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# Shoaling of sixth-order Stokes waves on a current

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**Abstract**

Jonsson and Arneborg (Jonsson, I.G., Arneborg, L., 1995. Energy properties and shoaling of higher-order Stokes waves on a current. Ocean Engng 22, 819-857.) combined energy flux and set-down to make shoaling predictions for fourth-order Stokes waves with and without a net volume flux. With basis in their expressions, sixth-order expressions are derived and combined to make shoaling predictions correct to sixth order with an arbitrary net volume flux. The new sixth-order results are compared with the fourth-order results and the practically exact results obtained by Sobey and Bando (Sobey R.J., Bando K., 1991. Variations on higher-order shoaling. J. Waterway, Port, Coastal Ocean Engng ASCE 117, 348-368) with a Fourier 18 model. The effects of introducing sixth-order theory rather than the fourth-order theory in shoaling calculations are in general found to be small. As expected the deviations increase with increasing wave-steepness, decreasing depth and opposing currents. Also as expected, the results obtained with the sixth-order expressions improve the results obtained with the fourth-order expressions when compared to the results of Sobey and Bando. As novelties, some considerations regarding the consistency of odd- and even-order shoaling calculations, and the magnitude of the bottom slope, are presented. Furthermore a comparison between the wave-induced current and the total current is given. © 2000 Published by Elsevier Science Ltd.

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## 1. Introduction

In 1985 Fenton presented new fifth-order expressions for the velocity potential, the surface elevation etc. derived from Stokes theory that corrected an error in the expressions of Skjellbreia and Hendrickson (1961) and further introduced a constant-over-depth current in the expressions.

Jonsson and Arneborg (1995) used the expressions of Fenton (1985) to derive fourth-order expressions for the energy density and the energy flux by means of depth integration and time averaging. They checked these results using some exact relations between integral properties found by Longuet-Higgins (1975) and Klopman (1990).

Shoaling of linear waves on a current was first treated by Jonsson et al. (1971). Higher-order shoaling has been treated by various authors using different wave theories and assumptions. A review of these works are given by Sobey and Bando (1991), and briefly summarized below. Le Méhauté and Webb (1964) applied third-order Stokes theory and assumed a zero set-down and a zero Eulerian current velocity. The latter assumption disagrees with the physical situation when pure waves propagate towards a beach with a zero mass flux. Sakai and Battjes (1980) used Cokelet's (1977) theory to make shoaling predictions, and were among the first to imply the assumption of a zero mass flux rather than a zero Eulerian current velocity. They did not predict the set-down, but introduced no errors by this, since the energy flux is independent of the chosen datum for a zero mass flux (see Jonsson and Arneborg, 1995). However, their shoaling curves have the physical depth, *h*, as ordinate rather than the more convenient undisturbed depth *D* (see Fig. 1). Stiassnie and Peregrine (1980) combined Cokelet's theory with the solitary wave theory in shallow water. Their model included the set-down and was based on a zero mass flux. Sobey

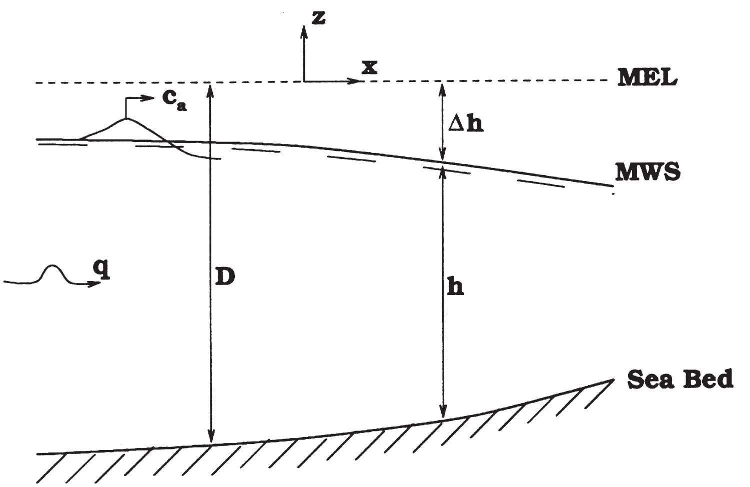


Fig. 1. Definition sketch for shoaling calculations.

and Bando (1991) used a Fourier 18 wave theory to predict wave height, wave length, return current velocity and set-down.

The algebraic results of Jonsson and Arneborg (1995), however, made it possible, probably for the first time, to perform higher-order shoaling calculations for Stokes waves on a current. Jonsson and Arneborg made these calculations correct to fourth order with the constant Mean Energy Level (MEL) as the datum for the energy flux. As mentioned by Jonsson and Arneborg this approach has some advantages when compared to more sophisticated theories: good accuracy, except for large steepnesses, ease of application and simple inclusion of an arbitrary current. Furthermore, Jonsson and Arneborg derived some of the expressions with sixth-order accuracy and they mentioned that all other integral properties can be calculated with this knowledge. Hence, some of the work needed to extend their work to sixth order has already been made.

The remaining work is described in this paper, presenting the governing equations correct to sixth order in Section 2. Odd-order shoaling calculations give rise to problems regarding the consistency of the order of the applied equations. These problems are described in Section 3 and are the reason for going directly from fourth order to sixth order. In Section 3 also the constraints on the slope of the bed are discussed. In Section 4 the results for the shoaling calculations are presented and compared to the fourth-order results of Jonsson and Arneborg (1995). For the case of a zero mass flux, a comparison with the results produced by Sobey and Bando (1991) with a Fourier 18 model is made.

## 2. The governing equations

Shoaling is the mechanism that describes the evolution of steady waves propagating perpendicular to the coast over a slowly varying bottom topography and with bottom contours that are straight and parallel to the (straight) coastline. The physical problem is sketched in Fig. 1.

$c_a$ is the absolute phase velocity of the wave and $q$ is the net volume flux, independent of the wave. We define the mean fluid transport velocity $c_S$ by $q=c_S h$. The still water depth $D$ is the depth one would measure if no waves and no current were present. $h$ is the physical water depth defining the mean water surface (MWS), and $\Delta h$ is the set-down which is the distance from the still water level to MWS. The still water level defines the constant Mean Energy Level (MEL), as introduced by Lundgren (1963) for pure waves, and by Jonsson et al. (1971) for waves on a current (see also Jonsson and Arneborg, 1995). This level can be used as datum for the energy flux in shoaling calculations, since this is constant contrary to the mean water surface. We choose a coordinate system with the $x$-axis in the still water level pointing in the direction of wave propagation, and the $z$-axis pointing upwards with the origin in the still water level.

Expressions of higher-order accuracy exists only for horizontal bottoms. However, as we assume very slow changes of the water depth, the expressions derived for constant depth can be used as a good approximation. The requirements to the bottom slope are discussed in Section 3.

Following the work of Fenton (1985), we use the wave steepness $\varepsilon = kH/2 = \pi H/L$ as the perturbation parameter, where $k = (2\pi/L)$, $H$ and $L$ are the wave number, wave height and wave length, respectively.

