unique (small) dimensionless parameter ϵ , the ratio of the mass densities of the two fluids (air and water). The smallness of ϵ allows to derive some important average properties of the wave system. The average square slope of the waves is, as observed, a small but not very small quantity, because it is of order $|ln(\epsilon^2)|^{-1}$. This supports the often used assumption of small nonlinearity in the wave-wave interaction. We introduce an equation to be satisfied by the two-point correlation of the height fluctuations. Lastly we reconsider the formation of swell, that is the relationship between the randomness of waves and the observation of quasi monochromatic water waves.

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1 Statement of the problem, scaling laws for wind excited waves

As ourselves, Pierre Coullet, our good friend, is a dedicated lover of the sea, in his case the Mediterranean. In this Festshrift we thought appropriate to write something on the never ending motion of the waves excited by the wind and doing all sorts of nonlinear things, as we explain below.

Wave turbulence makes a problem interesting both from the point of view of fundamental science and because of its obvious connections with real life phenomena playing a significant role in many human endeavors. After many scientists, we consider below the following problem: a constant wind blowing on the horizontal surface of an infinite Ocean (in the three dimensions), and exciting a system of random fluctuating waves that reach, after transients, a turbulent steady state, the "fully developed sea" of the oceanographic literature [1]. One studies statistical properties like the variance of the surface slope. This has been thought about for many years [2] - [3], mainly by using the weak interaction approximation, based on the (unexplained to the best of our knowledge) observation that the slope of waves is small on average. Below we approach this problem by using simple scaling ideas relying on a small parameter, the ratio of the mass density of the air ρ_{air} to the mass density of water ρ_w. Let $\epsilon = \frac{\rho_{air}}{\rho_w}$ be this small ratio, about 10^{-3}. Assuming incompressibility, (and neglecting, for the moment, surface tension and viscosity), the data with a physical dimension are the acceleration of gravity g and the (uniform) wind speed U. As shown by Newton, one can make out of these two quantities a length, $\lambda = \frac{U^2}{g}$ and a time, $\tau = \frac{\lambda}{U} = \frac{U}{g}$. Here we shall use λ and τ as units of space and time, that leads to a system of equations without the physical parameters g, U [4].

Consider the fluctuations $\delta h(\mathbf{x}, t)$ of the surface elevation which depends on space, i.e. on the horizontal coordinates $\mathbf{x}$ (boldface are for vectors in the Euclidean geometrical space), and on time t. From the data, the only scaling parameters are the length λ and the time τ. The fluctuations $\delta h(\mathbf{x}, t)$ may then be written as

$$\delta h(\mathbf{x}, t) = \lambda \delta H(\frac{\mathbf{x}}{\lambda}, \frac{t}{\tau}), \quad (1)$$

where δH is an universal stochastic function depending only on the dimensionless parameter ϵ, with capital letters for scaled quantities. We expect $\delta h(\mathbf{x}, t)$ to scale like λ, and to change with respect to space and time with the typical scales λ, and τ, eventually times factors depending on ϵ only. More precisely we expect that the typical length scale for the horizontal dependence is λ without factor depending on ϵ, at least for ϵ small, because the typical wavelength of the unstable fluctuations is of order λ. This is in qualitative agreement with the observation that, the larger the wind speed the larger is the average wave-length, as reported for instance in [2]. Note that, if this average wavelength were much less than λ (i.e. if the surface were flat at the scale λ), this system would be unstable against fluctuations at wave-length of order λ, which is impossible in a statistically steady state. Moreover this kind of assumption is fully consistent with the Pierson-Moskowitz spectrum where the peak wave period is of order τ although the amplitude, written in units of λ, is multiplied by a small numerical prefactor of order 10^{-2} explained below by a non-trivial dependence of the average wave-amplitude on ϵ [5]. Indeed the wave-height should scale like $f(\epsilon)\lambda$, the $f(.)$ function tending to zero when its argument tend to zero: without such a factor, i.e. if $f(0)$ were not zero, the waves would have a finite amplitude in absence of wind, which is against common sense, because there are no waves if no instability feeds the wave system. Therefore the "scaled" stochastic function $\delta H(\mathbf{X}, T)$, where $\mathbf{X} = \frac{\mathbf{x}}{\lambda}$ and $T = \frac{t}{\tau}$, is proportional to a (reducing) factor $f(\epsilon)$. We shall prove that $f(\epsilon)$ is logarithmic with respect to ϵ. Assuming invariance under translation in space and time, the pair correlation $S_{\delta H}(\mathbf{X}, T; \mathbf{X}+\mathbf{Y}, T+T') =< \delta H(\mathbf{X}, T)\delta H(\mathbf{X}+\mathbf{Y}, T+T') >$ is a function of $(\mathbf{Y}, T')$ and of ϵ only. This does not mean that it is isotropic, as one expects the direction of the wind to induce such anisotropy.

In this section we assume that a steady solution of the wave equation is reached, where the forcing by the wind balances the dissipation by wave-breaking. Our scaling approach does not require any detailed knowledge of the spectrum structure, as derived from cascade arguments describing the nonlinear transfert of invariants across the spectrum [6].

