THE PARTITION OF ENERGY IN PERIODIC IRROTATIONAL WAVES ON THE SURFACE OF DEEP WATER

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Introduction. In the theory of long waves, and also for infinitesimal "surface" waves, the total wave energy is half kinetic and half potential to a first order of approximation.[1] It was pointed out by Rayleigh (1911), in proceeding by successive approximations to the solution of the problem of oscillatory permanent waves on the surface of deep water, that the kinetic energy exceeds the potential by an amount proportional to a quantity of the fourth order, when quantities of order higher than the fourth are neglected.

The exact theory of oscillatory permanent waves was formulated and developed by Stokes (1847), who represented the solution in the form of a Fourier expansion. In this paper are presented general formulas for the kinetic and potential energy in terms of coefficients in the series employed by Stokes, from which the energy may be computed to whatever order these coefficients are known. A review of the general theory is given first, together with the results of a new method for performing the successive approximations that lead to the coefficients in Stokes' solution.

Review of the General Theory. The motion, two dimensional and irrotational, is taken in the xz plane, the x-axis being horizontal and the z-axis vertical, counted positive *upward*. The waves progress in the direction of x *positive*, and the progressive motion is reduced to steady motion by imposing a uniform horizontal velocity in the direction of x negative, of magnitude equal to the wave speed c. Thus, if we denote the velocity components of the progressive motion by u, w, then the corresponding components of the steady motion are $U = u - c$, and w. Further, if ϕ, ψ denote the velocity potential and stream function for the progressive motion, and Φ, Ψ the corresponding functions for the steady motion, then

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[1] See Lamb (1932), Art. 174, 230.

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial z} \qquad\qquad w = -\frac{\partial \phi}{\partial z} = +\frac{\partial \psi}{\partial x}$$

$$U = -\frac{\partial \Phi}{\partial x} = -\frac{\partial \Psi'}{\partial z} \qquad\qquad w = -\frac{\partial \Phi}{\partial z} = +\frac{\partial \Psi'}{\partial x}. \qquad (1)$$

Hence $\qquad\qquad \Phi = \phi + cx \qquad\qquad \Psi' = \psi + cz, \qquad\qquad (2)$

provided the origins for the several parameters involved are properly chosen.[2]

We take as the form of the solution that presented by Stokes (1880), in which the variables are inverted, the *independent* variables being Φ, Ψ' so that x, z are harmonic functions of these. It will prove convenient, in writing this solution and in subsequent calculations, to employ *nondimensional* quantities, defined as follows:

$$\xi = kx \qquad\qquad \lambda = k\phi/c \qquad\qquad \Lambda = k\Phi/c$$

$$\zeta = kz \qquad\qquad \omega = k\psi/c \qquad\qquad \Omega = k\Psi'/c,$$

where $k = 2\pi/L$, the wave length being L. The solution pertaining to the case of deep water is then

$$\xi = \Lambda + \Sigma A_r e^{r\Omega} \sin r\Lambda, \qquad \zeta = \Omega + \Sigma A_r e^{r\Omega} \cos r\Lambda, \qquad (3)$$

the summation extending over all *positive* integral values of the index r. In the steady motion the free surface is a streamline, which we take to be $\Omega = 0$, while for great depths $\Omega \to -\infty$. Thus, at the free surface[3]

$$\xi_o = \Lambda + \Sigma A_r \sin r\Lambda, \qquad \zeta_o = \Sigma A_r \cos r\Lambda, \qquad (4)$$

and for great depths $\xi \to \Lambda$ and $\zeta \to \Omega$.

The coefficients A_r in (3) are determined in such a manner as to satisfy the condition of uniform pressure at the free surface, which may be written in the nondimensional form

$$\left(\frac{Q}{c}\right)_o^2 + \frac{2}{\mu}\left(\zeta_o - \overline{\zeta_o}\right) = 1,$$

where $Q^2 = U^2 + w^2$; and

$$\mu = \frac{k}{g}c^2 = \frac{2\pi c^2}{gL} \qquad\qquad (5)$$

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[2] The symbol x in (2) refers to the horizontal co-ordinate measured relative to the steady motion; equations for the progressive motion may be found at any stage by writing $x - ct$ for x.

