Finite-Amplitude Rotational Waves in Viscous Shear Flows

By W. R. C. Phillips

Growing finite-amplitude initially spanwise-independent two-dimensional rotational waves and their nonlinear interaction with unidirectional viscous shear flows of various strengths are considered. Both primary and secondary instabilities are studied, but only secondary instabilities are permitted to vary in the spanwise direction. A generalized Lagrangian-mean formulation is employed to describe wave-mean interactions, and a separate theory is constructed to account for the back effect of the developing mean flow on the wave field. Viscosity is seen to significantly complicate calculation of the back effect. The primary instability is seen to act as a platform for, and catalyst to, secondary instabilities. The analysis leads to an eigenvalue problem for the initial growth of the secondary instability, this being a generalization of the eigenvalue problem constructed by Craik for inviscid neutral waves. Two inviscid secondary instability mechanisms to longitudinal vortex form are observed: the first has as its basis the Craik–Leibovich type 2 mechanism. The second, which is as yet unproven, requires that both the wave and flow field distort in concert at all levels of shear. Both mechanisms excite exponential growth on a convective rather than diffusive scale in the presence of neutral waves, but growing waves alter that growth rate.

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1. Introduction

Fluid-flow phenomena whose motions exhibit both mean and fluctuating parts are commonplace in nature and engineering, ranging from water waves propagating on a shear current to Tollmien–Schlichting waves in a transitory boundary layer. Important in each case is an understanding of the nonlinear processes that couple the mean and fluctuating motions, and moreover the secondary and possible tertiary phenomena attributable to the nonlinear rectification of those oscillatory motions, e.g., modifications of the mean flow as a result of the waves and the back effect, if any, of those mean-flow modifications on the wave field.

Crucial to such studies are quantities that follow individual fluid particles, a task for which the Eulerian equations of mean motion are poorly suited. Indeed, Eulerian-mean vorticity as defined by Reynolds averaging has no simple conservative properties even when viscosity is ignored and thus acts to conceal the role played by nonlinear rectification (in its guise as Stokes drift) in vortex line deformation.

The quest for a more rational way to separate wave from mean flow and to define wave-mean interactions culminated, following much effort [1–6, and others], with the generalized Lagrangian-mean (GLM) equations of Andrews and McIntyre [7]. These equations describe the back effect of oscillatory disturbances upon the mean state and are exact provided the mapping between the true Lagrangian and the reference GLM remains invertible. Of course GLM still describes mean motions and is therefore conceptually equivalent to Reynolds averaging, but it describes Lagrangian aspects of the motion from a Eulerian framework and is consequently able to capture structural aspects of the flow.

Of interest in the present work are mean structures that arise in unidirectional viscous shear layers (of various strengths) owing to the presence of growing or neutral finite-amplitude rotational waves (that are initially spanwise independent), and specifically the evolution equations that describe the etiology of such structures. Also of interest are connections between the GLM approach and more conventional approaches to describing primary and secondary instability in bounded shear layers, with a view to determine which is more efficient.

1.1. Previous work

Mean structures that arise in wave-mean interactions of this ilk were first investigated by Craik [8] and Leibovich [9], who sought to model Langmuir circulations. These are organized convective motions that form in the surface layer of open bodies of water when winds of moderate strength blow over them. The motions take the form of longitudinal vortices that align with the wind and act at the free surface to concentrate flotsam and various organic films into clearly visible streaks or windrows. Craik and Leibovich considered $O(\epsilon)$ neutral irrotational waves interacting with an $O(\epsilon^2)$ unidirectional Eulerian mean shear flow. They found the interaction unstable to longitudinal vortex form via an instability now known as CL2, or Craik–Leibovich type 2 (see Section 4).

CL2 continues to operate in stronger shear. But although only minor modifications to the theory are required in $O(\epsilon)$ shear, that is not the case for $O(1)$ shear flows, where the back effect of the mean-flow modification upon the wave field must be explicitly calculated [10]. In essence, waves do not drive CL2 [11] but act through the pseudomomentum as a catalyst: This means that the magnitude of the mean-flow modification is bound not by the strength of the waves but by the magnitude of the preexisting vorticity in the initial state. With sufficiently strong preexisting vorticity, therefore, the mean-flow modification acts to distort the waves. Of course the detailed kinematics of the instability mechanism are less clear with $O(1)$ shear than with $O(\epsilon^2)$, although the seminal idea of the CL2 instability remains within the theory and for this reason Phillips et al. [12] denote the former CL2-$O(\epsilon^2)$ and the latter CL2-$O(1)$.

To construct an inviscid theory for $O(1)$ shear flows in the presence of $O(\epsilon)$ rotational neutral waves, Craik employed the GLM-equations and found the resulting eigenvalue problem for longitudinal vortices far more complicated than its counterpart for weaker shear; requiring *inter alia* a further differential equation to account for wave distortion. That notwithstanding, Craik was able to obtain definite results analytically to demonstrate the existence of longitudinal vortex instability when the spanwise spacing of the vortices is small; and this technique was extended to a different, wider class of flows, by Phillips and Shen [13], who show the ubiquity of this instability. Detailed numerical results by Phillips and Wu [14] and Phillips et al. [12] concur and further indicate that wave distortion acts (i) to diminish catalytic action for all but the shortest waves; and (ii) to suppress the instability markedly if the waves are sufficiently long. Furthermore, by comparison with the data of Gong et al. [15], Phillips et al. determine that CL2-$O(1)$ is physically realizable.

Questions then arise regarding the influence of viscosity and growing waves on the instability, and these set the stage for the present study in which, as a precursor to future numerical work, we construct the relevant eigenvalue problems.

1.2. Scope of the present work

We begin with a brief review of GLM (Section 2) and then specialize the GLM-equations to the problem of $O(\epsilon)$ growing (or decaying) waves interacting with a unidirectional viscous shear flow whose strength may range from $O(\epsilon^2)$ to $O(1)$. The waves are initially two dimensional and the shear flow is assumed composed of two parts (Section 3): one determined by the primary instability, which by definition can have no spanwise dependence, the other by a secondary instability, which is allowed to vary in the spanwise direction (cf. [16]). In Section 4 we consider secondary instabilities that may arise in such circumstances; two are observed: CL2 (Section 4.1) and a possibly new instability to longitudinal vortex form, in which both the mean velocity and wave field distort in concert (Section 4.2).

