A Proposed Spectral Form for Fully Developed Wind Seas Based on the Similarity Theory of S. A. Kitaigorodskii[1]

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Abstract. The data for the spectrums of fully developed seas obtained by *Moskowitz* [1964] for wind speeds from 20 to 40 knots (10.29 to 20.58 m/sec), are used to test the similarity hypothesis and the idea proposed by *Kitaigorodskii* [1961] that, when plotted in a certain dimensionless way, the power spectrums for all fully developed seas should be of the same shape. Over the important range of frequencies that define the total variance of the spectrum within a few per cent, the transformed plots yield a nondimensional spectral form that is nearly the same over this entire range of wind speeds within the present accuracies of the data. However, since slight variations of the wind speed have large effects on the location of this nondimensional spectral form, inaccuracies in the determination of the wind speed at sea allow for some latitude in the final choice of the form of the spectrum. Also, since the winds used to obtain the nondimensional form were measured at a height greater than 10 meters, the problem of relating the spectral form to a standard anemometer height arises. The variability introduced by this factor needs to be considered. The results, when errors in the wind speed, the sampling variability of the data, and the anemometer heights are considered, suggest a spectral form that is a compromise between the various proposed spectrums and that has features similar to many of them. A spectral form is recommended for tentative application to the problem of wave forecasting by spectral techniques. Improved wind speed measurements (taken at several elevations and averaged over longer time intervals) and better wave data (taken for longer time intervals and analyzed so as to better fit the procedures) are needed so that the form of the spectrums of fully developed wind seas and seas limited by either fetch or duration can be determined with even greater precision.

Introduction. Although dimensional analysis is a useful tool in many problems, *Neumann and Pierson* [1957][2] have, in the paper just cited and in other papers, cast a skeptical eye on its application to wave theory. Some recent work by Kitaigorodskii provided such a thorough discussion of the application of the theory of similarity to the problem of determining the spectrums of wind waves and such explicit conclusions that his proposals were tested with the data obtained by *Moskowitz* [1964]. Also, the upper bound developed by Phillips has apparently been verified over a certain range of frequencies by a number of individual spectral estimates [*Phillips*, 1963; *Longuet-Higgins et al.*, 1963], and it provides a part of the theory needed to obtain a possible spectral form.

As the content of this paper develops, it will be seen that such procedures provide a very powerful tool when combined with the statistical tests applied by *Moskowitz* [1964] for the precise determination of the spectrums over an important range of frequencies for fully developed wind-generated seas. Also, the present data will provide a spectral form that can serve for many practical applications.

The work of Kitaigorodskii. Kitaigorodskii [1961] makes the assumption that the dominant part of the power spectrum of a wind-generated sea—that part that determines the total variance within a few per cent— is a function of only four variables:

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[2] The references cited in this paper are listed on page 5202 of this issue of the *Journal of Geophysical Research.*

$$S(f) = F(f, g, U_+, X) \qquad (1)$$

In (1), f is the frequency, g is gravity, U_+ is the friction velocity, and X is the fetch. A transformation can readily be made to duration if it, instead of the fetch, is the limiting factor for a partially developed sea.

The Charnock-Ellison friction velocity [*Ellison*, 1956] might conceivably be the definitive parameter in this problem, but recent work by *Schmitz* [1962] raises a question even here. Since, in any case, U_+ is not measured at sea, we shall, instead of introducing an assumption as to the form of the drag coefficient to determine U_+, use the wind U as obtained by anemometer on the weather ships that took the wave observations. It must be kept in mind that this may introduce other factors. In particular, the fact that the anemometers were not 10 meters above the surface complicates the interpretation of the results.

With this change, (1) becomes (2). It will be assumed that the properties deduced in terms of U_+ will also hold for U.

$$S(f) = F(f, g, U, X) \qquad (2)$$

A dimensionless spectrum, as a function of a dimensionless frequency and a dimensionless fetch, is then obtained according to the following equations.

$$S'(f) = F_1'(f', X') \qquad (3)$$

where

$$S'(f) = S(f)g^3/U^5 \qquad (4)$$

$$f' = Uf/g \qquad (5)$$

$$X' = gX/U^2 \qquad (6)$$

Since the data of Moskowitz are for fully developed seas, the dependence on X' vanishes and

$$S'(f) = F(f') \qquad (7)$$

Kitaigorodskii has deduced a number of properties of the spectrum from this analysis. The wave height must follow a U^2 law, for example. The law that $f_{max}\, U/g =$ constant must also hold. If U_2 is greater than U_1 for fully developed seas, then

$$S(f, g, U_2) \geq S(f, g, U_1) \quad \text{for all} \quad f \qquad (8)$$

The only presently proposed spectral forms that agree with this analysis are the spectrums proposed by *Roll and Fischer* [1956], the asymptotic form proposed by *Phillips* [1958], and some functions given by *Bretschneider* [1963].