The governing equations are:
conservation of mass

$$c_S h = \text{const} \tag{2.1}$$

conservation of energy flux with MEL as datum ($F_{MEL}$)

$$F_{MEL} = \text{const} \tag{2.2}$$

an equation for the bottom topography (see Fig. 1)

$$h + \Delta h = D \tag{2.3}$$

the equation for the set-down derived to sixth order by Jonsson and Arneborg (1995, Eq. (39))

$$\begin{aligned} k\Delta h = {} & \frac{k}{2g}c_S^2 + \varepsilon^2\left[C_0^2\frac{1}{4}A_{11}^2 + \sqrt{\frac{k}{g}}c_S\frac{D_2}{kh}\right] + \varepsilon^4\left[C_0^2\left(\frac{1}{2}A_{11}A_{31} + A_{22}^2\right) + \frac{1}{2}\left(\frac{D_2}{kh}\right)^2 \right. \\ & \left. + \sqrt{\frac{k}{g}}c_S\frac{D_4}{kh}\right] + \varepsilon^6\left[C_0^2\left(\frac{1}{4}A_{31}^2 + \frac{1}{2}A_{11}A_{51} + 2A_{22}A_{42} + \frac{9}{4}A_{33}^2\right) + \frac{D_2D_4}{(kh)^2} \right. \\ & \left. + \sqrt{\frac{k}{g}}c_S\frac{D_6}{kh}\right] \end{aligned} \tag{2.4}$$

and the dispersion relation (see below). In Expression (2.4), $g$ is the acceleration of gravity, and $A_{ij}$, $C_i$, $D_i$ are dimensionless coefficients by Fenton (1985), except for $D_6$ which was derived by Jonsson and Arneborg (1995)

$$\begin{aligned} D_6 = {} & (\coth kh)^{1/2}(560 + 596S - 3836S^2 - 8148S^3 - 6512S^4 - 1991S^5 \\ & + 1483S^6 + 1404S^7 + 244S^8)/[64(1-S)^6(3+2S)(4+S)] \end{aligned} \tag{2.5}$$

with $S = \text{sech}(2kh)$.

Since the equations for mass conservation and bottom topography are exact and the set-down is known to sixth order, we only need to derive sixth-order expressions for the dispersion relation and the energy flux with MEL as datum, to be able to make sixth-order shoaling calculations.

### 2.1. The dispersion relation

In Jonsson and Arneborg (1995) the dispersion relations for first- to fifth-order theory are given. These are repeated below. In the equations $\omega_a$ is the absolute angular frequency, which is related to $c_a$ by $c_a = \omega_a/k$.

First-order theory dispersion relation

$$c_S\sqrt{\frac{k}{g}} - \frac{\omega_a}{\sqrt{gk}} + C_0 = 0. \tag{2.6}$$

Second-order theory dispersion relation

$$c_S\sqrt{\frac{k}{g}}-\frac{\omega_a}{\sqrt{gk}}+C_0+\varepsilon^2\frac{D_2}{kh}=0 \tag{2.7}$$

Third-order theory dispersion relation

$$c_S\sqrt{\frac{k}{g}}-\frac{\omega_a}{\sqrt{gk}}+C_0+\varepsilon^2\left(C_2+\frac{D_2}{kh}\right)=0. \tag{2.8}$$

Fourth-order theory dispersion relation

$$c_S\sqrt{\frac{k}{g}}-\frac{\omega_a}{\sqrt{gk}}+C_0+\varepsilon^2\left(C_2+\frac{D_2}{kh}\right)+\varepsilon^4\frac{D_4}{kh}=0. \tag{2.9}$$

Fifth-order theory dispersion relation

$$c_S\sqrt{\frac{k}{g}}-\frac{\omega_a}{\sqrt{gk}}+C_0+\varepsilon^2\left(C_2+\frac{D_2}{kh}\right)+\varepsilon^4\left(C_4+\frac{D_4}{kh}\right)=0. \tag{2.10}$$

From this sequence it is quite easy to find the pattern when the order of accuracy of the dispersion relation is increased by one. When going from an odd order to the even order $i$ the term $\varepsilon^i D_i/kh$ is added, and when going to the next odd order (order $i$+1) the term $\varepsilon^i C_i$ is added. From this we induct, that the sixth-order dispersion relation can be found by adding the term $\varepsilon^6 D_6/kh$ to the fifth-order dispersion relation. This can also be shown by combining Eqs. (8) and (38) from Jonsson and Arneborg (1995) [Eq. (8) belongs to the sixth-order theory, cf. their discussion, p. 835]. Hence the sixth-order theory dispersion relation becomes

$$c_S\sqrt{\frac{k}{g}}-\frac{\omega_a}{\sqrt{gk}}+C_0+\varepsilon^2\left(C_2+\frac{D_2}{kh}\right)+\varepsilon^4\left(C_4+\frac{D_4}{kh}\right)+\varepsilon^6\frac{D_6}{kh}=0. \tag{2.11}$$

This result agrees with the result of Steenberg (1995) who used a variational principle on the averaged Lagrangian.

### *2.2. The energy flux*

Since the shoaling calculations are based on conservation of the energy flux, a constant horizontal level is needed as datum for the potential energy. For this level the Mean Energy Level (MEL) is used. The relation between the energy flux with MWS as datum and MEL as datum was found by Jonsson and Arneborg (1995), who inserted $z-\Delta h$ instead of $z$ in the definition of the energy flux. The result obtained, which is exact, is

$$F_{MEL}=F_{MWS}-\rho gh\Delta hc_S \tag{2.12}$$

where $\rho$ is the fluid density.

In both Sobey et al. (1987) and Klopman (1990) the following exact expression for the energy flux with MWS as datum is presented

$$F_{\mathrm{MWS}}=c_{\mathrm{a}}(3E_{\mathrm{K}}-2E_{\mathrm{P,MWS}})+\frac{1}{2}\langle u_{b}^{2}\rangle(I+\rho c_{\mathrm{a}}h)-2c_{\mathrm{a}}c_{\mathrm{E}}I \tag{2.13}$$

where $E_{\mathrm{K}}$ is the kinetic energy and $E_{\mathrm{P,MWS}}$ the potential energy with MWS as datum. $u_{\mathrm{b}}$ is the horizontal velocity at the bottom, $I$ the mean mass flux in a fixed frame, and $c_{\mathrm{E}}$ the mean Eulerian current velocity (below wave trough level). The symbol $\langle\,\rangle$ denotes time averaging over one absolute wave period.