Wave distortion owing to the secondary flow arising in $O(1)$ shear is considered in Section 5.3 and the resulting correction to the pseudomomentum field in Section 5.4. The ensuing formulation is significantly more complicated than its inviscid counterpart in $O(1)$ shear, because the second-order Rayleigh–Craik equation and its algebraic accomplice (which together account for wave distortion in Craik's theory) are replaced by two ordinary differential equations, one of fourth order and the other of second. The resulting eigenvalue problems are discussed in Section 6.

Finally, to give the analysis physical basis, we view it from the context of two disparate physical problems: first, the growth of Langmuir cells beneath wind-driven growing waves, where the waves typically are $O(\epsilon)$ and the shear can range from $O(\epsilon^2)$ to $O(1)$ [17, 34]; and second, to compare our findings with well-established previous results, we look at plane Poiseuille flow, where the shear is typically $O(1)$.

2. The generalized Lagrangian-mean formulation

2.1. Background

Andrews and McIntyre's [7] generalized Lagrangian mean equations are an exact and very general Lagrangian-mean description of the back effect of oscillatory disturbances upon the mean state. The Lagrangian-mean velocity so described, however, is not the "mean following a single fluid particle," but rather the velocity field describing trajectories about which the fluctuating particle motions have zero mean, when *any* averaging process, be it temporal, spatial, ensemble, or other, is applied. To express ideas like "steady mean flow," an Eulerian description of the Lagrangian mean, with position $\mathbf{x}$ and time t as independent variables, is employed. Hence the GLM description is really a hybrid Eulerian–Lagrangian description of wave mean-flow interactions. Andrews and McIntyre emphasize that the equations are *exact* and thus valid for waves of all amplitudes, although for practical purposes they have so far been restricted to waves of small amplitude, measured by a dimensionless parameter ϵ, so that any displacement $\boldsymbol{\xi}$ from the mean trajectory is $O(\epsilon)$ compared to the wavelength of the wavefield.

To define an exact Lagrangian-mean operator $\langle\ \rangle^{\mathrm{L}},(\overline{\ })^{\mathrm{L}}$, corresponding to any given Eulerian-mean operator $\langle\ \rangle,(\overline{\ })$, necessitates defining with equal generality an exact, disturbance-associated particle displacement field $\boldsymbol{\xi}(\mathbf{x},t)$. For any scalar or tensor field, φ, say, of any rank, it is then possible to write

$$\langle \varphi(\mathbf{x},t)\rangle^{\mathrm{L}} = \langle \varphi^{\xi}(\mathbf{x},t)\rangle \qquad \text{where} \qquad \varphi^{\xi}(\mathbf{x},t) = \varphi(\mathbf{x}+\boldsymbol{\xi},t).$$

Then provided the mapping

$$\mathbf{x} \mapsto \mathbf{x}+\boldsymbol{\xi} \tag{2.1}$$

is invertible, there is, for any given $\mathbf{u}(\mathbf{x},t)$, a unique "related velocity field" $\mathbf{v}(\mathbf{x},t)$, such that when the point $\mathbf{x}$ moves with velocity $\mathbf{v}$ the point $\mathbf{x}+\boldsymbol{\xi}$ moves with the actual fluid velocity $\mathbf{u}^{\xi}$, as

$$(\partial/\partial t+\mathbf{v}\cdot\boldsymbol{\nabla})[\mathbf{x}+\xi] = \mathbf{u}^{\xi}. \tag{2.2}$$

Further, provided $\langle\boldsymbol{\xi}(\mathbf{x},t)\rangle = 0$ and $\langle\mathbf{v}(\mathbf{x},t)\rangle = \mathbf{v}(\mathbf{x},t)$, then $\mathbf{v}$ is the Lagrangian-mean velocity, $\overline{\mathbf{u}}^{\mathrm{L}}$, which is related to the Eulerian-mean velocity by the generalized Stokes drift $\overline{\mathbf{d}}$, as $\overline{\mathbf{u}}^{\mathrm{L}} = \overline{\mathbf{u}}+\overline{\mathbf{d}}$. So, in terms of the Lagrangian-mean material derivative, $\overline{D}^{\mathrm{L}} = \partial/\partial t+\overline{\mathbf{u}}^{\mathrm{L}}\cdot\boldsymbol{\nabla}$, Equation (2.2) becomes

$$\overline{D}^{\mathrm{L}}\boldsymbol{\xi} = \mathbf{u}^{l}, \tag{2.3}$$

where the Lagrangian disturbance velocity $\mathbf{u}^{l}$ is given by $\mathbf{u}^{l}(\mathbf{x},t) = \mathbf{u}^{\xi}-\overline{\mathbf{u}}^{\mathrm{L}}$, such that $\overline{\mathbf{u}}^{l} = 0$.

2.2. *The generalized Lagrangian-mean equations*

For homentropic flows of constant density ρ in a nonrotating reference frame, the GLM-momentum and continuity equations are

$$\overline{D}^{\mathrm{L}}\left(\overline{u}_i^{\mathrm{L}}-\overline{p}_i\right)+\overline{u}_{k,i}^{\mathrm{L}}\left(\overline{u}_k^{\mathrm{L}}-\overline{p}_k\right)+\pi_{,i} = \chi_i, \tag{2.4}$$

$$\pi = \frac{\overline{p}_{,i}}{\rho}+\overline{\Phi}_i^{\mathrm{L}}-\frac{1}{2}\langle u_j^{\xi}u_j^{\xi}\rangle$$

and

$$\overline{D}^{\mathrm{L}}\varrho+\varrho\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}^{\mathrm{L}} = 0. \tag{2.5}$$

Here repeated indices imply summation and commas denote partial differentiation.