Application to new data. It is a simple matter to test these considerations by means of the data provided by Moskowitz. One way is to divide the tabulated values by the fifth power of the wind velocity and plot the results against the product of f and U^3. This can easily be converted to a dimensionless form by relabeling the scale after the appropriate powers of g are introduced and a consistent system of units is used. Figure 1 shows the results of this calculation for the range of frequencies where the data are believed reliable.

With the usual scatter in such data, the curves have been transformed to curves that are quite similar, one to the other. The curve for 20 knots (10.29 m/sec) comes quite close to the curve for 40 knots (20.58 m/sec), even though the area under the second curve is about sixteen times the area under the first curve in their untransformed form. For values of $f' > 0.18$, the agreement is more or less within the sampling variability of the data. The forward faces of the different curves do not fall exactly one on the other, but they are not too different. The scatter in the peak values exceeds the results expected if the wind speeds were exact and if the sampling variability of the averages was properly determined. The results might well be compared with some of the curves given by *Wiegel* [1963] and with the graphs of *Kitaigorodskii and Strekalov* [1962].

The reasons for this discrepancy are probably that the winds are not determined precisely enough and that a certain amount of blurring of the spectral values has occurred because of variations in the wind. It proved very difficult to

[3] The original data on which this work is based provided wave records calibrated in feet and winds measured to the nearest knot. Accuracies to the nearest foot in reading the wave records and to plus or minus one-half knot are the best to be expected. The results that follow are given in meters and meters per second (although other scales are provided as auxiliary scales in the illustrations). The number of significant figures carried is merely an indication of a conversion and not of the accuracy of the data.