Let us now comment on the neglect of the viscosity of air and of water, and of the surface tension σ. The latter introduces a new length scale, the capillary length $\lambda_c = \sqrt{\frac{\sigma}{g\rho_w}}$. Even though surface tension of real seawater is a highly variable quantity, the capillary length is somewhere between one centimeter and one millimeter, far smaller than the length scales we are concerned with. This does not mean however that its effects are negligible: surface tension could regularize the instabilities at the shortest scales, which otherwise could lead to the development of cone-like singularities on the free surface [7], while white caps are observed instead. Here we are interested on large scale phenomena only, then capillarity effects are not considered, and the capillary length scale will be taken as zero. We consider high wind velocity regime, more precisely the regime above the onset of wave-breaking [8] which occurs for U_{10} around $7m/s$, (U_{10} is the wind speed measured 10 meters above the sea surface), where gravity becomes the only governing parameter of wind-wave interaction. At inferior wind velocities, surface tension may either dominate gravity everywhere or just make impossible the steepening leading to wave breaking [9]. Similar statements could be made about the effect of viscosity: if the Reynolds number is large, viscosity is relevant at very small scales that can be taken as simply zero.

Let us now make a remark about the air/sea interaction: the air flow above the sea surface is highly turbulent and one expects the formation of a Prandtl logarithmic layer there. The scaling parameter of such a Prandtl layer is a flux of horizontal momentum per unit area of the surface. Compared to our scaling via the wind speed, this introduces logarithmic corrections that are, most likely, hardly detectable. However this scaling via a flux of momentum, as compared to scaling via the wind speed, is not completely without consequences. Actually this flux should be the same in the water below the surface, with an horizontal momentum of the same order of magnitude as the aerial flux. The flux of horizontal momentum scales like ρu^2 for a fluid of density ρ and speed u, therefore the average horizontal speed underwater is of order of $U\sqrt{\epsilon}$ (U wind speed), for example a wind speed of 60 km/h should generate an underwater current of about 1.9 km/h. This prediction fully agrees with the observations, $u_{sea}/U_{10} = 0.032$ [10] where U_{10} is the wind speed measured at $10m$ over the sea surface. This makes what is called the "3 per cent rule", i.e. that the surface current is approximately 3 per cent of the wind speed.

For wind-excited waves we consider wave-breaking as the dominant mechanism for dissipation. As this is a strongly dissipative and fully nonlinear process, it could look impossible a priori to estimate the power lost per unit area by wave-breaking, given for instance the pair correlation of the fluctuations of height. This pair correlation assumes implicitely a smooth wave propagation, namely a single-valued $\delta h(x, y, t)$ and a non self-crossing surface, although wave-breaking is a complex nonlinear process, requiring to account for physical effects like surface tension and viscosity and to cope with a multivalued $\delta h(x, y, t)$. Nevertheless wave-breaking of dominant waves (with a wave-length near the peak of the spectrum) should be a relatively rare event, as we shall explain. Let us precise that we may also include in the breaking process the micro-breakers, or small white caps, whose density increases with the wind speed, and saturates at U_{10} around 20m/s. In this regime, all waves have a white cap on their crest, whatever is their wave-length. Nevertheless, in the stationary state, the crest length of breaking waves remains a small fraction of the total crest length. Those white caps may replace the formation of conical or wedge-like singularity [7] of the inviscid equations. Note that white caps appear also after the breaking of very long waves. Here we focus on the energy dissipation due to wave-breaking of all waves.

Let us estimate the probability of such a process. In a single wave-breaking event, the average lost energy W_{br} scales as $\rho_w U^2 \lambda^2 \sigma_{\delta h}$, i.e. the energy corresponding to a "typical" volume, $\lambda^2 \sigma_{\delta h}$ of a wave having an horizontal surface λ^2, a height $\sigma_{\delta h}$, and propagating at velocity U (which is of order of the velocity reached by long waves, the peak waves, before breaking). By comparison, the kinetic energy of the wind in the same volume, is smaller by a factor ϵ. The balance of power may be obtained in the steady regime by taking the frequency of wave-breaking of order ϵ, since in this case the power lost per unit area is of the same order as the input from the wind, $\epsilon W_{br}/\lambda^2\tau$.

The result of these considerations is that, at a given time, the area covered by wave-breaking events is of order ϵ times the total area, independently on the wind speed. Note that this is not the proportion of area covered by foam, much bigger than the area covered by wave-breaking events in high wind conditions, because of the spreading of the foam by the wind, and because of the small white caps.