Observe that the nonlinear forcing of the mean flow is expressed in terms of the mean of the vector wave property $\mathbf{p}$. The ensuing vector $\overline{\mathbf{p}} = \bar{p}_i(\mathbf{x}, t)$ is the pseudomomentum per unit mass, whose ith component is

$$\bar{p}_i = -\langle \xi_{j,i} u_j^l \rangle. \tag{2.6}$$

Note that the pseudomomentum or quasi-momentum [18] should not be confused with the pressure $\wp$. Further, Φ is the force potential per unit mass and χ is a function that allows for dissipative forces. In the present work Φ is zero and the contribution due to the viscous force $\nu \nabla^2 \mathbf{u}$ is

$$\chi_i = \nu \left[\bar{u}^{\mathrm{L}}_{i,kk} + \langle \xi_{j,i} u^l_{j,kk} \rangle \right]. \tag{2.7}$$

The density $\varrho(\mathbf{x}, t)$ of the GLM-flow $\overline{\mathbf{u}}^{\mathrm{L}}(\mathbf{x}, t)$ is defined to satisfy (2.5) and is connected to the actual fluid density $\rho^{\xi}(\mathbf{x}, t) = \rho(\mathbf{x} + \boldsymbol{\xi}, t)$ by

$$\varrho = \rho^{\xi} J, \qquad J \equiv \det\{\delta_{ij} + \xi_{i,j}\},$$

where J is the Jacobian of the mapping $\mathbf{x} \mapsto \mathbf{x} + \boldsymbol{\xi}$ and δ_{ij} is the Kronecker delta. Note that restricting attention to flows of constant density ρ does not usually give rise to constant ϱ; but it does necessitate that ϱ be a mean quantity, to wit $\langle \varrho(\mathbf{x}, t) \rangle = \varrho(\mathbf{x}, t)$, thereby allowing the mass conservation equation (2.5) to be written as

$$\overline{D}^{\mathrm{L}} J + J \boldsymbol{\nabla} \cdot (\overline{\mathbf{q}} + \overline{\mathbf{p}}) = 0. \tag{2.8}$$

At this point it is helpful to write (2.4) in a form akin to Navier–Stokes, and we do so by introducing the dependent variable $\bar{q}_i = \bar{u}_i^{\mathrm{L}} - \bar{p}_i$ while noting that

$$\bar{q}_j \left(\bar{q}_j + \bar{p}_j \right)_{,i} + \left(\bar{q}_j + \bar{p}_j \right) \bar{q}_{i,j} = -\left(\bar{q}_j + \bar{p}_j \right) \left(\bar{q}_{j,i} - \bar{q}_{i,j} \right) + \left(\left(\bar{q}_j + \bar{p}_j \right) \bar{q}_j \right)_{,i}$$

to find

$$\bar{q}_{i,t} + \bar{q}_j \bar{q}_{i,j} - \bar{p}_j \left(\bar{q}_{j,i} - \bar{q}_{i,j} \right) + \Pi_{,i} = \chi_i, \tag{2.9}$$

where

$$\Pi = \frac{1}{2} \bar{q}_j \bar{q}_j + \bar{p}_j \bar{q}_j + \pi.$$

We also introduce the vorticity-associated vector field $\mho = \nabla \times \bar{\mathbf{q}}$; then noting (2.5) and taking the curl of (2.9), yields

$$\mho_{i,t} + \left(\bar{q}_j + \bar{p}_j\right) \mho_{i,j} = \mho_j \left(\bar{q}_i + \bar{p}_i\right)_{,j} - \mho_i \left(\bar{q}_j + \bar{p}_j\right)_{,j} + \varepsilon_{ijk}\, \chi_{k,j}, \quad (2.10)$$

where ε_{ijk} is the alternating tensor.

2.3. Small amplitude waves

Various simplifications occur when dealing with incompressible, Boussinesq flows in which ϵ (see Section 3) is characteristic of the initial disturbance. First, the Jacobian takes the form [7]

$$J = 1 - \frac{1}{2}\langle \xi_j \xi_k \rangle_{,jk} + O(\epsilon^3) \quad (2.11)$$

while the generalized Stokes drift becomes

$$\bar{d}_i = \langle \xi_j \breve{u}_{i,j} \rangle + \frac{1}{2}\langle \xi_j \xi_k \rangle \bar{u}_{i,jk} + O(\epsilon^3). \quad (2.12)$$

Second because the Eulerian fluctuating velocity is $\breve{\mathbf{u}} = \mathbf{u}(\mathbf{x}, t) - \bar{\mathbf{u}}(\mathbf{x}, t)$, the small-amplitude Lagrangian velocity perturbation follows as

$$u_j^l = \breve{u}_j + \xi_k \bar{u}_{j,k} + O(\epsilon^2); \quad (2.13)$$

and finally the viscous contribution (2.7) simplifies noticeably, as we see in Section 3.2.

3. Imposed shear of specified strength and $O(\epsilon)$ waves

We apply the GLM formulation to a class of unidirectional shear flows that have imposed on them, or are unstable to, small-amplitude waves that are independent of spanwise direction; and of particular interest is the instability of the ensuing wave-mean interaction to longitudinal vortex form. Our intent in the first instance is to restrict only the slope of the waves but remain as general as possible with regard to the level of the imposed shear. In consequence the ensuing equations are relevant to a range of bounded and unbounded flows, but is behooves us to discuss them with regard to specific physical problems, viz. Langmuir circulations beneath growing wind-driven surface gravity waves and plane Poiseuille flow.

Consider then the interaction between a unidirectional shear flow with characteristic velocity $\mathscr{V}$ and two-dimensional straightcrested waves of wavelength λ that propagate in (or opposite to) the direction of the basic flow. The amplitude of the waves is assumed to grow from infinitesimal to finite, but we require their slope ϵ to satisfy $\epsilon < O(1)$ at all times. Orbital velocities are thus characterized by $\epsilon\mathscr{C}$, where $\mathscr{C}$ is a typical phase speed. We next suppose that the characteristic thickness of the shear layer is $\mathscr{L}$ and make variables dimensionless with respect to $\mathscr{L}$ and $\mathscr{C}$. Finally we write $\mathscr{V}/\mathscr{C} = O(\epsilon^s)$ and $\mathscr{L}/\lambda = O(\epsilon^\beta)$, where $s \geq 0$ while β is real and of either sign. Then the level of shear is $O(\epsilon^s)$ and in the event viscosity plays a role, the Reynolds number $R \equiv \mathscr{L}\mathscr{C}/\nu$. Finally we invoke space coordinates (x, y, z) and choose a reference frame that moves in the x-direction with the phase speed of the waves c_r^{w}.