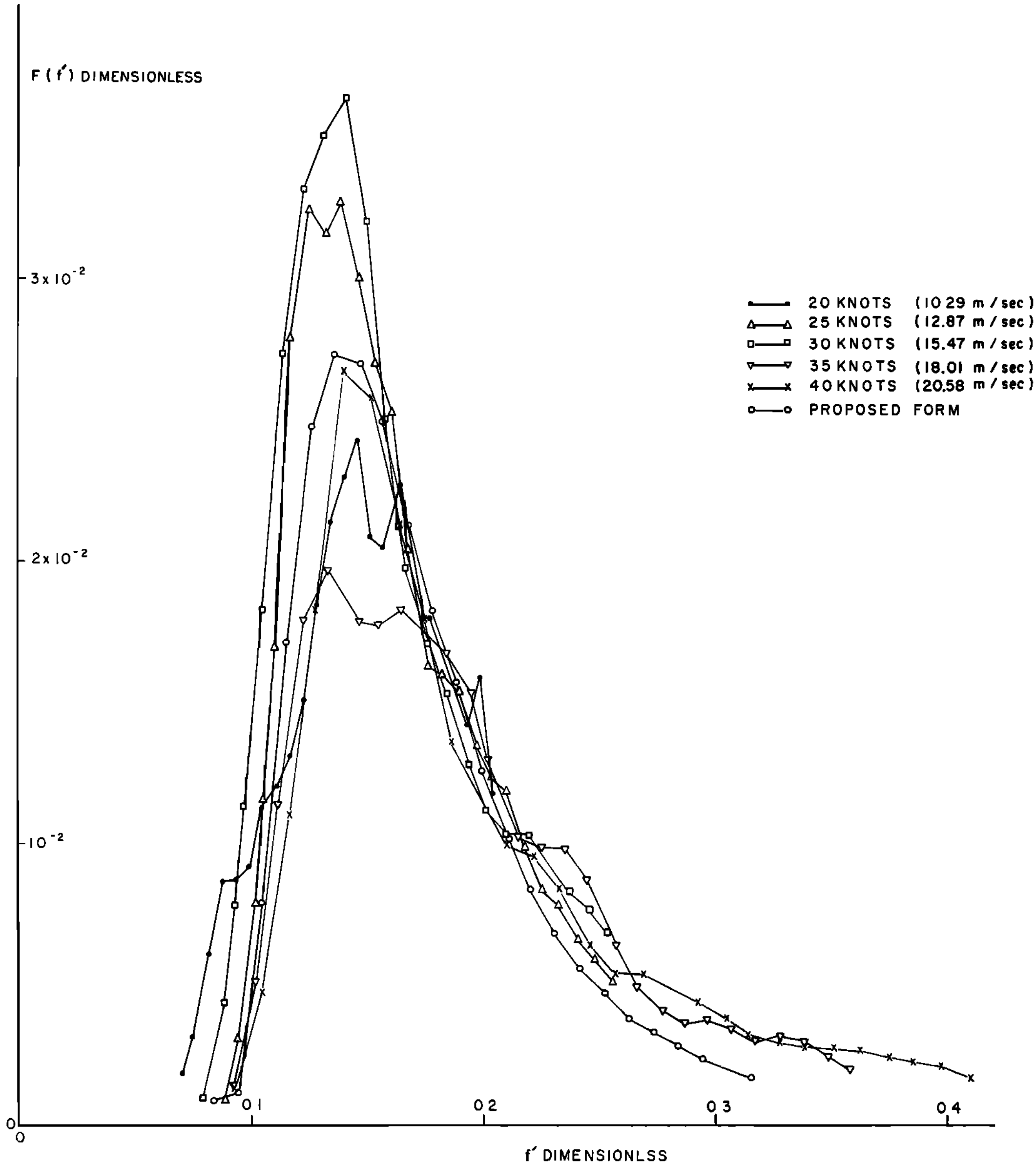

Fig. 1. Transformed plots of the spectrums for the five nominal wind speeds.

select the data for the subsets that constitute each of the average spectrums, as stronger winds before the observation and swell eliminated many spectrums for the low winds and wind shifts and rapid variations in wind speed eliminated many spectrums for the high winds. No effects of air-sea temperature difference could be detected, but they may have been present.

Location of peak and adjustment for wind speed variability. In the basic data, the peaks of the various spectrums were at the frequencies shown in Table 1. The value chosen for the 12.87 m/sec wind is the secondary of two peaks, and no value has been used for the 18.01 m/sec wind, as this averaged spectrum seems to be distorted in shape in comparison with the other four. With these qualifications, the product of the observed wind speed and the frequency at the

TABLE 1. Frequency at which the Peak Occurred in the Spectrum for Different Wind Speeds

Wind Speed U knots	Wind Speed U m/sec	Frequency at Peak (f), sec^{-1}	Product knot sec^{-1}	Product m sec^{-2}
20	10.29	25/180	500/180	257/180
25	12.87	19*/180	475/180	244/180
30	15.47	16/180	480/180	247/180
40	20.58	12/180	480/180	247/180
		[Average]	484/180	249/180

* Secondary peak.

peak is sensibly constant, varying from 1.40 to 1.36 with an average of 1.38 (m/sec²) The resolution of these spectrums is not great enough to refine this calculation, and so a value of 1.37 will be used. This in turn corresponds to a dimensionless frequency, $f_{max}' = 0.140$ in Figure 1.

At this frequency, or near it, the values of Sg^3/U^5 are shown in Table 2, along with their upper and lower confidence intervals as determined by 20 degrees of freedom per sample times the sample size. The confidence intervals do not overlap.

If the fifth root of the ratio of Sg^3/U^5 to the mean is computed, the factor value is obtained. Thus the fifth root of 2.41/2.75 is 0.974. If the nominal wind speed of 10.29 m/sec is multiplied by this factor the result is 10.04 m/sec. Similar values for the other winds are shown. Certainly, with the present accuracy of observed winds at sea, with the variable effects of stability, and in view of the fact that the winds reported for the spectrums that constitute the individual subsets were not exactly at the values shown for the nominal wind, one certainly could not argue with the assumption that the new wind value is as representative of the wind as the nominal wind value. This procedure quite evidently is highly sensitive to the correct value of the wind speed, as the fifth roots of the ratios involved lead to differences of at most 6% and usually 3 or 4%.

A longer averaging time for the wind observation, and perhaps measurements at two different heights, could improve the data in this respect. It would have been possible to define a slightly different friction velocity for each wind speed and obtain this same result. Before this could be accurately done, however, it would be necessary to reduce the sampling variability of the spectrums and improve the wind speed measurements.

The dimensionless spectrums for the corrected winds. If these new wind values are used instead of the nominal values to transform the spectrums to a nondimensional form, the result is as shown in Figure 2. These curves all come quite close together, as compared with the usual variability that occurs. Our procedure has suppressed a large amount of this variability.

At this point, we might remark that the results of Kitaigorodskii suggest that wave data should be taken in a special way. In order for plots such as this to be more easily comparable, the length of the records should be proportional to the wind speed. If, for example, a wave record of 20-min duration is taken when the wind speed is 10 m/sec, then when the winds are 20 m/sec the record ought to be 40 min long. In Figures 1 and 2, there are twice as many values on the transformed spectrum for a 10 m/sec nominal wind over a given interval of f' as there are for

TABLE 2. The Values of Sg^2/U^5 (Dimensionless) near the Peaks of the Different Averaged Spectrums, Their 90% Confidence Intervals, and the Derived Wind Speed Correction

