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Optimal Field Size for Wave Spectra Determination

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The nonhomogeneity of wave fields in shoaling water places intrinsic restrictions on the validity of high-resolution directional wave number spectra. A condition for the optimum sample size for shallow water directional spectra is derived, and the concomitant limiting resolution calculated.

A method for determining directional surface wave spectra by optical transforms has been discussed by *Stilwell* [1969]. Stilwell considered the case of homogeneous wave fields. Recently, however, *Klemas et al.* [1974] have been applying this method in shallow water, where the homogeneity condition is severely violated. The violation of homogeneity means that the accuracy of the resultant wave number spectrum is limited not only by Rayleigh's criterion for resolution but also by an additional constraint. This note is intended to point out that there is an intrinsic uncertainty for wave number spectra in shallow water or other nonhomogeneous regimes. This is embodied in an equation giving the optimal resolution for a wave number spectrum in a nonhomogeneous region.

Rayleigh [1879, 1880] considered the problem of optical instrumentation and showed that resolving power was directly related to the size of the elements in an optical system. This work was generalized by *Heisenberg* [1927], who gave a criterion relating spectral resolution to the spatial extent of a sample:

$$\Delta k \Delta x \geq ½ \tag{1}$$

where Δk is the wave number dispersion, defined as a root mean square (rms) value, and Δx is essentially the rms width of a sample window in configuration space.

APPLICATION TO PHOTOGRAPHIC TECHNIQUES

In the case of a rectangular photograph the Fourier transform of the area of the photograph acts as a filter on the Fourier transform of the pattern photographed. In one dimension the Fourier transform of the photograph is the classical one for diffraction from a rectangular slit:

$$F(k) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin kl/2}{k} \tag{2}$$

where l is the width of the photograph in the direction under consideration. This pattern taken in its entirety has an infinite width; however, in the optical Fourier transform technique, only the central lobe is used, i.e., the portion $-\pi < kl/2 < \pi$. The rms width of this portion is given by

$$(\Delta k)^2 = \int_{-2\pi/l}^{2\pi/l} k^2 F(k)\, dk \left(\int_{-2\pi/l}^{2\pi/l} F(k)\, dk\right)^{-1} \tag{3}$$

$$\Delta k = 2(\pi)^{1/2}/l \tag{4}$$

which is consistent with (1)

In shoaling water the wavelength of a given component of the spectrum will not be a constant, although its frequency will be. In linear theory, under the assumptions that permit ray tracing (the eikonal approximation), the wave number for a given period will be a function of the depth alone. Since the period of a wave is conserved, the variation of a wave number with depth may be calculated by differentiating the dispersion relation. In this study we are concerned with gravity waves, since these are the only waves whose wave numbers will be transformed by shoaling outside of the breaker zone; therefore

$$\omega^2 = gk \tanh kh \tag{5}$$

where ω is the angular frequency of the waves, g the acceleration of gravity, k the magnitude of the vector wave number, and h the depth (taken as positive). Differentiating (5) implicitly and simplifying, we find

$$\frac{dk}{dx} = \frac{-mk^2}{\frac{1}{2}\sinh 2kh + kh} \tag{6}$$

where m is the bottom slope and the x axis is oriented antiparallel to the gradient of the bottom.

As the magnitude of the wave number changes, refraction may alter its direction. In what follows, we shall assume that the isobaths are locally parallel. Referring to Figure 1, we see that if a wave is incident at an angle θ to the bottom gradient, the change in direction while the wave front advances a distance δx along the x axis is

$$\delta\theta = \delta t/\delta w\ \delta c \tag{7}$$

where $\delta t = \delta x \cos \theta/c$ is the time required to advance the distance δx, δw is an elemental length along the wave front, and c is the phase speed of the wave. If we take advantage of the geometric relationships in the figure and the fact that $ck = \omega$, the rate of change of angle with distance is

$$\frac{d\theta}{dx} = \frac{\sin 2\theta}{2k}\frac{dk}{dx} \tag{8}$$

By using equations (6) and (8) it is possible to calculate the variation of the x and y components of wave number k_1 and

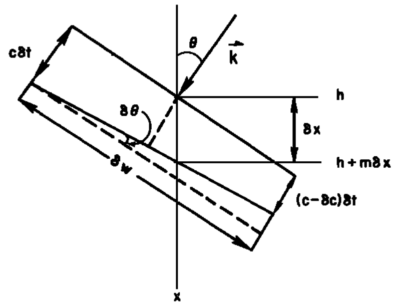

Fig. 1. Schematic refraction diagram.