We use uppercase letters to denote quantities pertaining to the primary flow, which by design is devoid of spanwise (y) dependence, and lowercase letters otherwise, while an overbar on the unscaled dimensionless variable denotes a streamwise average. Our unperturbed Eulerian shear flow in $[z_1, z_2]$ is then $\overline{\mathbf{U}}(z,t) + \mathbf{i}c_r^{\mathrm{w}} = \epsilon^s[U,0,0]$.

Envisage now an $O(\epsilon)$ wave field $\breve{\mathbf{U}}$ that interacts with the primary shear flow to excite streamwise-averaged spanwise-varying Eulerian velocity perturbations $\tilde{\mathbf{u}}$, whose strength relative to the primary shear flow is measured by the parameter Δ, and express the resulting flow field in GLM-variables. The outcome is the velocity-associated vector field $\bar{\mathbf{q}} = \overline{\mathbf{Q}} + \tilde{\mathbf{q}}$, which we expand as

$$\bar{\mathbf{q}}(y,z,t) \;=\; \epsilon^s\big\{\big[Q_1, 0, \epsilon^{2-s}Q_3\big] + \Delta\big[q_1, \epsilon^n q_2, \epsilon^n q_3\big] + \cdots\big\} \qquad (n \geq 0), \tag{3.1}$$

and an affiliated scalar field Π, which includes the pressure as $\epsilon^s[\mathscr{P}(x,z,t) + \Delta\wp(x,y,z,t) + \cdots]$. Note that the power n can have values other than zero and that n is related to s, as we see in Section 4.

In the first instance the waves produce $O(\epsilon^2)$ primary fields of pseudomomentum $\overline{\mathbf{P}}$ and Stokes drift $\overline{\mathbf{D}}$. So since the Eulerian and Lagrangian mean velocity fields are related through $\bar{\mathbf{q}} = \bar{\mathbf{u}} + \bar{\mathbf{d}} - \bar{\mathbf{p}}$, we see that $Q_3 = D_3 - P_3$, which explains the extra primary mean field component in (3.1) (in contrast to the primary Eulerian flow, which by design has only one component). Moreover the $O(\epsilon^s\Delta)$ axial velocity perturbation (owing to the interaction between the waves and mean flow) may in turn act to distort the wave field and produce an $O(\epsilon^{s+2}\Delta)$ spanwise-varying component of pseudomomentum [10]. So with no loss of generality we write $\bar{\mathbf{p}}$ or $\bar{\mathbf{d}}$ as $\overline{\mathbf{P}} + \tilde{\mathbf{p}}$, expand as

$$\bar{\mathbf{p}}(y,z,t) \;=\; \epsilon^2\{[P_1, 0, P_3] + \epsilon^s\Delta[p_1, \epsilon^n p_2, \epsilon^n p_3 + \cdots]\}, \tag{3.2}$$

and with (3.1) substitute into (2.8) and (2.9). But before doing so, it is instructive to first explore the kinematic limitations of GLM and determine how GLM is manifest in the presence of viscosity.

3.1. Conservation of mass

The validity and indeed strength of GLM stems from the mapping (2.1), but because (2.1) must remain invertible it is also, ironically, GLM's chief restriction. A second less problematic occurrence with GLM is that GLM flows are usually divergent. Both difficulties are evident kinematically from (2.8): to wit

$$-\overline{D}^{\mathrm{L}} \ln J = \epsilon^2 \frac{\partial D_3}{\partial z} + \epsilon^{s+2+n}\Delta\left(\frac{\partial d_2}{\partial y} + \frac{\partial d_3}{\partial z}\right). \qquad (3.3)$$

Observe that the mean field $\overline{\mathbf{q}}+\overline{\mathbf{p}}$ is divergence free to $O(\epsilon^2)$ only if $D_3 = 0$, which necessitates neutral waves, i.e., waves with a steady amplitude field [18, 19; see also Section 5]. It also requires the absence of critical layers, where D_3 (for monochromatic neutral waves) is unbounded [see (5.13) and (5.15)]. Of course singularities do not exist within the flow field at the critical layer; rather the Jacobian J is zero there, indicating that the mapping (2.1) is no longer invertible. Physically, critical layers are thin layers of fluid centered on levels at which the phase velocity of the disturbance is equal to the velocity of the basic flow; and the fact that $J = 0$ at critical layers means the averaging procedure breaks down there. Of course neutral waves give rise to streamlines that form closed "cats eyes" near critical layers, while streamlines due to marginally stable waves roll up [20]. In consequence $J = 0$ means that the averaging procedure gives simple results only for open nonfolded streamlines. Of course such restrictions need not negate the usefulness of GLM, but rather mean that we must restrict attention to flows, or at least regions of the flow, in which the streamlines are not folded.

Finally (3.3) notwithstanding, we are at liberty to write $\boldsymbol{\nabla}\cdot\overline{\mathbf{u}} = \boldsymbol{\nabla}\cdot(\overline{\mathbf{q}}+\overline{\mathbf{p}}-\overline{\mathbf{d}}) = 0$, and introduce the perturbation Stream function ψ as

$$q_2 = \frac{\partial \psi}{\partial z} + \epsilon^2(d_2 - p_2) \qquad \text{and} \qquad q_3 = -\frac{\partial \psi}{\partial y} + \epsilon^2(d_3 - p_3). \quad (3.4)$$

Thus for calculation purposes, at least for the class of problems under consideration, the effect of a divergent mean-flow field is minor.

3.2. The viscous contribution

Potentially daunting is the calculation of the Lagrangian-mean contribution to the viscous force. Leibovich [21] has considered this in the context of $O(\epsilon^2)$ shear in the presence of a wave field that is irrotational to $O(\epsilon)$ and found that χ_i reduces to $R^{-1}\{\nabla^2\bar{\mathbf{u}} + O(\epsilon^4\mathscr{L}^2/\lambda^2)\}$. Here we allow for rotational waves and all levels of shear with R constant. As it happens, again for the class of problems under consideration, the determination of χ_i is usually, but by no means always, straightforward.