Nominal Wind, m/sec	Average Reported Winds, m/sec	Sg^3/U^5 ($\times 10^{-2}$)	Upper ($\times 10^{-2}$)	Lower ($\times 10^{-2}$)	Factor	New Wind, m/sec
10.29	10.40	2.41	2.80	2.07	0.974	10.04
12.87	12.92	3.23	3.88	2.68	1.033	12.27
15.47	15.54	3.64	4.22	3.13	1.057	16.32
18.01	18.01	1.80	2.16	1.50	0.959	17.29
20.58	20.69	2.66	3.06	2.26	0.9955	20.48
Mean		2.75				

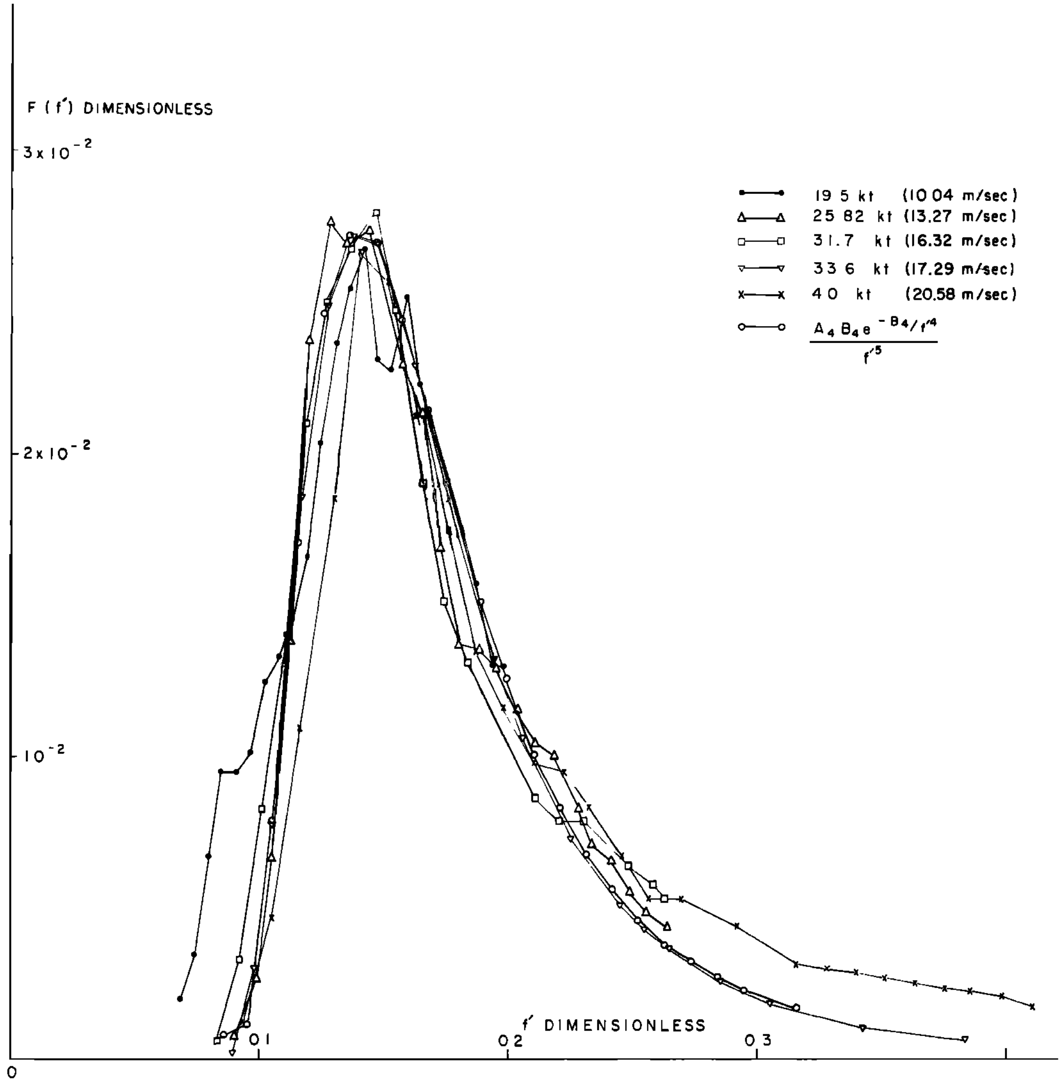

Fig. 2. Transformed plots of the spectrums for the new winds.

a 20-m/sec nominal wind. With the same Nyquist frequency, a record for a wind of 20 m/sec that is twice the length of a record for a wind of 10 m/sec can be analyzed at twice the resolution of the 10 m/sec record to provide a transformed spectrum with the same resolution as the 10 m/sec spectrum. A second important consequence, if the theory is correct, is that a carefully made series of observations offshore from a given land mass for approximately 10 m/sec winds could settle the whole problem of the effects of fetch and duration.