The horizontal velocity at the bottom is calculated using the result

$$\langle u_{\mathrm{b}}^{2}\rangle=2g\Delta h \tag{2.14}$$

derived in, for example, Jonsson and Arneborg (1995, Eq. (35)), in combination with Eqs. (2.4). The quantity $I$ is related to the volume flux velocity by

$$I=\rho c_{\mathrm{S}}h, \tag{2.15}$$

and $c_{\mathrm{E}}$ is related to $c_{\mathrm{S}}$ by Eq. (38) of Jonsson and Arneborg (1995)

$$c_{\mathrm{E}}=c_{\mathrm{S}}+\sqrt{\frac{g}{k}}\left(\varepsilon^{2}\frac{D_{2}}{kh}+\varepsilon^{4}\frac{D_{4}}{kh}+\varepsilon^{6}\frac{D_{6}}{kh}\right). \tag{2.16}$$

The kinetic energy can be found by using a result of Longuet-Higgins (1975)

$$E_{\mathrm{K}}=\frac{1}{2}(c_{\mathrm{a}}I-c_{\mathrm{E}}Q) \tag{2.17}$$

where $Q$ is the mean mass flux in the negative direction in a frame moving with the phase velocity $c_{\mathrm{a}}$ and is thus given by

$$Q=\rho h(c_{\mathrm{a}}-c_{\mathrm{S}}). \tag{2.18}$$

From the dispersion relation, Eq. (2.11) and the relation $\omega_{\mathrm{a}}=kc_{\mathrm{a}}$, $c_{\mathrm{a}}$ can be related to $c_{\mathrm{S}}$ by

$$\sqrt{\frac{k}{g}}(c_{\mathrm{S}}-c_{\mathrm{a}})=-C_{0}-\varepsilon^{2}\left(C_{2}+\frac{D_{2}}{kh}\right)-\varepsilon^{4}\left(C_{4}+\frac{D_{4}}{kh}\right)-\varepsilon^{6}\frac{D_{6}}{kh}. \tag{2.19}$$

Inserting Eqs. (2.15), (2.16), (2.18), (2.19) into Eq. (2.17) the kinetic energy is evaluated to

$$E_{\mathrm{K}}=\frac{1}{2}\rho hc_{\mathrm{S}}^{2}+\frac{1}{4}\rho\frac{g}{k^{2}}\varepsilon^{2}\left[1-2\varepsilon^{2}\left(C_{0}D_{4}+C_{2}D_{2}+\frac{D_{2}^{2}}{kh}\right)-2\varepsilon^{4}\left(C_{0}D_{6}+C_{2}D_{4}+C_{4}D_{2}+2\frac{D_{2}D_{4}}{kh}\right)\right]. \tag{2.20}$$

To fourth order this expression agrees with the expression found by Jonsson and Arneborg (1995). It is also consistent with the sixth-order expression found by Steenberg (1995) with $c_{\mathrm{S}}=0$, derived by depth integration and time averaging.

The potential energy with MWS as datum is given by

$$E_{P,MWS}=\left\langle \int_0^\eta \rho g z \,\mathrm{d}z \right\rangle = \frac{1}{2}\rho g \langle \eta^2 \rangle \quad (2.21)$$

in which the fifth-order result for $\eta$ of Fenton (1985) can be inserted. Correct to sixth order the result obtained is

$$E_{P,MWS}=\frac{1}{4}\rho\frac{g}{k^2}\varepsilon^2[1+\varepsilon^2(B_{22}^2+2B_{31})+\varepsilon^4 2(B_{31}^2+B_{22}B_{42}-(B_{53}+B_{55}))] \quad (2.22)$$

This result was through extensive algebraic manipulations shown to be identical with the expression of Steenberg (1995)

$$E_{P,MWS}=\frac{1}{4}\rho\frac{g}{k^2}\varepsilon^2\left[1-\varepsilon^2 2C_0D_4-\varepsilon^4 2\left(C_0D_6+\frac{1}{3}C_2D_4-\frac{1}{3}C_4D_2\right)\right] \quad (2.23)$$

who used depth-integration and time averaging of Fenton's (1985) results as well. The latter expression is used in the following.

Inserting Eqs. (2.4), (2.14), (2.15), (2.16), (2.20), (2.23) into Eq. (2.13) and using $c_a = \omega_a/k$ algebraic manipulations lead to

$$F_{MWS}=\rho g h c_S \Delta h+\varepsilon^2\frac{\omega_a}{k}\frac{g}{k^2}\rho\left[\frac{1}{4}(1+G)-\sqrt{\frac{k}{g}}c_S D_2\right]$$

$$+\varepsilon^4\frac{\omega_a}{k}\frac{g}{k^2}\rho\left[-\frac{3}{2}D_2C_2-\frac{1}{2}D_4C_0-\frac{D_2^2}{kh}+C_0^2\left(\frac{1}{2}A_{11}A_{31}+A_{22}^2\right)kh-\sqrt{\frac{k}{g}}c_S D_4\right] \quad (2.24)$$

$$+\varepsilon^6\frac{\omega_a}{k}\frac{g}{k^2}\rho\left[-\frac{1}{2}D_6C_0-\frac{7}{6}C_2D_4-\frac{11}{6}C_4D_2-2\frac{D_2D_4}{kh}\right.$$

$$\left.+C_0^2\left(\frac{1}{4}A_{31}^2+\frac{1}{2}A_{11}A_{51}+2A_{22}A_{42}+\frac{9}{4}A_{33}^2\right)kh-\sqrt{\frac{k}{g}}c_SD_6\right]$$

where $G\equiv 2kh/\sinh 2kh$. Now the energy flux to sixth order with MEL as datum can be calculated from Eq. (2.12) to yield

$$F_{\mathrm{MEL}}=\varepsilon^2\frac{\omega_a}{k}\frac{g}{k^2}\rho\left[\frac{1}{4}(1+G)-\sqrt{\frac{k}{g}}c_SD_2\right]$$

$$+\varepsilon^4\frac{\omega_a}{k}\frac{g}{k^2}\rho\left[-\frac{3}{2}C_2D_2-\frac{1}{2}C_0D_4-\frac{D_2^2}{kh}+C_0^2\left(\frac{1}{2}A_{11}A_{31}+A_{22}^2\right)kh-\sqrt{\frac{k}{g}}c_SD_4\right] \quad (2.25)$$

$$+\varepsilon^6\frac{\omega_a}{k}\frac{g}{k^2}\rho\left[-\frac{1}{2}C_0D_6-\frac{7}{6}C_2D_4-\frac{11}{6}C_4D_2-2\frac{D_2D_4}{kh}\right.$$

$$\left.+C_0^2\left(\frac{1}{4}A_{31}^2+\frac{1}{2}A_{11}A_{51}+2A_{22}A_{42}+\frac{9}{4}A_{33}^2\right)kh-\sqrt{\frac{k}{g}}c_SD_6\right].$$

This result agrees to fourth order with the result of Jonsson and Arneborg (1995, Eq. (40)), and to sixth order with the result of Steenberg (1995), who calculated the energy flux based on action flux [using Eq. (33) in Jonsson, 1998]. In this context, it should be noted that in contrast to what was stated by Jonsson and Arneborg (1995), the fourth-order deep-water, no-current energy flux found from Eq. (2.24) or Eqs. (2.25) ($\Delta h$=0) is indeed consistent with Eq. (2.8) with $U$=0 in Holliday (1973). This is further detailed in Jonsson and Steenberg (1999).