We begin with a Taylor expansion of the bracketed portion of (2.7), to wit

$$\nabla^2\bar{u}_i + \langle \xi_j \nabla^2 u_{i,j} \rangle + \langle \xi_{j,i} \nabla^2 \breve{u}_j \rangle + \left\langle \xi_{j,i} \nabla^2 \left(\xi_k \bar{u}_{j,k} \right) \right\rangle + O(\epsilon^{2+s+\beta}); \quad (3.5)$$

then on replacing $\bar{\mathbf{u}}$ with $\bar{\mathbf{q}}$ (which we can do formally because $\bar{\mathbf{q}} = \bar{\mathbf{u}} + O(\epsilon^2)$), it is evident that the first—and usually dominant—term in (3.5) becomes

$$\nabla^2\bar{\mathbf{q}} = O\left(\epsilon^s\left[1, \Delta, \epsilon^{2-s}\right]\right).$$

Now although the second term reduces (since $\mathbf{u} = \bar{\mathbf{u}} + \breve{\mathbf{u}}$ and $\bar{\boldsymbol{\xi}} = 0$) to $\langle \xi_j (\nabla^2 \breve{u}_i)_{,j} \rangle$, the order of $\nabla^2\breve{\mathbf{u}}$ is determined by the rotational level of the waves, as

$$\nabla^2\breve{\mathbf{u}} = O(\epsilon^{1+m+2\beta});$$

here we have set $m = 0$ if the waves are rotational and $m = 1$ if they are irrotational to $O(\epsilon)$. In consequence the second and third terms in (3.5) are both $O(\epsilon^{2+m+2\beta})$ and are negligible relative to the first whenever $2 + m + 2\beta > s$. Lastly, the fourth term in (3.5) is

$$\langle \xi_{j,i} \nabla^2 \xi_k \bar{u}_{j,k} \rangle = O\left(\epsilon^{2+s}\left[\frac{\mathscr{L}}{\lambda}, 1, \frac{\lambda}{\mathscr{L}}\right]\right)$$

and is also negligible relative to the first term provided $\epsilon^2 < \mathscr{L}/\lambda < \epsilon^{-2}$. Indeed terms three and four may be ignored when the waves are irrotational for all admissible s, and for $s \le 1$ when the waves are rotational, provided $-\frac{1}{2} < \beta < 1$. But both terms must be retained when the waves are rotational and $s = 2$, assuming of course rotational waves exist at $s = 2$. Thus for completeness we write, for both rotational and irrotational waves, that

$$\chi_i = R^{-1}\left[\nabla^2\bar{q}_i + \bar{F}_i + \bar{G}_i + O(\epsilon^{2+s+2\beta})\right], \quad (3.6)$$

where

$$\bar{F}_i = \nabla^2\left(\bar{p}_i - \bar{d}_i\right) = \epsilon^2\left(F_i + \epsilon^{s+\delta n}\Delta\mathscr{F}_i + \cdots\right)$$

say, and

$$\overline{G}_i = \langle \xi_j \nabla^2 \breve{u}_{i,j} \rangle + \langle \xi_{j,i} \nabla^2 \breve{u}_j \rangle = \epsilon^{2+m+2\beta}\left(G_i + \epsilon^{s+\delta n} \Delta \mathscr{G}_i + \cdots \right),$$

where $\delta = 0$ for $i = 1$ and unity otherwise.

Of course (3.6) recovers Leibovich's result for irrotational waves with $s = 2$, because then $\bar{p}_i = \bar{d}_i + O(\epsilon^4)$ and $\overline{F}_i + \overline{G}_i = O(\epsilon^4 \mathscr{L}^2 / \lambda^2)$, but leads to a far more complicated result when $s = 2$ and the waves are rotational.

3.3. *The primary flow field*

It has long been known that finite amplitude waves act to distort the mean flow from its unperturbed state [22] and we term the unperturbed flow plus this $O(\epsilon^2)$ spanwise independent correction the primary flow field. We determine it by substituting (3.1) and (3.2) into (2.9). Then, on noting that the primary flow field must identically satisfy (2.9) and because Π_1 is here equal to the mean streamwise pressure gradient, the x-momentum equation takes the form

$$\frac{\partial Q_1}{\partial t} + \epsilon^2 D_3 \frac{\partial Q_1}{\partial z} + \frac{\partial \mathscr{P}}{\partial x} = R^{-1}\left[\frac{\partial^2 Q_1}{\partial z^2} + \epsilon^{2-s} F_1 + \epsilon^{2-s+m+2\beta} G_1 \right], \quad (3.7)$$

while the z-momentum equation becomes

$$\frac{\partial Q_3}{\partial t} + \epsilon^2 \frac{1}{2} \frac{\partial Q_3^2}{\partial z} - \epsilon^s P_1 \frac{\partial Q_1}{\partial z} + \epsilon^{s-2} \frac{\partial \Pi}{\partial z} = R^{-1}\left[\frac{\partial^2 Q_3}{\partial z^2} + F_3 + \epsilon^{m+2\beta} G_3 \right]. \quad (3.8)$$

Observe that although the Stokes drift does not explicitly appear in (2.9), it is evident from (3.3) and (3.7) that D_3, at least, plays an important role. To wit, it is D_3 that acts to distort the mean flow from its unperturbed viscous form, a role associated in Eulerian formulations with the more familiar Reynolds stress.

Of course in the absence of waves and with homogeneous Dirichlet boundary conditions, (3.7) describes plane Poiseuille flow; alternatively, in the absence of a pressure gradient and with Neumann boundary conditions (3.7) reduces to a stress Rayleigh problem with relevance to wind-driven mean flows. Furthermore, because the wavespeed can exceed $U_{\max}$ in both Poiseuille flow [23] and (in the water) in wind-driven flows [17, 34], critical layers can be avoided.

3.4. The secondary flow field

Although by definition the secondary flow field varies in the spanwise direction, it is unwise to assume the total contribution of this component is spanwise dependent. Indeed, a consequence of the nonlinear term in (3.9) is that the spanwise-dependent portion can act to support a spanwise-independent flow, which well exceeds the $O(\epsilon^2)$ modification discussed in Section 3.3, thus vastly altering the base flow defined by (3.7) [24–26].