Moskowitz calculated each of his average spectrums at triple the original resolution. Nothing significant appeared to result except in the last spectrum. In it the forward face of the spectrum was shifted forward, with a compensating loss after the peak. If this higher-resolution spectrum is plotted instead of the one shown in Figure 2, it comes much closer to the spectrums for 12.87 and 15.47 m/sec. It might also be remarked here that the shape of the curve for 10.29 m/sec in Figure 1 (and for 10.04 m/sec in Figure 2) leads us to believe that there was a slight contribution left over at low frequencies from previously higher winds.

Curve fitting. There are many possible ways to proceed, given Figure 2. One way would be to correct the curve for 20.58 m/sec with the higher-resolution data and omit the low-frequency part of the curve for 10.29 m/sec. Then the five curves could be averaged across lines of constant value for the vertical scale so as to locate a central value for f'. A smoothed curve drawn through these points could then be read at a sufficiently high resolution for f' so that upon transforming the smooth curve back to a spectrum with appropriate dimensions a spectral form for any wind speed would be obtained. (The difference between a wind speed of 20.58 and 20.48 m/sec did not seem to warrant recalculating the curve for 20.58 m/sec.) This would be an objective way to determine $F(f')$. However, the calibration of the shipborne wave recorder leads to some questions concerning the high-frequency record of these curves, and it was thought more appropriate to try to find an analytical form for $F(f')$. This was done by assuming certain analytical forms with undetermined dimensionless parameters and forcing the analytical form to have a maximum of 2.75×10^{-2} at the value $f' = 0.140$.

It is evident from the work of Kitaigorodskii that the spectrum must be of a form such that f^5 occurs in the denominator. Since the spectrums are nested, there must be some other function of the dimensionless frequency f' involved. The form suggested by *Roll and Fischer* [1956] (equation 9) with arbitrary constants in the exponent and the coefficient is one possibility. Also, *Bretschneider* [1963] has suggested spectrums of the form shown by (11). Another possibility would be (10). The exponential term can be thought of as a high-pass filter acting on the limiting form proposed by *Phillips* [1958], and the question is which of these possible forms would give the best fit to the curves presented in Figure 2.

$$F(f') = (AB_2 e^{-B_2/f'^2})/f'^5 \tag{9}$$

$$F(f') = (AB_3 e^{-B_3/f'^3})/f'^5 \tag{10}$$

$$F(f') = (AB_4 e^{-B_4/f})/f'^5 \tag{11}$$

There is increasing theoretical support for the appearance of f'^5 in the denominator. The original theory of *Phillips* [1958] suggests that the waves, if high and if having a spectrum with $f'^{4.5}$ *or* f'^4 in the denominator, would break and dissipate their energy. The analysis of Kitaigorodskii supports this. *Hamada* [1964] has also shown that a form such as f'^6 would cause vortexes and dissipation by viscosity, again forcing a return to the f'^5 form.

These three possible forms are plotted in Figure 3, fitted so as to pass through the peak of the set of curves given in Figure 2. The form shown by (11) is the steepest, and the other forms open out as the exponent of f' in the denominator of the exponential term is decreased. Since these three forms are so very close together, the final proof as to which one is superior cannot now be given. However, for the purpose of forecasting the properties of the larger waves in a wind-generated sea, it would appear that the form given by (11) is slightly better. This form is shown by open circles in Figures 1 and 2. It comes very close to the two curves for the 12.87 (13.27) and the 15.47 (16.32) m/sec winds. Moreover, when the 20.58 m/sec curve is corrected for the effect of resolution, it comes close to this curve on the forward face of the spectrum. The steepness of the forward (low-frequency) face cannot be determined more sharply than this because of wind speed variations within the sample. Also, on the high-frequency side of the maximum out to a value of approximately 0.2 the fit is not too bad. Since the curve for 17.29 m/sec does not fit too well with all the other curves, and since it is believed that the curve for 10.04 m/sec has contributions from higher winds at low frequencies, this form seems slightly better than the other two.

For values on the f' scale greater than 0.27 the curves for 20.58 m/sec and 17.29 m/sec seem to rise, so that they agree better with the form given by (9). However, it is suspected that this, again, may be due to the calibration of the shipborne recorder. By increasing the noise level for the spectrums in the computations described by *Moskowitz et al.* [1962, 1963] this part of the curve could easily be decreased, whereas the changes near the peak would be relatively unimportant.