## 3. Perspectives of higher-order shoaling calculations

When higher-order Stokes theories are applied to shoaling calculations, problems arise with respect to consistency of order of the equations applied and the magnitude of the bed slope. Some of these will be discussed below.

### 3.1. Regarding odd- and even-order theories in shoaling calculations

As in any perturbation theory the governing equations for shoaling calculations must be consistent regarding the order of the equations.

The set-down can be calculated by

$$\Delta h = -\frac{1}{g}\left\langle \frac{\partial}{\partial t}\phi_{\mathrm{MWS}} \right\rangle \tag{3.1}$$

see, for example, Jonsson and Arneborg (1995, Eq. (16)). Here $\phi_{\mathrm{MWS}}$ is the velocity potential with the MWS as datum. With fifth-order accuracy for $\phi_{\mathrm{MWS}}$, the set-down would be obtained with fifth-order accuracy (which yields the same result as fourth-order accuracy, due to the lack of odd-order terms in $\varepsilon$ in the expression for the set-down [see Eq. (2.4)]). Accordingly, the energy flux determined by the calculations presented in the previous section would also be obtained with fifth- (fourth-) order accuracy.

However, Jonsson and Kofoed-Hansen (1993) managed to derive an expression for the set-down correct to sixth-order, taking basis in the fifth-order theory of Fenton (1985), by using the result (2.14)

$$\Delta h = \left\langle \frac{u_{\mathrm{b}}^2}{2g} \right\rangle \tag{3.2}$$

see Jonsson and Kofoed-Hansen (1993, Eq. (25)). Jonsson and Kofoed-Hansen succeeded as the mean Eulerian current velocity $c_{\mathrm{E}}$ was used in their derivation rather than the mass flux velocity $c_{\mathrm{S}}$ and therefore $D_6$ was not needed. However, as the mass flux velocity is specified independently of the wave, this must be used in shoaling calculations.

In Jonsson and Arneborg (1995) the volume flux $q$=$c_{\mathrm{S}}h$ correct to sixth order was derived by time averaging the depth integrated horizontal velocity

$$q=c_S h=\left\langle \int_{-h}^{\eta} u\,\mathrm{dz} \right\rangle \tag{3.3}$$

using the horizontal velocity correct to fifth-order. The result from this calculation enabled Jonsson and Arneborg to derive a relation between the mean Eulerian velocity $c_E$ and the mass flux velocity $c_S$ correct to sixth-order, and thus to determine the coefficient $D_6$ (see Jonsson and Arneborg, 1995, Eqs. (38) and (37)). Finally Jonsson and Arneborg derived an expression correct to sixth-order for the set-down, based on the mass flux velocity $c_S$ (see Jonsson and Arneborg, 1995 Eq. (39)).

Jonsson and Arneborg furthermore mentioned that with the knowledge of the coefficeient $D_6$ it is possible to derive all integral properties correct to sixth-order. This fact is consistent with the entire idea of perturbation techniques, i.e. that $n$'th-order expressions can be calculated from $(n-1)$'th- and lower-order expressions.

The governing equations for shoaling calculations of odd orders can be found via truncation of the equations to the needed order or by calculating the integral quantities via Eq. (3.1). However, as the integral properties can, via Eq. (3.2), be determined correct to an order which is one order higher than the basis, one might argue that one should take this higher-order accuracy into account in shoaling calculations of odd orders. Thus there is a dilemma about which order of accuracy to choose for the integral quantities for shoaling calculations of odd orders. We are not able to solve this dilemma, and thus we recommend to avoid odd orders in shoaling calculations.

According to the above discussion, conventional linear (first-order) shoaling calculations without a current are not an example of the above dilemma. Energy flux is used with second-order accuracy, the dispersion relation is used with first-order accuracy, and finally the set-down is used with zeroth-order accuracy. Using different order in the expressions for integral properties is in fact not very consistent.

Contrary to this, the shoaling calculations for waves on a current of Jonsson et al. (1971) are an example of the dilemma mentioned above. Second-order expressions for the energy flux and the set-down are used together with a first-order dispersion relation. However, as the dispersion relation and the integral properties do not have the same accuracy, one might argue that these calculations are inconsistent.

The governing equations for shoaling calculations of even orders do not give rise to the dilemma mentioned above. As integral properties have contributions of even order in $\varepsilon$ only, the integral properties derived via Eqs. (3.1) or (3.2) yield the same result, provided that we have an even-order (in our case sixth-order) expression for $\phi$. Furthermore even-order shoaling calculation seems consistent as we have the same accuracy in all expressions, and thus we recommend to perform even-order shoaling calculations only. Examples are the fourth-order shoaling calculations of Jonsson and Arneborg (1995) and the sixth-order shoaling calculations presented in this paper.

### *3.2. Regarding the bottom variation*

As mentioned above the bottom variation must be small in order to use the expressions (2.4), (2.11) and (2.25), which strictly speaking only are valid on constant water depth. The expressions are derived by applying a perturbation technique on the governing equations for the potential flow steady wave problem, i.e. the Laplace equation with kinematic and dynamic boundary conditions and a periodicity condition. On a sloping bottom the kinematic boundary condition at the bottom takes the form

$$\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial z}\right)\cdot\left(\frac{\mathrm{d}h}{\mathrm{d}x},1\right)=0 \Leftrightarrow \frac{\partial\phi}{\partial x}\frac{\mathrm{d}h}{\mathrm{d}x}+\frac{\partial\phi}{\partial z}=0, \quad z=-h(x). \tag{3.4}$$

Also the periodicity condition is changed, whereas the other equations remain unchanged.

Applying the perturbation approach to $n$'th order on the kinematic boundary condition at the bottom we find

$$\frac{\mathrm{d}h}{\mathrm{d}x}\sum_{i=0}^{n}\varepsilon^i\frac{\partial\phi^i}{\partial x}+\sum_{i=0}^{n}\varepsilon^i\frac{\partial\phi^i}{\partial z}=O(\varepsilon^{n+1}),\ z=-h(x) \tag{3.5}$$

where $\phi^i$ is the $i$'th-order component of the velocity potential. To avoid contributions from the first term in the $n$'th-order solution, we must require that d$h$/d$x$ is of order $O(\varepsilon^{n+1})$ when the volume flux derived from $\phi^0$ is nonzero and that d$h$/d$x$ must be of order $O(\varepsilon^n)$ when the volume flux is zero.

As we cannot expect contributions due to a nonzero bottom slope to average out when calculating integral properties, we must have the same requirements to the bottom slope when considering integral properties correct to $n$'th order.