Let us now estimate the ϵ-dependence of the magnitude of wave fluctuations. Given the statistics of the wave fluctuations, and because wave-breaking is a rare event, it depends on the probability of large fluctuations. Those large fluctuations are rare because of the smallness of ϵ. Said otherwise, the scaling laws for the magnitude of the fluctuations have to be amended in order to take into account this smallness: otherwise, if the magnitude of the fluctuations was such that the typical wave-length and wave-height were both λ, the probability of occurence of wave breaking would be of order one too, although we argued it is of order ϵ. Therefore the amplitude of the wave must be small compared to λ, to make exceptional the nonlinear evolution towards wave-breaking. This implies that, predominantly, the wave system is described by the linear approximation of the wave equations. Therefore, at a given location, and according to an idea that have been stated by Planck for the classical (Rayleigh) part of the black-body spectrum, the fluctuations of the free surface are predominantly Gaussian because they are made of a *linear* superposition of waves with a continuous distribution of frequencies. In this respect it is worth recalling what was stated by Hasselmann (under equation 4.31 in [3]). He wrote " the interaction will also destroy the initial Gaussian property of the sea, (but) it follows from our derivation...that the influence of the latter process (of destruction of Gaussianity by wave interaction) on energy transfer is negligible". This is certainly correct, under the conditions of validity of the weak turbulence approximation, which needs, as we shall show, the existence of a small physical parameter, the ratio of the mass densities. In the original work of Hasselmann, this was not necessary because he dealt with an initial value problem of wave dynamics without input and loss term but with an amplitude small by assumption. This weak amplitude approximation becomes problematic when input (of the wind) and loss (by wave-breaking) is included, with a loss of control a priori of the wave amplitude, unless a small parameter is included. It is also possible that the wave-interaction brings some long range phase coherence of the waves making invalid the assumption of Gaussian waves, see section 2.

The most interesting quantity from the point of view of wave-breaking is not so much the amplitude of the wave, but its slope. Being related linearly to $\delta h(x, y, t)$ it has also a Gaussian distribution. Therefore the gradient of the height along x, the wind direction, i.e. the derivative $\frac{\partial \delta h}{\partial x} = \delta h_{,x}$, has a probability distribution

$$\mathcal{P}(\delta h_{,x}) = \frac{1}{(2\pi)^{1/2}\sigma_x} e^{-\frac{\delta h_{,x}^2}{2\sigma_x^2}}, \tag{2}$$

where σ_x^2 is the variance of the slope along the x direction ($\sigma_x > 0$).

Wave-breaking is a nonlinear phenomenon [7] characterized by a free surface becoming first vertical and then overturning. Before this happens, the slope has to reach values of order one. Indeed such a phenomenon is not described by the linear approximation for wave propagation, since it assumes the height to be a single valued smooth function of the horizontal coordinates. Nevertheless we may assume that, before a wave locally overturns, its slope gets to finite (non-small) values that are at the border of applicability of the linear approximation. Assuming that the waves propagate predominantly in the wind direction, the probability of overturning may be approximated by the area under the tail of the density $\mathcal{P}(\delta h_{,x})$,

$$\mathcal{P}_{br} = 2\int_{\alpha}^{\infty} \mathcal{P}(\delta h_{,x}) \mathrm{d}(\delta h_{,x}) = 2\mathrm{erfc}(\frac{\alpha}{\sqrt{2}\sigma_x}), \tag{3}$$

where α is a parameter of order unity defining the limit slope (above which the wave breaking occurs with a finite probability), and $\mathrm{erfc}(z) = \frac{z}{\sqrt{\pi}}\int_z^{\infty} e^{-\varsigma^2} d\varsigma$. In the limit of large z, $\mathrm{erfc}(z) \simeq \frac{exp(-z^2)}{z\sqrt{\pi}}$. According to the arguments presented before, the probability $\mathcal{P}_{br}$ must be of order ϵ to ensure the balance between energy input and dissipation by wave-breaking. In the limit $\epsilon << \sigma_x << \alpha$, this gives

$$\sigma_x \sim \frac{\alpha}{\sqrt{2}\ln(1/\epsilon)}, \tag{4}$$

which is of order ~ 0.1 for the case of wind-waves, if one takes $\alpha = 1$.

This dependence is quite weak, in agreement with the observation that the slope is small on average, but not very small [11]. In this respect the density of breaking events is far more sensitive to the smallness of ϵ than the wave amplitude itself. Indeed it seems difficult to change ϵ, so the prediction of an ϵ dependence of the probability of wave-breaking is hard to test. However one may think to a value of ϵ close to 1, with water and oil for instance (E.J. Wesfreid private communication), which would then yield a wave system where breaking is almost everywhere, a clearcut prediction of the present theory. Along the same line of thinking, the dependence of wave-breaking on the spectral power in a transcendental way could explain how wave-breaking seems to depend critically on the history of the wind-sea interaction. This would be because even a small change in σ_x in the course of the evolution of the wave system changes by factors of order 1, if not larger, the value of the exponential giving the probability of wave-breaking. In particular this could explain why "young seas" (in a growing wind) show far more wave-breaking events than established ones, and why swell, i.e. sea states found by relaxation without wind of a steady wave state, show almost no wave-breaking.