To determine the secondary flow we again substitute (3.1) and (3.2) into (2.9), but this time subtract (3.7), which leads to the $O(\epsilon^s\Delta)$ streamwise evolution equation for q_1,

$$\frac{\partial q_1}{\partial t} + \epsilon^{s+n}\Delta\left(q_2\frac{\partial q_1}{\partial y} + q_3\frac{\partial q_1}{\partial z}\right) + \epsilon^2 D_3\frac{\partial q_1}{\partial z} + \epsilon^{s+2+n}\Delta\left(p_2\frac{\partial q_1}{\partial y} + p_3\frac{\partial q_1}{\partial z}\right)$$
$$+ \epsilon^{s+n} q_3\frac{\partial Q_1}{\partial z} + \epsilon^{s+2+n} p_3\frac{\partial Q_1}{\partial z} = R^{-1}\left(\nabla^2 q_1 + \epsilon^2\mathscr{F}_1 + \epsilon^{2+m+2\beta}\mathscr{G}_1\right), \tag{3.9}$$

while the same expansions and (2.10) yield the $O(\epsilon^{s+n}\Delta)$ streamwise component of the vorticity-associated vector field,

$$\frac{\partial \mho_1}{\partial t} + \epsilon^{s+n}\Delta\left(\frac{\partial \mho_1 q_2}{\partial y} + \frac{\partial \mho_1 q_3}{\partial z}\right) + \epsilon^{s+2+n}\Delta\left(\frac{\partial \mho_1 p_2}{\partial y} + \frac{\partial \mho_1 p_3}{\partial z}\right)$$
$$+ \epsilon^2\frac{\partial}{\partial z}(\mho_1 D_3) + \epsilon^{2-n}\frac{\partial q_1}{\partial y}\frac{\partial P_1}{\partial z} - \epsilon^{s+2-n}\frac{\partial Q_1}{\partial z}\frac{\partial p_1}{\partial y}$$
$$+ \epsilon^{s+2-n}\Delta\left\{\frac{\partial q_1}{\partial y}\frac{\partial p_1}{\partial z} - \frac{\partial q_1}{\partial z}\frac{\partial p_1}{\partial y}\right\}$$
$$= R^{-1}\left(\nabla^2\mho_1 + \epsilon^2\mathscr{H}_1 + \epsilon^{2+m+2\beta}\mathscr{I}_1\right), \tag{3.10}$$

where

$$\mho_1 = \frac{\partial q_3}{\partial y} - \frac{\partial q_2}{\partial z}, \qquad \mathscr{H}_1 = \varepsilon_{1jk}\mathscr{F}_{k,j}, \qquad \mathscr{I}_1 = \varepsilon_{1jk}\mathscr{G}_{k,j}.$$

Finally in terms of more familiar Eulerian variables, the streamwise velocity perturbation is $u_1 = q_1 + \epsilon^2(p_1 - d_1)$, while the streamwise component of vorticity is

$$\Omega_1 = \mho_1 + \epsilon^2\frac{\partial(d_2 - p_2)}{\partial z} - \epsilon^2\frac{\partial(d_3 - p_3)}{\partial y} = -\nabla^2\psi + O(\epsilon^2). \tag{3.11}$$

These indicate that we may identify q_1 with u_1, and $\mho_1$ with Ω_1, only if the appropriate components of wave distortion are negligible.

But the set (3.9), (3.10), and (3.11) is incomplete without knowledge of the pseudomomentum and in turn the wavefield; we determine the former in Section 5, but first it behooves us to discuss situations in which q_1 and $\mho_1$ grow with time.

4. Secondary nonlinear instabilities

Our objective is to elicit secondary instabilities that lead to the growth of q_1 and $\mho_1$ with time and we begin with the premise that likely instabilities occur when (3.9) and (3.10) are coupled. Two scenarios, those for which $\partial P_1/\partial z$ is generally nonzero and for which $\partial P_1/\partial z \approx 0$, are evident. Crucial in both instances are nonlinearities owing to the waves interacting both with themselves and the shear flow; measures of these nonlinearities are given by the generalized Stokes drift and the pseudomomentum discussed in Section 5.4.

4.1. Case (i): $\partial P_1/\partial z \neq 0$

When $\partial P_1/\partial z \neq 0$, Equations (3.9) and (3.10) are coupled via $q_3 \partial Q_1/\partial z$ and $\partial q_1/\partial y \, \partial P_1/\partial z$, so to explore such coupling we require $n = (2-s)/2$ and rescale time as $\tau = \epsilon^{(s+2)/2} t$. Then provided $\mathscr{L}/\lambda > O(\epsilon^{-1/2})$

$$\frac{\partial q_1}{\partial \tau} + \Delta\left(q_2 \frac{\partial q_1}{\partial y} + q_3 \frac{\partial q_1}{\partial z}\right) + \epsilon^{(2-s)/2} D_3 \frac{\partial q_1}{\partial z} + q_3 \frac{\partial Q_1}{\partial z}$$
$$= \epsilon^{-(s+2)/2} R^{-1} \nabla^2 q_1 + O(\epsilon^{(2-s)/2} R^{-1}) \tag{4.1a}$$

and

$$\frac{\partial \mho_1}{\partial \tau} + \Delta\left(\frac{\partial \mho_1 q_2}{\partial y} + \frac{\partial \mho_1 q_3}{\partial z}\right) + \epsilon^{(2-s)/2} \frac{\partial}{\partial z}(\mho_1 D_3) + \frac{\partial q_1}{\partial y} \frac{\partial P_1}{\partial z}$$
$$- \epsilon^s \frac{\partial Q_1}{\partial z} \frac{\partial p_1}{\partial y} + \epsilon^s \Delta\left\{\frac{\partial q_1}{\partial y} \frac{\partial p_1}{\partial z} - \frac{\partial q_1}{\partial z} \frac{\partial p_1}{\partial y}\right\}$$
$$= \epsilon^{-(s+2)/2} R^{-1} \nabla^2 \mho_1 + O(\epsilon^{(2-s)/2} R^{-1}). \tag{4.1b}$$