Considering the maximal deep water steepness $\varepsilon_{\max}=\pi(H/L)_{\max}\approx 0.443162$ [see Eq. (4.1)], we get that the bottom slope must be smaller than $O(0.0171)$ and $O(0.00336)$, respectively, for fourth- and sixth-order theory shoaling calculations. For less steep waves, say $\varepsilon=0.1$ ($H/L\approx 3\%$), the bottom slope must be less than $O(0.01\times 10^{-3})$ for fourth-order and $O(0.1\times 10^{-6})$ for sixth-order shoaling calculations, respectively. These constraints limit the applicability of the theory. However, as higher-order shoaling theories should be applied on cases with high wave-steepnesses, and as bottom slopes are quite small in general, the higher-order shoaling theories can be used in many cases.

The above discussion presents a dilemma for a given accuracy regarding the order of the governing expressions. For a given bottom slope we will have that small waves are represented accurately by the asymptotic expansion, but the error due to a neglection of the bottom slope is relatively large. When the waves become steeper, the asymptotic error becomes larger, whereas the error due to the neglection of the bottom slope becomes relatively smaller. It is not possible to make both errors small.

### *3.3. Regarding the accuracy of the physical water depth*

As mentioned in Jonsson and Arneborg (1995), there is a "snag" in using a perturbation theory in the derivation of the governing equations for shoaling calculations. The expressions assume that the physical water depth $h$ is exact, but as the set-down $\Delta h$ is calculated correct to sixth order, and the still water depth $D$ is prescribed, the equation for the bottom topography, Eq. (2.3), gives that the physical water depth is correct to sixth-order only. The result is that higher-order errors are introduced in the shoaling calculations. As also mentioned in Jonsson and Arneborg, this inconsistency cannot be avoided by deriving new expressions by inserting $h=D-\Delta h$ correct to the appropriate order in the original expressions and then truncate to the appropriate order, because $h$ also appears in the expression for the set-down. However, as we accepted to neglect higher-order terms in the governing equations, we have already introduced higher-order errors at this point. It seems unlikely that the higher-order errors mentioned above contribute more significantly than the higher-order errors in the governing equations. However, the errors increase with increasing steepness and decreasing water depth as most of the coefficients grow with decreasing water depth.

## 4. Results

To perform the shoaling calculations, the five Eqs. (2.1)–(2.4) and (2.11) with Eq. (2.25) where solved iteratively using explicit iteration. This method is not the most stable one, but it did not break down for a realistic set of parameters. For comparison and program testing the fourth-order results were reproduced. No deviations from the results of Jonsson and Arneborg (1995) were found.

The calculations are limited by the applicability of Stokes theory as in Jonsson and Arneborg (1995), and the highest wave height for waves of permanent form. Furthermore the Froude number, based on $c_S$, was not allowed to exceed unity. The maximal wave height for waves of permanent form was determined from the following formula by Fenton (1990), based on numerical results found by Williams (1981):

$$\frac{H_{\max}}{h}=\frac{0.141063\frac{L}{h}+0.0095721\left(\frac{L}{h}\right)^2+0.0077829\left(\frac{L}{h}\right)^3}{1+0.0788340\frac{L}{h}+0.0317567\left(\frac{L}{h}\right)^2+0.0093407\left(\frac{L}{h}\right)^3}. \tag{4.1}$$

These limitations are the same as used by Jonsson and Arneborg.

### 4.1. Presentation of the results

The results are given as evolution curves for dimensionless wave height $H/H_0$ and dimensionless wave length $L/L_0$, with the dimensionless still water depth $D/L_0$ as horizontal axis. $H_0$ is the deep-water wave height, and $L_0=(g/2\pi)T_a^2$ is the linear deep-water wave length in the absence of a current, $T_a=2\pi/\omega_a$ being the absolute wave period. In the figures, both the fourth- and sixth-order results are plotted. Transition to a dotted curve indicates, that the maximal wave height has been exceeded and the solution has become unphysical. The termination of a curve indicates that the theory has broken down due to the limitation of Stokes theory, that the Froude number has exceeded unity or that the iteration method has broken down.

### 4.2. Results for pure waves

In Fig. 2 the dimensionless wave height is shown. Curves are drawn for the following four values of the linear deep-water steepness

$$\frac{H_0}{L_0}=\left\{0,0.06,0.10,0.95\cdot\left(\frac{H_0}{L_0}\right)_{\max}\right\}$$

where the value of $(H_0/L_0)_{\max}$ was found by use of Eq. (4.1) in the limit $h\rightarrow\infty$, giving $(H_0/L)_{\max}=0.141063$. Following multiplication with $L/L_0$ determined by the appropriate deep-water dispersion relation, gave the linear deep-water steepness needed. A zero linear deep-water steepness corresponds to conventional linear theory.

The figure shows that the drop in wave height from linear theory is reduced when the deep-water wave steepness is increased. For the steeper waves we still observe a fast increase in the wave height at finite depth. The difference between the fourth- and sixth-order solutions is largest for the steepest waves, which is natural, since the perturbation parameter $\varepsilon=\pi H/L$ is proportional to the steepness. The sixth-order wave height is larger than the fourth-order one for the curves shown, except for some insignificant deviations and for the results of the wave having the steepness 0.06 just before termination.

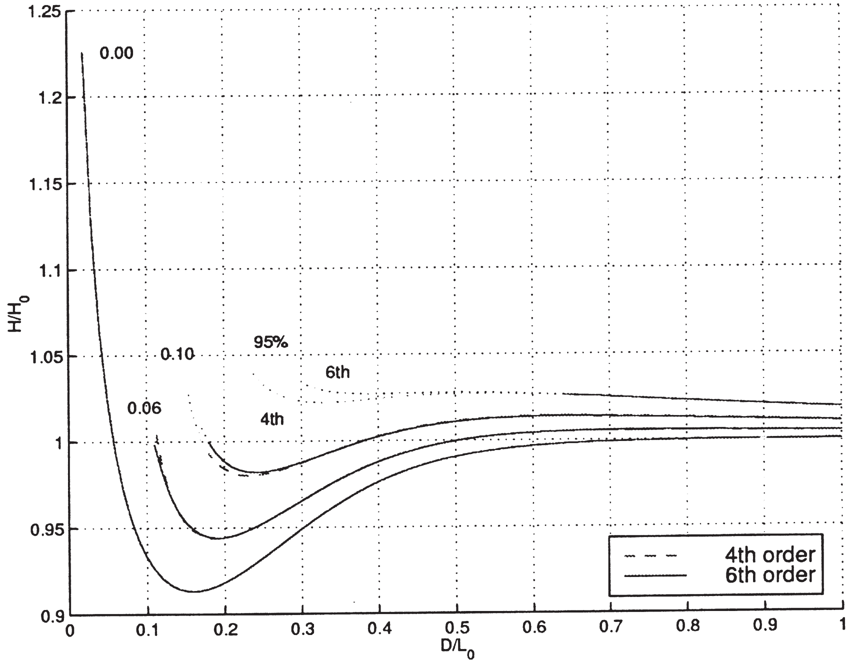


Fig. 2. Wave height shoaling curves for pure waves. Numbers on the curves indicate the linear deep-water steepness $H_0/L_0$. 95% denotes 95% of the maximal linear deep-water steepness.