Let us sketch now a more quantitative approach. As has been shown since a rather long time [3], small non linearities yield a kinetic equation for wave turbulence. In this theory the interaction between waves of various wavenumber and frequencies yields a kind of Boltzmann-like theory. In the Hasselmann-expansion, written with respect to the elevation amplitude as small parameter, the nonlinear interaction between the waves appear as a series involving successively four, five, etc. wave-interactions [12]. Actually the relation (4) proves that the ratio between the five- and four wave-interaction terms is always small. Therefore, using our scalings, and the variance of the slope as small parameter, we infer that formally the Hasselmann-expansion can be continued order by order (even for non small waves), at the price of fastly growing complexity. There is no reason that this expansion becomes ill-defined at any finite order. This does not mean however that, even by including the kinetic terms at all orders, all the physics is captured. This is the phenomenon of expansion beyond all orders [13]. Indeed the occurence of wave-breaking depends on effects transcendentally small with respect to the expansion parameter, here $(\ln(1/\epsilon))^{-1/2}$. The "standard" way of getting such transcendentally small terms is by looking at the general large order term in the expansion, getting its leading order part and then summing the largest terms of each order, something perhaps doable, but surely very cumbersome in the present case.

Let us now include in a qualitative sense the dissipation due to wave-breaking, in the light of the above estimations summarized in equations (3)-(4). Using the scales variables, the steady state spectrum $N(\mathbf{K})$ of the surface elevation, which is the spatial Fourier transform of the single time correlation function $<\delta H(\mathbf{X},T)\delta H(\mathbf{X}+\mathbf{Y},T)>$, writes

$$N(\mathbf{K}) = \frac{1}{(2\pi)^2}\int \mathrm{d}\mathbf{Y} e^{i\mathbf{Y}\cdot\mathbf{K}} < \delta H(\mathbf{X},T)\delta H(\mathbf{X}+\mathbf{Y},T) > , \tag{5}$$

where $\mathbf{K}=\mathbf{k}\lambda$. From now on we use mostly the spectral density of wave action,

$$\mathcal{N}(\mathbf{K}) = \frac{N(\mathbf{K})}{\Omega(K)}, \tag{6}$$

where $\Omega(K)=\sqrt{\mathbf{K}}=\tau\sqrt{gk}$. In real variables the action spectrum has dimension $[L]^4[T]$ (with $[L]$ length scale and $[T]$ times), and obeys the stationary Hasselmann's equation [3],

$$S_{nl}[\mathcal{N}(\mathbf{K})] + S_{in}[\mathcal{N}(\mathbf{K})] + S_{diss}[\mathcal{N}(\mathbf{K})] = 0. \tag{7}$$

In equation (7) the term S_{nl} represents the exchange by nonlinear interaction between waves, S_{in} the input by the wind, and S_{diss} for the dissipation by whitecaps and wave-breaking of dominant waves.

When the non linear transfer across the spectrum is by four-wave interaction, the dominant effect at small amplitudes, S_{nl} writes

$$\begin{aligned} S_{nl}[\mathcal{N}(\mathbf{K})] = \int & \mathrm{d}\mathbf{K}_1\mathrm{d}\mathbf{K}_2\mathrm{d}\mathbf{K}_3 |T(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)|^2 \times \\ & \delta(\mathbf{K}+\mathbf{K}_1-\mathbf{K}_2-\mathbf{K}_3)\delta(\Omega+\Omega_1-\Omega_2-\Omega_3)\times \\ & \mathcal{N}(\mathbf{K}_1)\mathcal{N}(\mathbf{K}_2)\mathcal{N}(\mathbf{K}_3)\mathcal{N}(\mathbf{K})\times \\ & \left(\frac{1}{\mathcal{N}(\mathbf{K}_1)}+\frac{1}{\mathcal{N}(\mathbf{K})}-\frac{1}{\mathcal{N}(\mathbf{K}_2)}-\frac{1}{\mathcal{N}(\mathbf{K}_3)}\right), \end{aligned} \tag{8}$$

where $\delta(.)$ is the Dirac distribution. The transition matrix $T(.)$ has a rather complex explicit form [14]-[6], and is equal to K^3 times a numerical function of the ratios K/K_i, with $i=1,2,3$, and of the angles between the four vectors $(\mathbf{K},\mathbf{K}_1,\mathbf{K}_2,\mathbf{K}_3)$. Moreover $\Omega_i=\tau\sqrt{gk_i}$.

In equation (7), the input from the wind is given by the rate of instability times the spectral density of wave action, namely by $S_{in}[\mathcal{N}(\mathbf{K})]=\epsilon\mathcal{N}(\mathbf{K})K_x$, where K_x is the Cartesian component of $(\mathbf{K})$ along the wind. This simple approximation could be refined by multiplication by a function $G(\mathbf{K})$ representing the detailed dependence of the rate of instability as a function of the wave number of the waves. We just take $G=1$ to make the exposition simpler.

An essential point of our formulation is that the input is small, proportional to ϵ, in the dimensionless variables. This smallness is a consequence of the scaling laws, it not assumed from the beginning. Moreover this term is proportional to K_x to represent the angular dependence of the growth rate of the Kelvin-Helmholtz instability, and it is also proportional to the intensity of the fluctuations because this is a linear instability.