k_2, respectively. By definition,

$$k_1 = k \cos \theta \quad (9)$$

$$k_2 = k \sin \theta \quad (10)$$

$$\frac{dk_1}{dx} = -k \sin \theta \frac{d\theta}{dx} + \cos \theta \frac{dk}{dx} \quad (11)$$

The variation of k_2 that would interest us is that with respect to y. This, however, is zero. Substituting (6) and (8) into (11)

$$\frac{dk_1}{dx} = \left(\frac{\sin \theta \sin 2\theta}{2} - \cos \theta\right) \frac{mk^2}{\frac{1}{2} \sinh 2kh + kh} \quad (12)$$

If the bottom slope is approximately constant, the rms value of the deviation from the mean value of k_1 for a wave of a given period in a distance l can be calculated from (12). It is

$$\delta k_1 = \tfrac{1}{3}(3)^{1/2}\left(\frac{\sin \theta \sin 2\theta}{2} - \cos \theta\right) \frac{mk^2\, l}{\sinh 2kh + 2kh} \quad (13)$$

Equations (4) and (13) describe two qualitatively different causes of variation in the determination of the x component of wave number for a wave of a given frequency and original direction. Clearly, if an aerial photograph of the sea surface is too large, (13) will restrict the accuracy of the resultant spectrum, whereas if it is too small, (4) will be the limiting factor. Therefore the optimum value of l is that for which $\Delta k_1 = \delta k_1$. This will be a function of the vector wave number in question as well as of the mean depth and bottom slope of the area observed. The optimal value of l is given by

$$l_{\mathrm{opt}} = \frac{2}{k}\left(\frac{3\pi(\sinh 2kh + 2kh)}{m(\sin \theta \sin 2\theta - 2 \cos \theta)}\right)^{1/2} \quad (14)$$

In the shallow water wave approximation for waves approaching at $\theta = 0$ this reduces to

$$l_{\mathrm{opt}} = 2\left(\frac{-2(3\pi h)^{1/2}}{mk}\right)^{1/2} \quad (15)$$

(Recall that m is negative, so that l_{opt} is real.) Table 1 provides illustrative values of l_{opt} in dimensionless form for $\theta = 0$, and the tabulation below gives the dependence of optimal field size $l_{\mathrm{opt}}(\theta)$ on θ (in degrees), the angle between the bottom gradient and the vector wave number.

θ	$l_{\mathrm{opt}}(\theta)/l_{\mathrm{opt}}(0)$	θ	$l_{\mathrm{opt}}(\theta)/l_{\mathrm{opt}}(0)$
0	1	50	1.94
10	1.02	60	2.83
20	1.10	70	5.00
30	1.24	80	13.8
40	1.49	90	∞

Equation (14) can be combined with equation (4) to provide a specification of the greatest spectral resolution possible in given circumstances:

$$\frac{\Delta k}{k} = (\pi)^{1/2}\left(\frac{m(\sin \theta \sin 2\theta - 2 \cos \theta)}{(3\pi)^{1/2}(\sinh 2kh + 2kh)}\right)^{1/2} \quad (16)$$

TABLE 1. Values of Dimensionless Optimal Field Size kl_{opt} as a Function of Bottom Slope m and Dimensionless Depth kh

$-m$	kh = 1	kh = 0.5	kh = 0.1	kh = 0.05
10^{-4}	5.88×10^2	3.65×10^2	1.57×10^2	1.11×10^2
10^{-3}	1.86×10^2	1.16×10^2	4.96×10	3.54×10
10^{-2}	5.88×10	3.65×10	1.57×10	1.11×10
10^{-1}	1.86×10	1.16×10	4.96	3.54

In some cases, e.g., $\Delta k/k \gtrsim 1$, it is clear that the notion of a wave number spectrum is almost meaningless, nor can any substantial gain be made by the use of better filters or data windows in view of the constraint embodied in inequality (1). On the other hand, equation (16) does not prevent one from analyzing for the spectrum in those areas of k vector space where it is unwarranted to do so. Thus care must be taken to limit the analysis to the domain for which the results will be physically meaningful.

CONCLUSIONS

We have shown that a trade-off must be made between the increase of resolution from a larger sample size and the intrinsic variation of the nonhomogeneous wave field. The field size for an optimal trade-off and the intrinsic limit of resolution have been calculated. These results will apply not only to the Stilwell method but with suitable modification for the form of the data window function, to all other means of analyzing wave spectra in shoaling water. From a broader perspective a similar optimization and calculation of limiting resolution should be employed in the analysis of any other nonhomogeneous or nonstationary process.

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