The dimensionless wave length is plotted in Fig. 3. Note that the quantity $L/L_0$ does not tend to unity at deep water, because $L_0$ is the *linear* deep-water wave length. The wave length decreases monotonically with decreasing depth. However, the steeper waves show this decreasing behaviour at larger depths than the waves with small steepnesses (compare the slope of the curves at deep water). This is due to the wave induced return current (i.e. an opposing current), which has a shortening effect on the wave length. This is enhanced when the depth is decreased due to the increase in the return current velocity. In general the sixth-order wave length is larger than the fourth-order one.

### 4.3. Results for waves on a current

In Figs. 4 and 5 the dimensionless wave height and wave length are plotted for waves having the linear deep-water steepness $H_0/L_0$=0.10 and with the dimensionless volume fluxes

$$q^* = \frac{c_s h}{c_0 L_0} = \{-0.10, -0.05, -0.02, 0, 0.02, 0.05, 0.10\}$$

$c_0=(g/2\pi)T_a$ being the linear deep-water phase velocity without a current.

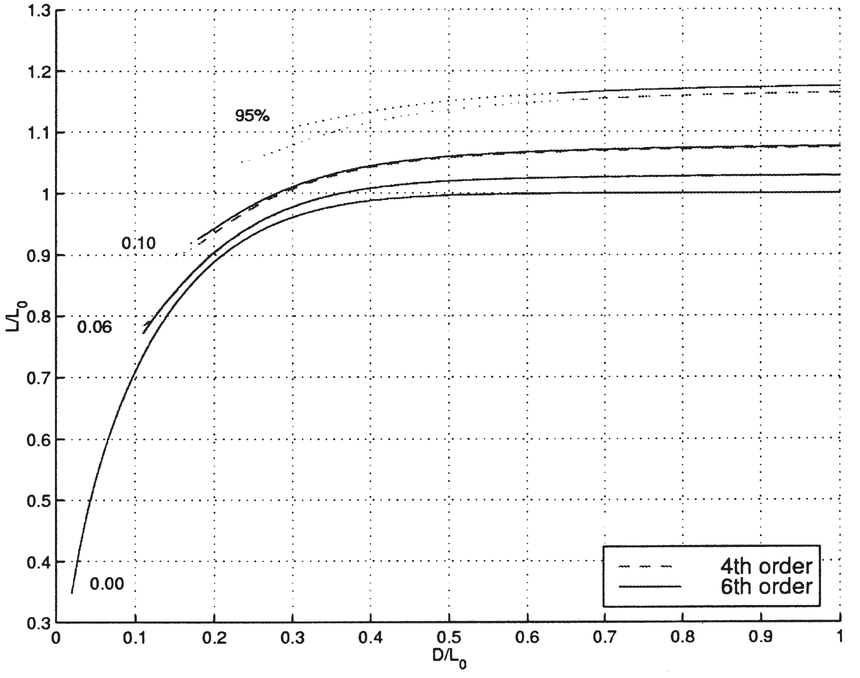


Fig. 3. Wave length shoaling curves for pure waves. Numbers on the curves indicate the linear deep-water steepness $H_0/L_0$. 95% denotes 95% of the maximal linear deep-water steepness.

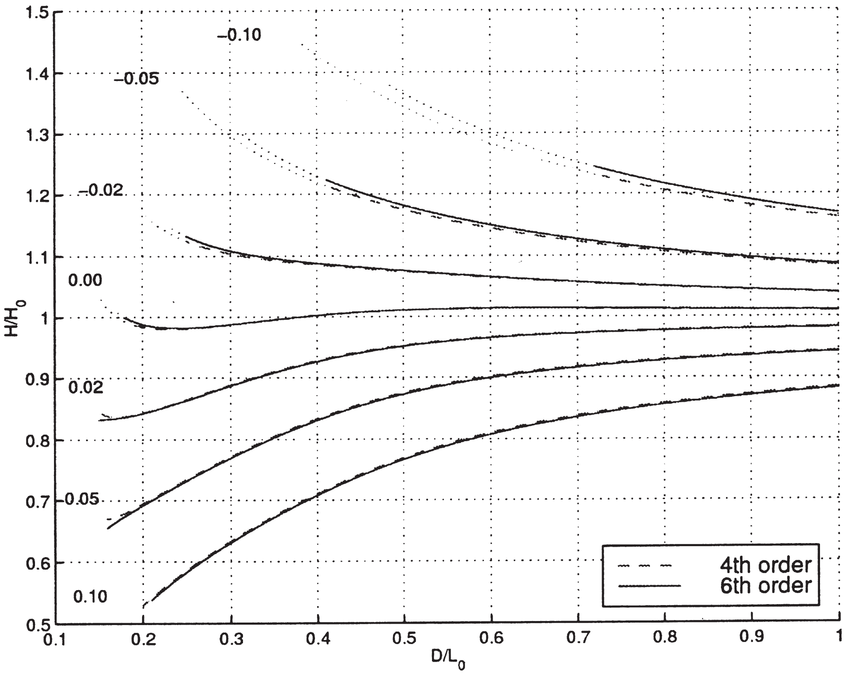


Fig. 4. Wave height shoaling curves for waves having the linear deep-water steepness $H_0/L_0$=0.10. Numbers on the curves indicate the dimensionless volume flux $q^*=c_sh/(c_0L_0)$.

For opposing currents the wave height increases with decreasing depth due to the compressing effect of the countercurrent (except for very small countercurrents). This effect is enhanced when the depth is reduced, since the mean Eulerian velocity below trough level, $c_E$, increases with decreasing depth. For following currents the wave height decreases due to the opposite stretchening effect (except for very small following currents). The effect of the added sixth-order terms is not large, and most significant for opposing currents. This is due to the steepening effect of an opposing current on the wave profile. For opposing currents the sixth-order wave height is larger than the fourth-order wave height, while the opposite holds for the following currents. For the following currents $q^*$=0.02 and $q^*$=0.05 the sixth-order curves are almost smooth before termination, while the fourth-order curves shows a small upward bend. This effect can also be observed in Fig. 5.

The compressing effect of an opposing current can be observed from the evolution curves for the wave length too (Fig. 5). For opposing currents the wave length decreases, while it increases for strong following currents. For the small following current $q^*$=0.02 the wave length increases at deep water and decreases at shallower water. The first increase is due to the stretchening effect of the current, which is overruled at shallower water by the usual shoaling effect.

The curve with no volume flux, $q^*$=0, has the same decreasing behaviour as the curves for opposing currents. This is due to the wave induced return current.