[3] The subscript $_o$ will be used to identify quantities evaluated at the free surface. A single summation sign Σ will stand throughout for $\Sigma_{r=1}^{\infty}$ or $\Sigma_{s=1}^{\infty}$.

is the square of the ratio of the wave speed to the speed of propagation of "infinitesimal" surface waves in deep water (g denoting the acceleration of gravity). The quantity $\overline{\zeta_0}$ is the elevation of the mean level of the free surface,

$$\overline{\zeta_0} = \frac{1}{2\pi} \int_0^{2\pi} \zeta_0 d\xi_0 = \frac{1}{2} \Sigma r A_r^2. \tag{6}$$

In the method utilized by Stokes, each coefficient A_r is expressed ultimately as a series of powers of the leading coefficient $A_1 = b$. This determination, which proceeds by successive approximations, has been carried to the tenth order by Wilton (1914), whose results (through eighth order) are shown in Table I.[4]

TABLE I. Coefficients A_{rs} (Through Eighth Order) in the Series $A_r = \Sigma A_{rs} b^s$. Where No Entry Is Made the Corresponding A_{rs} Is Null

r \ s	1	2	3	4	5	6	7	8
1	1	—	—	—	—	—	—	—
2	—	1	—	1/2	—	29/12	—	1123/72
3	—	—	3/2	—	19/12	—	1183/144	—
4	—	—	—	8/3	—	313/72	—	103727/4320
5	—	—	—	—	125/24	—	16603/1440	—
6	—	—	—	—	—	54/5	—	54473/1800
7	—	—	—	—	—	—	16807/720	—
8	—	—	—	—	—	—	—	16384/315

From an inspection of Table I it is evident that $A_r = \Sigma A_{rs} b^s$ is of order r at least; in fact, A_r/b^r is a series of even powers of b. No general formula for A_{rs} is known, except for A_{rr}, which Wilton (1914) has shown to be $A_{rr} = r^{r-2}/(r-1)!$.

The *height* h of the wave (elevation of crest above trough) is given by

$$kh = \zeta_0(\text{crest}) - \zeta_0(\text{trough}),$$

where $k = 2\pi/L$. Since $\Lambda = 0$ at a crest and π at a trough, we find from (4), for the ratio of height to length,

$$\frac{h}{L} = \frac{1}{\pi} (A_1 + A_3 + A_5 + \ldots),$$

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[4] The origin for Λ is conventionally taken to coincide with a trough, and with this convention the A_r-coefficients are found to be alternately positive and negative. However, we here place this origin at a crest, so that the A_r-coefficients are *all positive*.

or, writing $\pi h/L = ka$,

$$ka = A_1 + A_3 + A_5 + \ldots, \tag{7}$$

where $a = h/2$ is the wave *amplitude*.

The expansion (3) represents the solution of the wave problem for all waves whose height is less than that of the *highest* wave. Michell (1893) gave an exact theory for the highest wave, which Stokes had shown to have pointed crests, the front and rear slopes of the profile meeting there at an angle of 120 degrees. According to Michell's analysis the ratio of height to length for the highest wave is 0.142 approximately, or nearly 1/7. Havelock (1918), by an elegant extension of Michell's method, obtained an expansion that represents the solution of the oscillatory wave problem for all waves up to and including the highest, thus bringing into harmony the results of Stokes and those of Michell. Further, he found, for the highest wave, the ratio of height to length $h/L = 0.1418$ and the value of the parameter in Stokes' solution $b = 0.2919$.

In the method of approximation employed by Stokes, the dimensionless quantity μ is obtained as a series of even powers[5] of b,

$$\mu = 1 + b^2 + \frac{7}{2} b^4 + \frac{229}{12} b^6 + \frac{6175}{48} b^8 + \ldots$$

The convergence of this series is not rapid for values of b corresponding to waves close to the highest wave. Michell found for the highest wave $\mu = 1.200$.

By defining a quantity β according to

$$\mu = 1 + \beta^2$$

it is possible to perform the successive approximations leading to the A_r-coefficients in such a manner that each A_r is expressed ultimately as a series of powers of β. The result of such a determination (which is not more laborious than that of Stokes) is shown in Table II.[6]

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[5] The value 7427/48 given by Wilton for the coefficient of b^8 is erroneous.

[6] Table II and Table I may be shown to be in agreement by expressing β in terms of b and converting the β-series into b-series.

TABLE II. Coefficients A_{rs} (Through Eighth Order) in the Series
$A_r = \Sigma A_{rs}\beta^s$. Where No Entry Is Made the Corresponding
A_{rs} Is Null

r\s	1	2	3	4	5	6	7	8
1	+1	—	−7/4	—	+113/96	——	−949/384	——
2	—	+1	—	−3	——	+13/3	——	−1051/144
3	—	—	+3/2	—	−151/24	——	+7741/576	——
4	—	—	—	+8/3	——	−1031/72	——	+172457/4320
5	—	—	—	—	+125/24	——	−24511/720	——
6	—	—	—	—	——	+54/5	——	−149647/1800
7	—	—	—	—	——	——	+16807/720	——
8	—	—	—	—	——	——	——	+16384/315

It is to be noted particularly that each of the β-series in Table II is an *alternating* series, and therefore each may be expected to converge more rapidly than the corresponding b-series in Table I.