Note that because n varies with s it is evident from (3.1) that transverse and axial velocity perturbations may differ in order. Accordingly, wave distortion may be ignored for shear of $O(\epsilon)$ or less but plays an important role through p_1 at $O(1)$, where a further equation (see Section 5) must enter to complete the set (4.1). That notwithstanding, wave distortion in the y- and z-directions is $O(\epsilon^{3+s/2}\Delta)$ and may be neglected for all $s \in [0,2]$, allowing us to write

$$\frac{\partial q_2}{\partial y} + \frac{\partial q_3}{\partial z} = 0 \qquad \text{and thus} \qquad \mho_1 = -\nabla^2 \psi. \tag{4.1c}$$

In the presence of irrotational neutral ($D_3 = 0$) waves and $O(\epsilon^2)$ shear, P_1 reduces to D_1 and (4.1) reduce to the set of equations studied by Craik [8] and Leibovich [9], who determined that, subject to homogeneous Dirichlet or Neumann boundary conditions, q_1 and $\mho_1$ can grow exponentially fast. The resulting instability is known as Craik–Leibovich type 2 or CL2, to which, for reasons discussed in Section 1, we append (usually) the level of shear. To occur, CL2-$O(\epsilon^2)$ requires the presence of a wavy disturbance having a sheared pseudomomentum, together with preexisting vorticity imparting a Eulerian-mean shear in the same sense as the pseudomomentum. Here a kinematic description of (an inviscid flow subject to) the instability is possible: viz. that the Stokes drift gradient causes mean vortex lines (which move with the fluid) to tilt streamwise wherever the Eulerian-mean shear is laterally distorted, giving rise to a longitudinal component of vorticity and ultimately vortices that grow exponentially fast.

CL2 is thus an inviscid instability, although the $O(\epsilon^{(s+2)/2})$ growth rate predicted by inviscid theory will be annihilated by viscous damping unless $R \geq O(\epsilon^{-(s+2)/2})$. Craik and Leibovich did not study growing waves, but as is evident from (4.1) such waves have the greatest influence on the instability in weak (i.e., $s = 2$) shear; of course CL2 remains the underlying instability mechanism, but because D_3 is a function of time, details of the secondary flow and its growth rate will doubtless be affected. It is shear currents of this order that most commonly occur in the open ocean and it would seem that growing waves, due perhaps to a freshening breeze, may play an important role in the formation of Langmuir circulations hitherto absent, as in the observations reported by Smith [27].

Exponential growth can also occur when $s = 1$ or $s = 0$ [10]. However, although requirements for instability are as above for CL2-$O(\epsilon)$, the Craik–Phillips–Shen criterion [10, 13] must be satisfied to excite CL2-$O(1)$: viz. that from the reference frame of the wave, and in the direction of increasing mean flow, the gradient of the mean flow (normalized by the mean flow) must exceed the gradient of the wave amplitude (normalized by the wave amplitude). Phillips and Shen [13] have further shown that CL2-$O(1)$ is ubiquitous to a wide range of physically occurring bounded and unbounded flows; and, by comparison with the data of Gong et al. [15], Phillips et al. [12] have determined that CL2-$O(1)$ is physically realizable. Such knowledge begs the question whether the CL2-$O(1)$ instability has a role in the well-known secondary instability studied by Orszag and Patera [16]. Recall that both are catalyzed by two-dimensional finite-amplitude waves; both require concurrent stretching and tilting of vortex lines that lead to longitudinal vortices, which grow exponentially fast on a convective scale; and finally, both are ubiquitous. Indeed, we might well infer that CL2-$O(1)$ and Orszag–Patera are identical instabilities viewed from different reference frames. But not quite yet: Orszag and Patera determine that centrifugal effects play little role in their instability, whereas CL2-$O(\epsilon^2)$, for example, is formally equivalent—albiet in an averaged sense for an $O(\epsilon^2)$ mean curvature, not for an $O(\epsilon)$ local curvature—to the Taylor–Görtler instability [10]. So before making the above inference we must first dispel the notion that CL2-$O(1)$ too is centrifugal.

This we do by considering the root mean square kinetic energy K of each of the Fourier modes in the expansion

$$\mathbf{u}(\mathbf{x},t) = \sum_{|m| \le M} \sum_{|n| \le N} \mathrm{e}^{i(m\alpha^* x + nl^* y)} \mathbf{U}_{m,n}(m\alpha^*, nl^*, z, t)$$

so that

$$K(m\alpha^*, nl^*) = \left\{ \int_{z_1}^{z_2} \left[U_{m,n}^2 + V_{m,n}^2 + W_{m,n}^2 \right] dz \right\}^{1/2},$$

where α^* and l^* are the fundamental wavenumbers in the x- and y-directions respectively. Then in view of (5.1) and (5.2) it follows that because, for example, $\mathbf{U}_{0,1} = O(\epsilon^s\Delta, \epsilon^{s+1}\Delta)$, then

$$K(0,l^*) = O(\epsilon^s\Delta, \epsilon^{s+1}\Delta), \qquad K(\alpha^*,0) = O(\epsilon),$$

$$K(\alpha^*,l^*) = O(\epsilon^{s+1}\Delta).$$

Thus, in $O(\epsilon^2)$ shear, the greatest kinetic energy is to be found in the α^* and higher-order modes, as would be expected with a centrifugal instability. But because the extent of the mean-flow modification in CL2 is bound by the level of shear and not by the strength of the waves, the measure Δ may exceed ϵ; indeed a useful estimate in the fully nonlinear state is that $\Delta \equiv \epsilon^{s+1/2}$. In consequence $K(0,l^*)$ dominates when $s = 0$; and because $K(0,l^*)$ comprises modes that are streamwise independent, this form of the instability is not likely to be centrifugal. Thus the Orszag–Patera and CL2-$O(1)$ instabilities have much in common and may well be related.

Hence we may conclude that CL2 is a robust instability whose details vary according to the level of shear and type of waves (i.e., growing or neutral) and that CL2-$O(1)$ and the Orszag–Patera instability have much in common.