The energy loss by wave-breaking is equal to $b'\mathcal{P}_{br}\mathcal{N}(\mathbf{K})$ because it is proportional to the small probability of this process, and it should be proportional to the spectrum itself to get rid of any possibility of negative spectrum. The numerical factor b' is discussed below. Using the expression (3), with $\sigma_x^2=\int \mathrm{d}\mathbf{K}K_x^2N(\mathbf{K})$, the balance equation (7) leads to the following integral equation for the surface elevation spectrum in steady situations,

$$\mathcal{S}_{nl}[\mathcal{N}(\mathbf{K})] + \left(\epsilon K_x - \frac{1}{\alpha b}\sigma_x e^{-\alpha^2/2\sigma_x^2}\right)\mathcal{N}(\mathbf{k}) = 0, \tag{9}$$

The constant $b=\frac{1}{\sqrt{2b'}}$ is the duration of the breaking process, of order unity in units of τ. There are two unknown parameters, α and b. Contrary to similar equations in the literature [6], the breaking-wave loss term in equation (9) is not proportional to some (arbitrary) *power* of the amplitude of the fluctuations, but depends transcendentally on this amplitude. Here the loss term reflect the wave-breaking process, whereas wave-breaking is absent at any order in the expansion of the kinetic equations in powers of the amplitude. Although g and U have been scaled out, the small dimensionless parameter ϵ remains. The integral equation (9) yields the scaling laws for the amplitude of the fluctuations by noticing that the first term has "conservation laws", it is zero when integrated over $\mathbf{K}$ times various functions of this wave-number, a familiar property of Boltzmann and Boltzmann-like equations,

$$\int \mathrm{d}\mathbf{K}F(\mathbf{K})\mathcal{S}_{nl}[\mathcal{N}(\mathbf{K})] = 0 \tag{10}$$

with $F(\mathbf{K})=1$, K, and $\Omega(K)$. Therefore one finds three relations to be satisfied by the steady spectrum :

$$\int \mathrm{d}\mathbf{K}F(\mathbf{K})\left(\epsilon K_x - \frac{1}{\alpha b}\sigma_x e^{-\alpha^2/2\sigma_x^2}\right)\mathcal{N}(\mathbf{K}) = 0, \tag{11}$$

This yields the same scaling relation as derived before, if one assumes that the integration over $\mathbf{K}$ and the multiplication by $F(.)$ change the scaling in the same way in the two terms of equation (11), the one representing the input by the instability (proportional to $\epsilon k_x U$) and the one representing the loss by wave-breaking, proportional to the exponential. Notice too that, thanks to the prefactor σ_x in the dissipation part, the dissipation becomes more efficient as the amplitude of the spectrum grows, ensuring the decay of the large frequency modes.

Equation (9) together with the conservation relations (10)-(11) summarize the main results of this section. In the above analysis we propose a statistical description of the *stationary* state of wind-driven seas, leaving intentionally aside the dynamical evolution of the water waves under the effect of wind. The reason is that the dynamical Hasselmann equation

$$\frac{\partial \mathcal{N}}{\partial \tau} = S_{nl}[\mathcal{N}(\mathbf{K})] + S_{in}[\mathcal{N}(\mathbf{K})] + S_{diss}[\mathcal{N}(\mathbf{K})], \qquad (12)$$

may become invalid. As discussed below this may occur if a finite time singularity occurs, leading to a singular spectrum with a Dirac distribution peak.

2 Smoothness of solutions of Hassemann's equations and the swell problem

We discuss the swell formation, namely the relaxation of the wave system after the wind has stopped blowing. We question whether the evolution of turbulent surface waves with initial spectrum $\mathcal{N}_0(\mathbf{k})$, with peak frequency $\omega_0 = \sqrt{gk_0}$, is well described by using the dynamical Hasselmann's equation (HE) written as

$$\frac{\partial \mathcal{N}}{\partial \tau} = S_{nl}[\mathcal{N}(\mathbf{K})], \qquad (13)$$

where the kinetic operator S_{nl} is given by equation (8), and the scaled variables are $\tau = \omega_0 t$, and $\mathbf{K} = \mathbf{k}/k_0$.

The HE equation yields a seemingly well defined way of predicting the behavior of system of water waves (including under the destabilizing action of the wind). Below we argue that this view can be challenged: because of the possible loss of regularity of its solution, HE may become mathematically ill-posed. This is related, although in a somewhat indirect way, to some of the assumptions behind HE. In the first part of this paper we have shown that the variance of the slope is small as a result of the smallness of ϵ, the ratio of mass densities of air to water. It follows that the nonlinearity, measured by the wave steepness $ak_0 \sim \sigma_x$ (a being the standard deviation of the wave-height $a =< \delta h^2 >^{1/2}$ and k_0 the peak wavenumber), is also small

$$ak_0 << 1, \qquad (14)$$

typically about 0.1 in rough sea conditions. However there is another assumption needed for deriving HE from the basic fluid equations, namely the one of waves with random phases. This cannot be true always: it is manifestly not correct for a set of purely monochromatic waves, having a spectral width Δk (the width of the peak near its center k_0) equal to zero. More generally, the waves cannot be considered anymore with random phases [15] if the relative spectral width $\frac{\Delta k}{k_0}$ is smaller than or of the same order of magnitude as the nonlinearity, see the paragraph below equation (18). Therefore the validity of HE requires that the relative spectral width $\frac{\Delta k}{k_0}$ is noticeably larger than the wave steepness,

$$\frac{\Delta k}{k_0} > ak_0. \qquad (15)$$

While the inequality (15) is often considered as valid (in rough sea conditions, the relative spectral width is about 0.4), it may fails either because of particular initial conditions or because after some time, eventually after a finite time, the wave spectrum becomes narrowly centered around a value k_0. Something quite similar happens in the kinetic theory of Bose gases: at low energy, the momentum distribution becomes singular after a finite time [16] with a width tending to zero.