Analytical form of the spectrum. For these reasons, then, it would seem that a form given by (11) could serve the purposes of wave forecasting until more data of this nature, analyzed by these techniques, and better wind speed observations reduce the scatter of the results to a

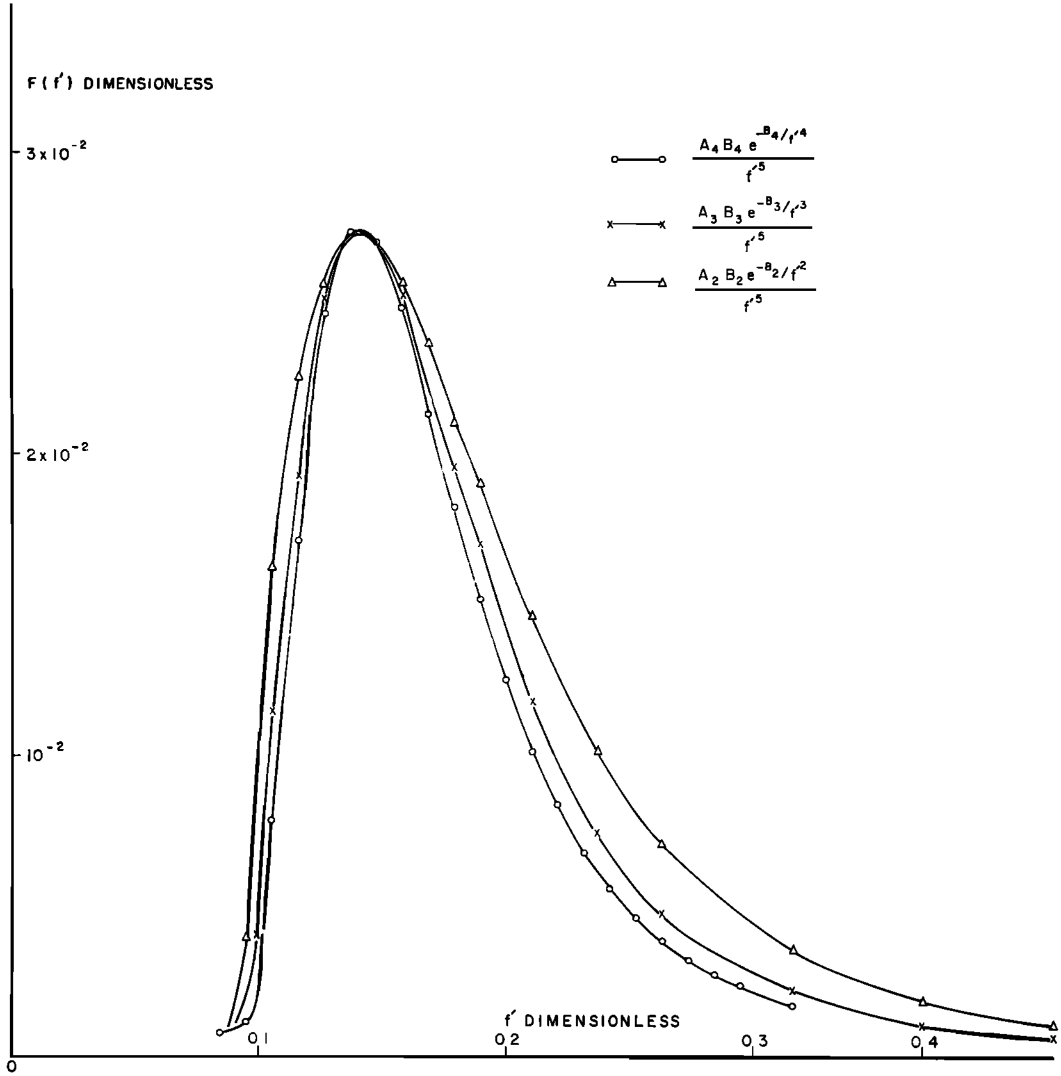

Fig. 3. Graphs of the possible spectral forms.

point where the nondimensional form can be determined with greater precision.

From the above nondimensional curves, fitted to the maximum point, one can obtain the spectrum given by (12). This spectral form is given in terms of ω, ω_0, and two dimensionless parameters, α and β. In this equation $\alpha = 8.10 \times 10^{-3}$, $\beta = 0.74$, and $\omega_0 = g/U$, where U is the wind speed reported by the weather ships. Any consistent set of units can be used.

$$S(\omega)\, d\omega = (\alpha g^2/\omega^5)e^{-\beta(\omega_0/\omega)^4}\, d\omega \tag{12}$$

The spectrum is defined only from zero to infinity, and the area under the spectrum is defined to be equal to the variance of the wave record. With improved data, the actual location of the peak may shift, and the value of α has considerable sampling variability along with its basic lack of precision due to inaccuracies in the wind speed.