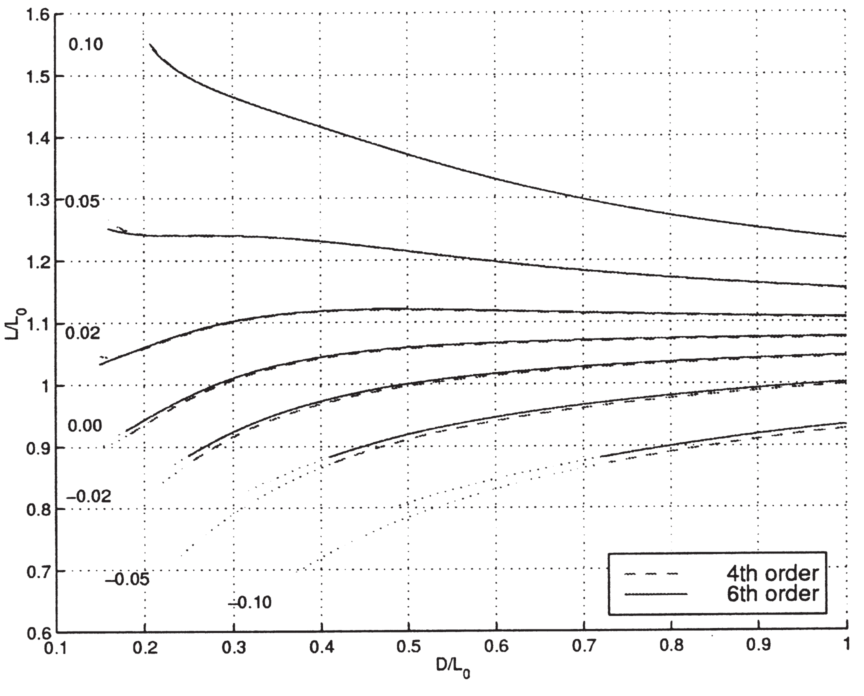


Fig. 5. Wave length shoaling curves for waves having the linear deep-water steepness $H_0/L_0$=0.10. Numbers on the curves indicate the dimensionless volume flux $q^*=c_s h/(c_0 L_0)$.

Like in Fig. 3 the sixth-order wave length is larger than the fourth-order wave length except for the small bends before termination of the two curves of $q^*$=0.02 and $q^*$=0.05. The differences between the fourth- and sixth-order curves are hardly discernible for following currents.

The dimensionless return currents for waves of linear deep-water steepness $H_0/L_0$=0.10 with the volume fluxes $q^*=\{-0.10, 0, 0.02\}$ are depicted in Fig. 6. The return current is incorporated in the following dimensionless wave induced return volume flux

$$q^*_{\mathrm{ret}}=\frac{U_{\mathrm{ret}}h}{c_0 L_0} \tag{4.2}$$

where $U_{\mathrm{ret}}$ is the return current defined by $U_{\mathrm{ret}}\equiv c_S-c_E$ (always positive). Because of the way $q^*_{\mathrm{ret}}$ is formulated in Eq. (4.2) it is possible to compare the results with the dimensionless total volume flux $q^*=c_S h/c_0 L_0$ directly.

As generel tendencies we see that the return current increases with decreasing current. The wave on the following dimensionless total volume flux $q^*$=0.02 generates a return current of about one-third of the value of $q^*$. For steeper waves ($H_0/L_0=0.95(H_0/L_0)_{\max}$), a return current of 75% was found at deep water (not shown). The return current for $q^*$=0 is a little larger seen from the figure, but note that the

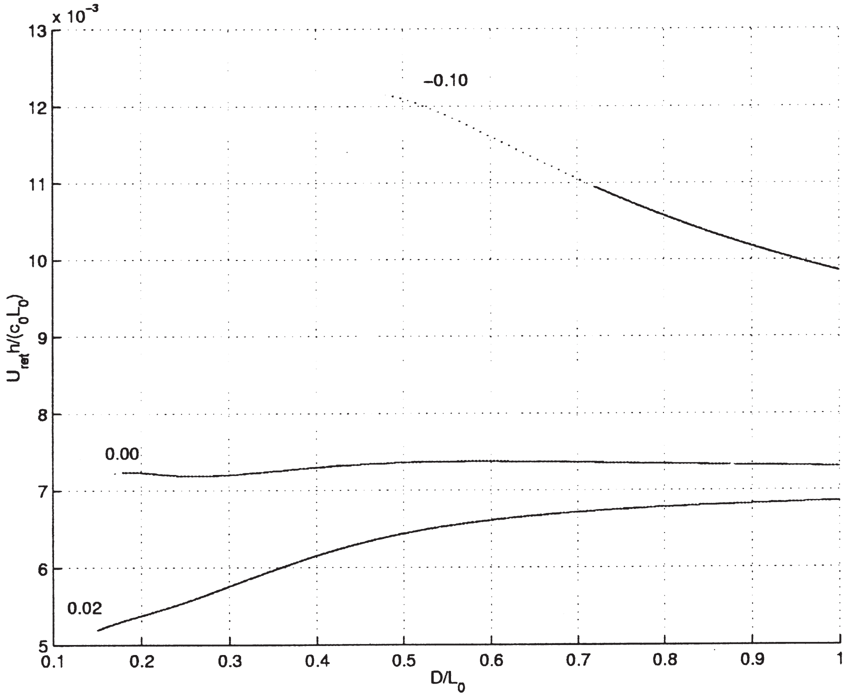


Fig. 6. Dimensionless wave induced 'return volume flux' for waves with linear deep-water steepness $H_0/L_0$=0.10. Numbers on the curves indicate the dimensionless volume flux $q^{*}=c_s h/(c_0 L_0)$.

ratio of $q^*_{\mathrm{ret}}$ and $q^*$ is infinite. Even for large currents the ratio can be quite big and even bigger for steep waves.

### 4.4. Comparison with more comprehensive methods

For the case of pure waves, the results have been compared with the results of Sobey and Bando (1991) produced with the Fourier 18 model of Sobey (1989) which is considered as practically exact. They started their shoaling calculations at the finite depth $\omega_a^2 D/g$=4, i.e. $D/L_0$=$2/\pi$ and with the wave steepnesses $2\pi H_{\mathrm{ref}}/L_0$=0.02, 0.05, 0.1, 0.2, 0.5 and 0.75. $H_{\mathrm{ref}}$ is the initial wave height in the shoaling calculations and thus differs from our $H_0$. We only show the comparison with the last three steepnesses, since there were practically no deviations for the first three. The steepness $H_{\mathrm{ref}}/L_0$ for these three waves are 0.032, 0.080 and 0.119.

In Fig. 7 the dimensionless wave heights are plotted. Both the Fourier 18 solution, the fourth-order solution and the sixth-order solution are shown.