The relation between β and b may be found by equating the expressions for A_1 in the two tables,

$$b = \beta - \frac{7}{4}\beta^3 + \frac{113}{96}\beta^5 - \frac{949}{384}\beta^7 + \ldots \ldots,$$

or reverting,

$$\beta = b + \frac{7}{4}b^3 + \frac{769}{96}b^5 + \frac{6439}{128}b^7 + \ldots \ldots$$

Similarly, by applying (7) for the wave amplitude,

$$ka = \beta - \frac{1}{4}\beta^3 + \frac{3}{32}\beta^5 + \frac{1543}{5760}\beta^7 + \ldots \ldots$$

or, writing $\alpha = ka$, and reverting,

$$\beta = \alpha + \frac{1}{4}\alpha^3 + \frac{3}{32}\alpha^5 - \frac{1543}{5760}\alpha^7 \ldots \ldots$$

Since $kc^2/g = \mu = 1 + \beta^2$, we find the following series for the wave speed,

$$c^2 = \frac{g}{k}\left(1 + \alpha^2 + \frac{1}{2}\alpha^4 + \frac{1}{4}\alpha^6 - \frac{22}{45}\alpha^8 \ldots \ldots\right). \quad (8)$$

The quantity $\alpha = \pi h/L = ka$ is the product of wave number and wave amplitude (one-half wave height). Since the maximum value of h/L, corresponding to the highest wave, is 0.1418, the maximum value of

α is 0.4455. For this value the successive partial sums of the series in (8) are 1.000, 1.198, 1.218, 1.220, 1.219, ..., the proper value being 1.200 for the highest wave. Although the convergence of (8) is not very strong for the highest wave, the error in neglecting terms of order higher than the second, that is, the error in the approximate formula

$$c^2 \approx \frac{g}{k}(1 + \alpha^2) = \frac{g}{k}(1 + k^2 a^2),$$

is less than two per cent for waves of arbitrary height.

Kinetic Energy. The kinetic energy of the *progressive* motion in one wave length is

$$e = \frac{1}{2}\rho \int \int (u^2 + w^2)\, dx dz,$$

the integration extending from $x = 0$ to L (crest to crest) and through the entire depth of fluid (ρ denoting the fluid density, supposed uniform). However, it follows directly from (1) that $u^2 + w^2 = \partial\,(\phi,\,\psi)/\partial(x,\,z)$, and hence

$$e = \frac{1}{2}\rho \int \int d\phi d\psi = \frac{1}{2}\rho \oint \phi d\psi.$$

The line integral may be written in terms of dimensionless quantities,

$$\frac{2k^2}{\rho c^2}e = \oint \lambda d\omega,$$

and evaluated as follows.

Equations (2) in nondimensional form,

$$\Lambda = \lambda + \xi, \qquad \Omega = \omega + \zeta,$$

together with (3), show that

$$\lambda = -\Sigma A_r e^{r\Omega}\sin r\Lambda, \qquad \omega = -\Sigma A_r e^{r\Omega}\cos r\Lambda.$$

It is evident that λ becomes vanishingly small at great depths ($\Omega \rightarrow -\infty$) and vanishes everywhere along the lateral boundaries of the region of integration, that is, along $\Lambda = 0$ and $\Lambda = 2\pi$. There remains in the line integral only a contribution along the free surface ($\Omega = 0$), where

$$\lambda_0 = -\Sigma A_r \sin r\Lambda, \qquad \omega_0 = -\Sigma A_r \cos r\Lambda,$$

the limits of integration being $\Lambda = 0$ and $\Lambda = 2\pi$.

Proceeding in this way it is easily found that[7]

$$\frac{2k^2}{\rho c^2} e = \pi \Sigma r A_r^2 = 2\pi \bar{\zeta}_o \, ,$$

where $\bar{\zeta}_o$ is the elevation of the mean level of the wave profile. Making use of (5) we have finally

$$\frac{2k^3}{\pi \rho g} e = \mu \Sigma r A_r^2 \, ;$$

expanding the right-hand member as a β-series,

$$\frac{2k^3}{\pi \rho g} e = \beta^2 - \frac{1}{2} \beta^4 - \frac{4}{3} \beta^6 - \frac{251}{144} \beta^8 \ . \ . \ . \quad (9)$$