Comparison with original data. Figure 4 shows the graphs of the original spectrums obtained by Moskowitz with their confidence limits and plots of (12) for the nominal wind speed and for the modified wind speed. For a wind of 10.04 m/sec the proposed spectral form

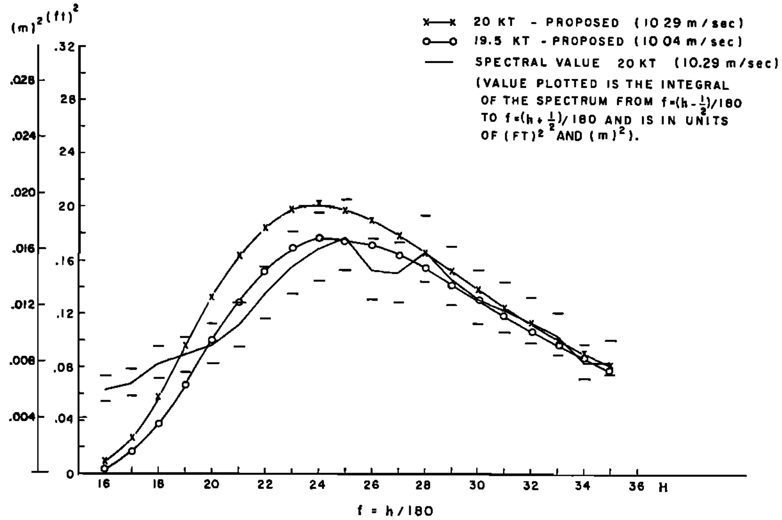

Fig. 4*a*. Comparison of proposed spectrum with original data for 10.29 m/sec winds.

fits the data obtained by Moskowitz over a substantial range of frequencies, with the exception of low frequencies, which, as mentioned before, may still have contributions left over from higher winds. For a wind speed of 13.27 m/sec the fit is again quite good over a substantial range of frequencies. For 16.32 m/sec the fit is not quite so good. The proposed form decreases a little too rapidly at frequencies of 10/180 and 11/180 to agree with the data. The curve is a little too high just past the peak and too low for high frequencies. The adjustment from 15.47 to 16.32 m/sec does, however, improve the fit, illustrating how sensitive these curves are to small changes in the wind speed. The curve for 18.01 (or 17.29) m/sec does not agree with the data at all. The curve for 20.58 m/sec agrees fairly well. The circles correspond to the values that would be used instead of the spectral values as originally obtained if three times the resolution had been used for these points. Here the forward face of the 40-knot spectrum agrees very well with the data and the fit is good all the way out to a frequency of 0.10. From there on, the observed values are higher than the theoretical values.

The agreement between the analytical form

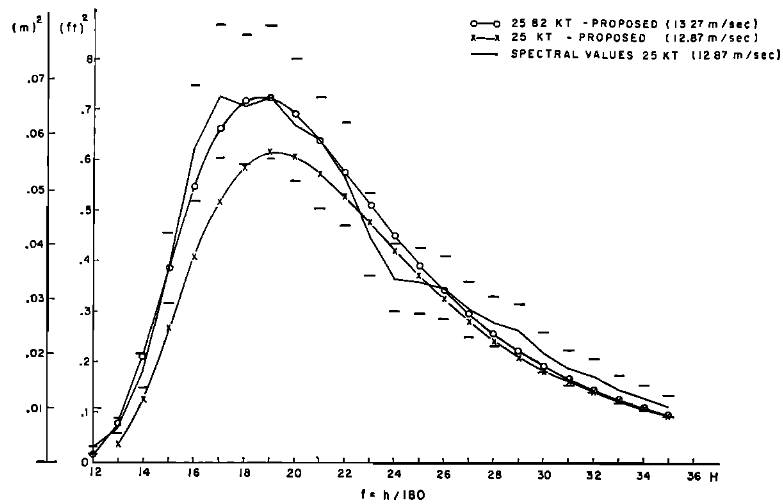

Fig. 4*b*. Comparison of proposed spectrum with original data for 12.87 m/sec winds.

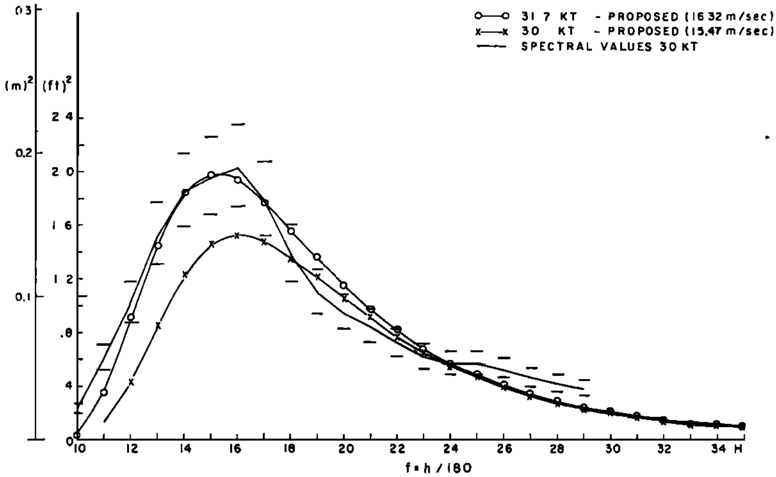

Fig. 4*c*. Comparison of proposed spectrum with original data for 15.47 m/sec winds.

and the averaged spectrums is quite good. It shows that a family of spectrums, obtained theoretically, can agree well with observed spectrums and that this family can depend on the one parameter observed by a ship, namely the wind speed at some elevation above the sea surface.