If one excepts the practically impossible exact modelization of the wave equations with many interacting waves, the approximation opposite to HE relying on a weak nonlinearity is the NLS or envelop equation, that we shall discuss now. This approach assumes that the waves are weakly nonlinear and narrow-banded,

$$\frac{\Delta k}{k_0} \sim ak_0 << 1. \qquad (16)$$

The outcome is an envelope equation. In this theory the wave height is written like

$$\delta h(\mathbf{x}, t) = \frac{1}{2}\left(\Psi(\mathbf{x}, t)e^{i(\omega_0 t - \mathbf{k}_0\cdot\mathbf{x}))} + \mathrm{cc}\right), \qquad (17)$$

and the envelope equation is the equation of motion of the complex amplitude Ψ varying slowly in time and space, the variation rate in space and time being of the same order as the nonlinear interaction. For water waves this amplitude equation is of the mixed type, focusing in the direction perpendicular to the wave speed and defocusing parallel to the wave direction. It reads explicitely

$$\frac{i}{\sqrt{gk_0}}\left(\frac{\partial \Psi}{\partial t} + \frac{\sqrt{g}}{2k_0^{3/2}}\mathbf{k}_0 \cdot \nabla\Psi\right) =$$
$$\frac{1}{8k_0^2}\left(\frac{\partial^2 \Psi}{\partial x^2} - 2\frac{\partial^2 \Psi}{\partial y^2}\right) - \frac{k_0^2}{2}|\Psi|^2\Psi, \qquad (18)$$

The second term on the left-hand side of equation (18) describes the advection with the group velocity $v_g = \frac{\sqrt{g}}{2\sqrt{k_0}}$. This amplitude equation is a direct consequence of the classical Stokes calculation of the first nonlinear correction to frequency -wavelength relation, $\omega(k) = \sqrt{gk}(1 + \frac{|\Psi|^2}{2})$. Note that the condition of validity of the Hasselmann's equation can be derived from the envelope equation (18) which introduces the length scale $L \sim \sqrt{\frac{1}{K_0^4|\Psi_0|^2}}$ defining the range of correlation length, given the amplitude of the waves, where one cannot consider anymore the waves as having random phases. Therefore the condition required for Hasselmann's kinetic theory to be valid is $\Delta k > 1/L$, or equation (15).

Consider the stability of an uniform solution of this equation, $\Psi = a\exp(-\frac{i}{2}(k_0 a)^2 \omega t)$, with $\omega = \sqrt{gk_0}$. Making a small perturbation in amplitude and phase, the solution becomes

$$\Psi = a(1+r)exp(-\frac{i}{2}(k_0 a)^2 \omega t) + i\phi,$$

and assuming that the small quantities r and ϕ vary like $e^{i(-\Omega t + q_x x + q_y y)}$, one finds

$$(\Omega - q_x v_g)^2 = D(2\delta\omega + D), \tag{19}$$

where $\delta\omega = \omega(k_0 a)^2$ is positive, and $D = \frac{v_g}{4k_0}(2q_y^2 - q_x^2)$. The relation (19) shows that the homogeneous solution of the equation (18) is unstable if $D < 0$ and $2\delta\omega > |D|$, this is the well-known Benjamin-Feir (BF) instability [17] of monochromatic water waves of small, but finite, amplitude. The direction of the unstable modulation (q_x, q_y) is within the angular domain ± 35 degrees of q_x, with maximum growth rate on the hyperbola

$$q_x^2 - 2q_y^2 = 4k_0^2 (k_0 a)^2, \tag{20}$$

of the (q_x, q_y) plane.

We amphizise that the BF instability *does not show up in Hasselmann's kinetic theory* because HE involves the frequency-wave number relation of linear waves only. However the envelop equation (18) cannot be correct anymore for long time modelization of the wave dynamics, Because of the BF instability. Various ideas have been suggested to change the equation in order to stabilize the large wave numbers domain [18]. However this is rather questionable because the envelope theory is no more valid in this range of wave numbers, then such changes are inconsistent with the long wave approximation. In these works the range of wave numbers where stabilization occurs is $k \sim k_0$ and/or $k > k_0$, precisely where the separation between carrier wave and modulation is blurred, although it is necessary for the validity of the envelope theory.

Physically one expects that the BF instability will transfer the energy of the carrier waves to sideband waves from where (in the spectral space) it will spread by a mechanism described by HE-like theory. Therefore we suggest that the correct theory for water wave dynamics should *mix together Hasselmann's kinetic theory and the envelope theory*, a question that we shall deal with now.