For the least steep wave, the three solution curves agree very well. For the two other steepnesses quite large deviations are observed at shallower water. The wave height predictions are however improved by adding the sixth-order terms. Only for the wave of steepness $H_{\mathrm{ref}}/L_0$=0.080 the wave height predicted by the sixth-order

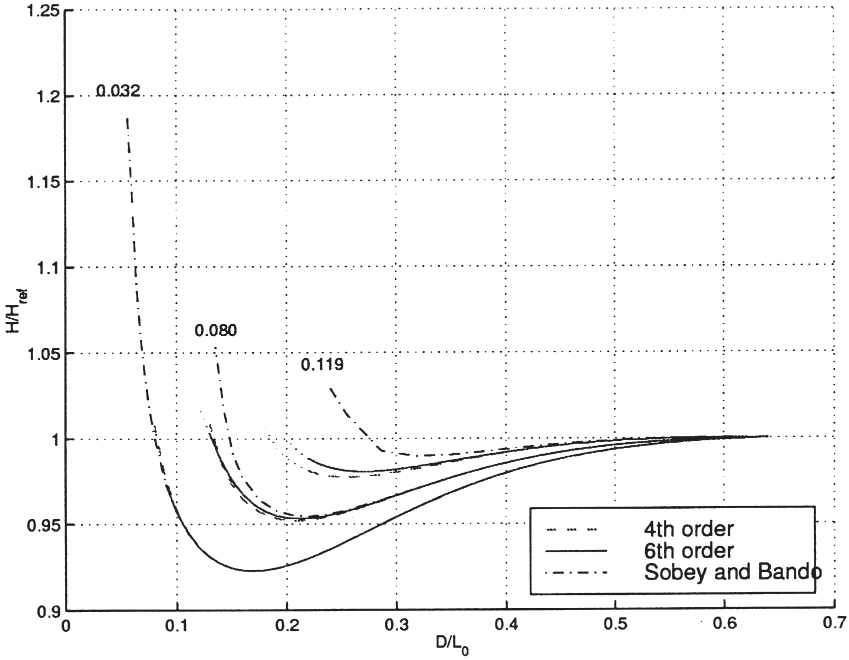


Fig. 7. Comparison of the fourth- and sixth-order shoaling solutions with numerical results of Sobey and Bando (1991) with respect to wave height. Volume flux is zero and numbers on the curves are values of $H_{ref}/L_0$.

solution becomes worse than the fourth-order solution, and this happens only just before the wave breaks. By calculating the quantity $H_{ref}/L_0$ for the curve of steepness $H_0/L_0$=0.10 in Fig. 2 it can be shown that the curve of steepness $H_{ref}/L_0$=0.119 would be lying between the curves for the two steepest waves in Fig. 2. Hence the depicted curves for $H_{ref}/L_0$=0.119 can be considered more or less as a worst-case regarding the deviations between the results.

The results for wave length in Fig. 8 show very good agreement for the two waves of smallest steepness. For the steepest wave the improvement of the sixth-order solution almost eliminates the error of the fourth-order solution. Only very close to the depth of breaking a slight deviation is observed.

## 5. Conclusion

Jonsson and Arneborg (1995) presented a sixth-order expression for the set-down of Stokes waves based on the volume flux velocity. Using this result and the sixth-order dispersion relation found by induction, a sixth-order expression for the energy flux with the Mean Energy Level (MEL) as datum has been derived. The expression has been compared with the result of Steenberg (1995), who determined the energy

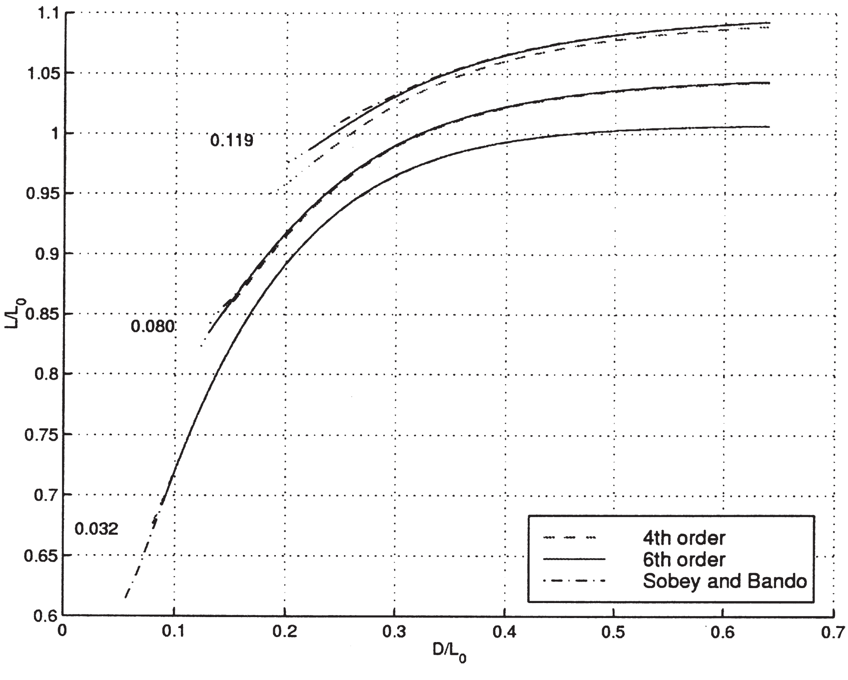


Fig. 8. Comparison of the fourth- and sixth-order shoaling solutions with numerical results of Sobey and Bando (1991) with respect to wave length. Volume flux is zero and numbers on the curves are values of $H_{ref}/L_0$.

flux with basis in action flux, and the two expressions have been found to be identical.

By combining the expressions mentioned above with the demand of constant mass flux and constant energy flux with MEL as datum, and an equation for the bottom topography, sixth-order shoaling calculations for waves with and without a current have been carried out, thus extending the fourth-order approach by Jonsson and Arneborg (1995).

It has been found that it is possible to make even-order shoaling calculations consistent whereas odd-order shoaling calculations entail a dilemma regarding the order of the equations used. The limitations for the bottom slope have also been discussed and we found that the magnitude of the bottom slope must be of order $O(\varepsilon^{n+1})$ for an $n$'th-order shoaling problem with an arbitrary volume flux and $O(\varepsilon^{n})$ with a zero volume flux.

We found the same general tendencies in the results as Jonsson and Arneborg (1995): Increasing steepness reduces the conventional 'shoaling drop' in wave height for a zero volume flux. For this case and opposing currents the wave length decreases with decreasing depth, while it for strong following currents shows an increasing behaviour.

The deviations between the fourth- and sixth-order results are found to be small in general. The largest deviations occur as expected for steep waves and for opposing currents. In general the sixth-order wave length is larger than the fourth-order one. For opposing currents the sixth-order wave height is larger than the fourth-order prediction, and slightly smaller for following currents.

For pure waves, the results have been compared with the results of Sobey and Bando (1991) produced with a Fourier wave theory model, which is considered as practically exact. For waves of small steepness the solutions agree, but for waves of increasing steepness, the wave height is increasingly underestimated by the sixth-order theory. The sixth-order wave length predictions are almost identical with the results of Sobey and Bando. For the steepest wave investigated, the improvement by the added sixth-order terms appears clearly, however, the underestimation of the wave height is still pronounced.

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