The effect of anemometer heights. The anemometers on the *Weather Explorer* and *Weather Reporter* were located as far as possible above the disturbing influence of the ships at an elevation of about 19.5 meters above the sea surface. The anemometer masts were located about three quarters of the way aft. The effect of the ship in disturbing the wind is not known, but indications are that it is not too great.

The wind speeds used by Moskowitz and in the equations just given are for this height. Results on wave spectrums by other authors are in general for wind speeds measured at lower heights, and the variation of wind speed with height needs to be considered in order to interpret these results completely.

The analysis of this problem requires a hypothesis concerning the nature of the stress on the sea surface and the variation of the drag

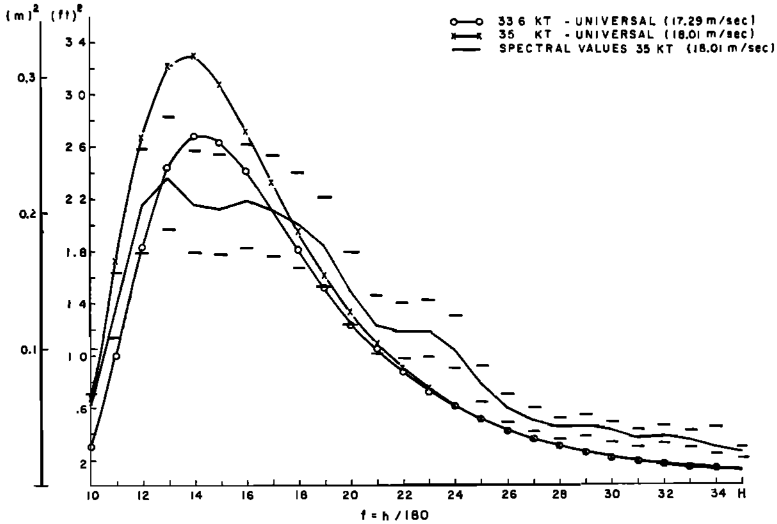

Fig. 4*d*. Comparison of proposed spectrum with original data for 18.01 m/sec winds.

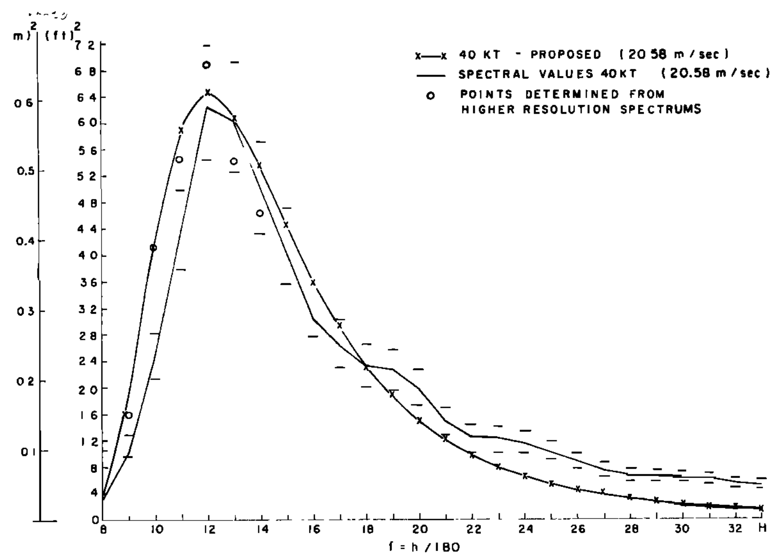

Fig. 4*e*. Comparison of proposed spectrum with original data for 20.58 m/sec winds.

coefficient as a function of wind speed. This subject is treated in the following paper. The spectral form given by (12) will describe the spectrum of a fully developed wind sea for a wind measured at 19.5 meters. This spectral form also yields wave properties in remarkable agreement with the results of other investigators if the variation of the wind with height is considered.

Conclusions. Within the present limitations of the data, the spectrums of fully developed wind-generated seas for winds measured at 19.5 meters are given very nearly by (12). The proposals of *Kitaigorodskii* [1961], when systematically applied to more data, should provide continuously refined and increasingly more accurate estimates of wind sea spectrums.

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