There is a mathematically related problem in the kinetic theory of Bose-Einstein condensates: one may either describe the quantum gas by means of the Boltzmann-Nordheim kinetic theory or by using the Gross-Pitaevskii equation, the latter looking like the amplitude equation (18).

Hasselmann's kinetic equation has similarities and differences with the Boltzmann-Nordheim kinetic equation for Bosons. We shall take advantage of this to gain some insight on the behavior of its solutions. First, like the Bose gas, the wave dynamics has invariants like mass, energy and momentum. There is nevertheless a rather deep difference coming from the H-theorem. Hasselmann's equation has a Liapunov function, the entropy (in the sense of the "H" function of Boltzmann kinetic theory)

$$\mathcal{H} = \int \mathrm{d}\mathbf{k} \ln(1/\mathcal{N}_k). \tag{21}$$

This function can only increase under the dynamics defined by Hasselmann's equation. It can increase to infinity: take for instance a Boltzmann like distribution $\mathcal{N}_k = e^{-\beta\omega(k)}$ the integral defining the entropy diverges algebraically, although the total energy converges. Therefore it is reasonable to assume that a solution with finite entropy could always evolve toward a solution of larger entropy and ultimately of infinite entropy, like the Boltzmann distribution that is not pathological at all ! Therefore, for this class of problem with an infinite number of degrees of freedom per unit area, the most likely result of an evolution at time infinity is toward a solution of infinite entropy. This makes a deep difference with the situation of a gas at finitenumber density where the integral defining the entropy is of the form

$$\mathcal{H}_{gas} = \int \mathrm{d}q\sqrt{q}[(1+n_q)\ln(1+n_q) - n_q \ln n_q]$$

which converges for the Boltzmann distribution.

Because of the logarithm of the probability density in the definition of the entropy in equation (21), this diverging entropy is realized for distributions spreading as widely as possible in the momentum space. This spreading process does not stop as time goes on, but continues indefinitely (at least for Hasselmann's equation), according to the ideas of [4].

Another significant property of the kinetic equation for Bosons is the occurence of finite time singularities that generate, although in a rather complex way, a condensate, namely a momentum distribution with a Dirac delta peak. Before the singularity time the momentum distribution shows a self similar behavior with powers of time which can be understood as a nonlinear eigenvalue problem. After the singularity the relevant solution of the Boltmann-Nordheim kinetic equation becomes a set of coupled equations for a smooth momentum distribution and the amplitude of the Dirac part. This momentum distribution reads

$$n(\mathbf{k}, t) = n_0(t)\delta(\mathbf{k} - \mathbf{k}_0) + \tilde{n}(\mathbf{k}, t), \tag{22}$$

where $\tilde{n}(\mathbf{k}, t)$ is the smooth part of the momentum distribution. The solution of this set of coupled dynamical equations for $n_0(t)$ of the condensate and $\tilde{n}(\mathbf{k}, t)$ tends at time infinity toward the Bose-Einstein equilibrium distribution. This assumes that the condensate part is homogeneous in space, something that is not true in an infinitely extended system because the phase coherence so implied takes an infinite time to settle. Because of this non uniform phase, the dynamics of the condensate must be described by a more complex equation than the one for $n_0(t)$, it must be described by a modified Gross-Pitaevskii equation, so that the evolution of the condensate and the smooth momentum distribution is described by coupling the Gross-Pitaevskii equation for the condensate with the kinetic equation for the smooth part of the momentum distribution. This coupling adds an *inelastic term* to the Gross-Pitaevskii equation that is itself given by an integral quadratic with respect to the smooth momentum distribution.

The analogy between the Boson case and the water waves is as follows: there is in both cases an "envelop" like equation, Gross-Pitaevskii for Bosons and the mixed NLS equation (18) for waves. The gas of thermal particles is described by the Boltzmann-Nordheim equation for Bosons and by Hasselmann's equation for waves. In both cases there is a need to couple the two equations. For water waves this is because the BF instability requires a coupling to random waves (those with large wavenumbers), although the HE may require a regularization after a finite time singularity.

No finite time singularity of the solution of Hasselmann's equation has yet been found, to the best of our knowledge. It does not mean that it does not exist, but it does not prove the opposite. The numerical problem is noticeably more difficult for HE than for the Boltzmann-Nordheim kinetic equation because a finite time singularity cannot occur at zero number for gravity waves (although in the Bose gas the value of the momentum where the singularity appears first can always be set to zero by a suitable Galilean transform). Indeed in the HE the coefficient of wave interaction $|T|^2$ tends rapidly to zero when the peak wavenumber K_P decreases (like $K_P^3 K^3$ for a four-wave interaction between two peak waves and two background waves[6]). Therefore the basic mechanism for the finite time singularity is not there at *zero* momentum: an accumulation of waves near zero wave number does not make grow the kinetic term, because at the same time the interaction decays rather quickly at zero momenta. The same does not happen for the Boson case, where the interaction does not depend on the momenta.