Potential Energy. The potential energy of the wave in one wave length is

$$v = \frac{1}{2} \rho g \int^L (z_o - \bar{z_o})^2 dx \, ,$$

or reducing to nondimensional quantities,

$$\frac{2k^3}{\pi \rho g} v = \frac{1}{\pi} \int_0^{2\pi} (\zeta_o - \bar{\zeta_o})^2 d\xi = 2\bar{\zeta_o^2} - 2\bar{\zeta_o}^2 \, .$$

In terms of the A_r-coefficients $\bar{\zeta_o}$ is given by (6). The evaluation of $\bar{\zeta_o^2}$ is somewhat more laborious but after some reduction is found to give[8]

$$2\bar{\zeta_o^2} = \Sigma A_r^2 + \Sigma\Sigma r A_p A_q A_r \, . \atop (r = p + q)$$

Thus, for the potential energy,

$$\frac{2k^3}{\pi \rho g} v = \Sigma A_r^2 + \Sigma\Sigma r A_p A_q A_r - \frac{1}{2} (\Sigma r A_r^2)^2 \, ; \atop (r = p + q)$$

expanding the right-hand member as a β-series,

$$\frac{2k^3}{\pi \rho g} v = \beta^2 - \beta^4 - \frac{5}{6} \beta^6 - \frac{257}{144} \beta^8 \ . \ . \ . \ . \quad (10)$$

Partition of Energy. The difference between kinetic and potential energy may be expressed as a β-series[9] by subtracting (10) from (9):

[7] This result may be obtained directly through an application of the *momentum theorem* developed by Starr (1947).

[8] We make use of the identity $\Sigma\Sigma r A_p A_q A_r \equiv \Sigma\Sigma r A_p A_q A_r$. The summation ex-
$\quad\quad (r = |p - q|) \quad\quad (r = p + q)$
tends over all positive integral values of p and q.

$$\frac{2k^3}{\pi \rho g}(e - v) = \frac{1}{2}\beta^4 - \frac{1}{2}\beta^6 + \frac{1}{24}\beta^8 - \frac{55}{144}\beta^{10} + \ldots \quad (11)$$

This result corroborates the computations of Rayleigh (1911), who found that the series for $e - v$ starts with a term of the fourth order.

Dividing (11) by (10) we obtain a series giving the ratio of $e - v$ to v,

$$\frac{e - v}{v} = \frac{1}{2}\beta^2 + \frac{11}{24}\beta^6 + \frac{31}{32}\beta^8 + \ldots$$

This may be developed as an α-series by means of the series preceding (8); thus,

$$\frac{e - v}{v} = \frac{1}{2}\alpha^2 + \frac{1}{4}\alpha^4 + \frac{7}{12}\alpha^6 + \frac{2033}{1440}\alpha^8 + \ldots, \quad (12)$$

where α is the *amplitude parameter ka*. The maximum value of α, corresponding to the highest wave, is 0.4455, for which the successive convergents of (12) are, in per cent, 9.93, 10.92, 11.37, 11.59, ... Although the convergence is not very rapid, it appears that the maximum value of $(e - v)/v$ is approximately $\frac{1}{8}$.

Water of Limited Depth. The foregoing considerations are applicable without alteration when the depth of water is limited, provided only that appropriate coefficients in the Fourier expansions are employed. Stokes (1880) has carried the successive approximations (leading to the A_r-coefficients) through the third order in this case, but so far as the writer is aware, these computations have not been extended subsequently. Further, there does not exist in the literature, to the writer's knowledge, an analysis of the highest waves of permanent type when the depth of water is limited, corresponding to the analysis given by Michell for deep water. For these reasons it has not seemed worthwhile to extend the energy computations to the case of limited depth.

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[9] A tenth-order term is included in (11) even though such terms are not present in (9) and (10). This is possible without extending Table II, because the tenth-order term in (11) is independent of terms higher than the eighth order in the A_r-coefficients.

SUMMARY

Periodic irrotational waves of permanent type on the surface of deep water are studied for the purpose of analyzing the partition of total wave energy into kinetic and potential. Employing Stokes' representation of the solution of the wave problem, successive coefficients in the Fourier expansion are computed to the eighth order in terms of $\beta = \sqrt{1 - \mu}$, where $\mu = 2\pi c^2/gL$ is the square of the ratio of the wave speed to the speed of propagation of infinitesimal surface waves. An approximate frequency equation is given, which may be written $\mu \approx 1 + (\pi h/L)^2$ (where h is the height of the wave) and which is in error by not more than two per cent for all waves including the highest.

The kinetic energy e and potential energy v are expressed as power series in β, the determination in each case being carried to the eighth order. A β-series is then derived for the ratio $(e - v)/v$, from which it is shown that the maximum value of this ratio, corresponding to the highest wave, is approximately $\frac{1}{8}$.

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