The situation is different for a possible collapse of the waves *at a non zero momentum* $\mathbf{k}_0$, because the relevant interaction coefficient will be T with its four vectors at about the same finite value $\mathbf{k}_0$, a non zero constant. However the local problem in wave-number space will remain non isotropic because the frequencies near $\mathbf{k} = \mathbf{k}_0$ will have to expanded like

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_0)\left(1 + \frac{\mathbf{q}\cdot\mathbf{k}_0}{k_0^2} + \frac{q_x^2 - 2q_y^2}{k_0^2} + ..\right), \qquad (23)$$

where x is the coordinate along $\mathbf{k}_0$ and y is perpendicular to it, although $\mathbf{q} = \mathbf{k} - \mathbf{k_0}$ is the small variation of $\mathbf{k}$ near the collapse value $\mathbf{k}_0$. The correction to $\omega(\mathbf{k})$ linear with respect to $\mathbf{q}$ does not contribute to the argument of the Dirac distribution term (on frequencies) in Hasselmann's kinetic operator because it cancels automatically by the momentum condition, although the second order correction cannot be transformed into a condition for the modulus of $\mathbf{q}$ only. Therefore, even for $\mathbf{q}$ small, Hasselmann's kinetic operator near a non zero $\mathbf{k}_0$ remains anisotropic so that an eventual finite time singularity cannot be analyzed the same way as the one of the Boltzmann-Nordheim kinetic operator by using its Carleman reduction to isotropic momentum distributions. In other words a collapse of the spectrum at a non zero momentum $\mathbf{k}_0$ could exist for water waves but its numerical observation requires to solve a triple integral in the HE equation, whereas a double integral is enough to describe the collapse for bosons.

Let us assume that some solutions of HE have a finite time singularity at non zero wave number, presumably the typical situation of wind-driven waves which have no isotropic spectrum. The obvious question now is what happens beyond the singularity. Actually one has to change theory even before the exact singularity. As discussed above, once the width Δk of the peak near the singularity of the momentum distribution becomes less than ak^2 (see equation (15)) one has to consider, instead of the standard HE, a set of evolution equations coupling the envelope equation and the kinetic operator, as done in [19] for the Bose condensation. It seems reasonable to assume that, as for the kinetic equation for Bosons, the right set of equations describing the evolution near the singularity and beyond, couples an envelope equation having an inelastic term depending on the continuous momentum distribution with a kinetic equation having a new term proportional to the modulus square of the solution of the modified envelope equation. The envelope will continue to show the Benjamin-Feir instability, but in a restricted range of wave numbers because the phase of the envelop enters in a non trivial way into the coupling between the smooth momentum distribution and the envelope.

The occurrence of swell, that is of long waves after a wind storm, could be explained by this mechanism of finite time singularity of solutions of HE, as it is reported that swell is made of quasi-monochromatic waves, exactly what is described by the envelop theory. In the model suggested, the growth of this singular part of the spectrum would result from the nonlinear interaction of random waves, something that could look somewhat counter intuitive.

As a side remark, notice that distributions with a fixed value of the wave number along a given direction and an arbitrary value of the momentum perpendicular to this direction, say along the y axis, are stable against the evolution by Hasselmann's equation. Such a distribution has the form $n(\mathbf{k}) = n_0\delta(k_x - k_{x,0})f(k_y)$ where $f(.)$ is an a priori continuous function of its argument. This set of functions is stable under the action of the non linear collision operator because when all vectors $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ have the same component $k_{x,0}$ along x, then $\mathbf{k}$ has also $k_{x,0}$ as x-component as soon as the relation

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3,$$

is satisfied.

The function $f(k_y)$ is the solution of a nonlinear integral equation derived from the Hasselmann's kinetic operator by putting singular distributions of the type just considered.

3 Conclusion

To summarize, using the existence of a small physical parameter in the wind-sea interaction, the ratio of the mass densities of the two fluids, we have derived scaling laws for observable quantities like the mean square slope of the sea surface. This gives a basis for the derivation of the Hasselmannn-type equation for the spectrum, that would be valid not only for the case of weakly nonlinear turbulence, where it is generally addressed, but even for storms with for very large wind velocities. We propose a fully explicit mathematical model, equation (9), for the steady spectrum of surface elevation perturbed by a constant wind. This is valid for large wind speeds, where dissipation is mostly due to wave-breaking.

Furthermore we have considered the smoothness of solutions of Hasselmann's equation (13) without wind and without wave breaking, this being pertinent for the formation of swell. Based on the mathematically related situation of the kinetic theory for Bosons, we looked at the possibility of a finite time singularity of the equation of evolution of the distribution in momentum space. If this scenario is correct, the basis of the wave-turbulence equations needs to be revised, and a way would have to be found to extend the time evolution beyond the singularity time, something that has been already done for the kinetic theory of Bosons. In particular this requires to use beyond the singularity a set of dynamical equations coupling the kinetic wave equations and the envelope of coherent waves. Notice also that the property of superfluidity could have an equivalent in wave dynamics: once condensation has occurred, namely after the singularity of HE, the "condensate", namely the monochromatic wave is rigid in the sense that its phase tends to become homogeneous (with a wavevector $\mathbf{k_0}$) and does not change under the interaction with the random wave part of the spectrum. Indeed all this remains to be confronted with experimental results although the solution of the spectral equation remains to be studied in details in the limit ϵ small.

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