4.2. *Case* (ii): $\partial P_1 / \partial z \approx 0$

Consider now the case for which $\partial P_1 / \partial z$ is in general zero. Here (3.9) and (3.10) are indirectly coupled through $q_3 \partial Q_1 / \partial z$ and $\partial Q_1 / \partial z \, \partial p_1 / \partial y$, via the distortion of pseudomomentum by q_1. We thus set $n = 1$ for all s and $\tau = \epsilon^{s+1} t$ giving, for $\mathscr{L} / \lambda > O(\epsilon^{-1/2})$,

$$\frac{\partial q_1}{\partial \tau} + \Delta \left(q_2 \frac{\partial q_1}{\partial y} + q_3 \frac{\partial q_1}{\partial z} \right) + \epsilon^{1-s} D_3 \frac{\partial q_1}{\partial z} + q_3 \frac{\partial Q_1}{\partial z} = \epsilon^{-s-1} R^{-1} \nabla^2 q_1 + O(\epsilon^{1-s} R^{-1}) \tag{4.2a}$$

$$\frac{\partial \mho_1}{\partial \tau} + \Delta \left(\frac{\partial \mho_1 q_2}{\partial y} + \frac{\partial \mho_1 q_3}{\partial z} \right) + \epsilon^{1-s} \frac{\partial}{\partial z} (\mho_1 D_3) - \frac{\partial Q_1}{\partial z} \frac{\partial p_1}{\partial y} + \Delta \left\{ \frac{\partial q_1}{\partial y} \frac{\partial p_1}{\partial z} - \frac{\partial q_1}{\partial z} \frac{\partial p_1}{\partial y} \right\} = \epsilon^{-s-1} R^{-1} \nabla^2 \mho_1 + O(\epsilon^{1-s} R^{-1}) \tag{4.2b}$$

along with (4.2c). The set (4.2) is reminiscent of equations describing the Benney–Lin [28] and Craik–Leibovich type 1 [29, 33] instabilities, but these instabilities are fundamentally different, because each assumes an imposed spanwise-periodic wave field and initially grows algebraically in time. Here, the imposed wave field is two dimensional and the equations are coupled not through the dependent variables but rather through wave distortion: that is, the velocity field and wave field distort in concert. In the process, mean vortex lines associated with the primary flow are distorted spanwise and produce streamwise vorticity that may grow, at least initially (provided $D_3 = 0$), exponentially fast. Moreover, the instability can occur for all s if $D_3 = 0$, but is restricted to moderate $O(\epsilon)$ or strong $O(1)$ shear if $D_3 \neq 0$. Finally although the instability is inviscid it is subject to annihilation by viscosity unless $R \geq O(\epsilon^{-s-1})$. Of course (as with the $s = 0$ case in Section 4.1) a further equation must enter to complete the set (4.2) and we proceed to derive such in Section 5.

But we have as yet no proof that this instability exists: scenarios in which $\partial P_1 / \partial z \simeq 0$ require a unique relationship between the wave field and the mean flow. In the case of monochromatic waves, for example, that relationship must not only (in the case of inviscid flow) satisfy the Rayleigh equation but must also ensure that the right-hand side of (5.14) is constant. In short, admissible wave and mean fields must first be determined as eigensolutions of (5.4) and (5.14). This is a nontrivial but tractable problem numerically, although of course the existence of such eigensolutions is no guarantee that all, or indeed any, are unstable to longitudinal vortex form.

5. Monochromatic wave fields

To close (4.1) or (4.2), we must specify the wave field and determine the degree to which it distorts. To help fix ideas we consider monochromatic waves that if distorted do so in a spanwise periodic manner as, in Eulerian variables,

$$\begin{aligned}\breve{\mathbf{u}} = {} & \epsilon \operatorname{Re}\{g_w(t)\mathrm{e}^{\mathrm{i}\alpha x}[\phi'(z), 0, -\mathrm{i}\alpha\phi(z)]\} \\ & + \epsilon^{s+1}\Delta \operatorname{Re}\{g_w(t)\, g_v(t)\mathrm{e}^{\mathrm{i}\alpha x}[\mathscr{U}_1(z)\cos ly, \mathscr{U}_2(z)\sin ly, \mathscr{U}_3(z)\cos ly]\} \\ & + O(\epsilon^{s+2}, \epsilon^{s}\Delta^2). \end{aligned} \tag{5.1}$$

During the initial stage of the instability, and so long as Δ is small enough that linearization with respect to it yields a good approximation of the equations governing the spanwise instability, it follows that $g_v(t) \sim \mathrm{e}^{\sigma^v t}$; accordingly $g_w(t) \sim \mathrm{e}^{\sigma^w t}$. More specifically, while the $O(\epsilon)$ waves are allowed to grow or decay as αc_i^w, the $O(\epsilon^s\Delta)$ axial velocity perturbation grows (or decays) as αc_i^v with phase speed c_r^v; so

$$\sigma^w = \alpha(c_i^w - \mathrm{i}c_r^w) \qquad \text{and} \qquad \sigma^v = \alpha(c_i^v - \mathrm{i}c_r^v).$$

Then $\phi(z)$, c_i^w, and α in (5.1) denote the eigenfunction, eigenvalue, and wavenumber of the primary wave field, which satisfy the Orr–Sommerfeld equation (5.4).

In due course the interaction of this x-periodic wave field with the mean shear gives rise to streamwise-averaged perturbations (to the mean flow) having both spanwise-independent and spanwise-periodic perturbations. When $s = 0$ the latter take the form [cf. (3.1)]

$$\tilde{\mathbf{u}} = \Delta \operatorname{Re}\Big\{\mathrm{e}^{\sigma^v t}\mathrm{e}^{\mathrm{i}ly}\Big[\hat{u}(z), -\epsilon\mathrm{e}^{\alpha c_i^w t}\mathrm{i}\hat{v}(z), \epsilon\mathrm{e}^{\alpha c_i^w t}\hat{w}(z)\Big]\Big\} \tag{5.2}$$

(such that $l\hat{v} + \hat{w}' = 0$ to satisfy continuity); observe that while the axial flow grows as $\mathrm{e}^{\alpha c_i^v t}$, the transverse flow grows as $\mathrm{e}^{\alpha(c_i^v + c_i^w)t}$.

Finally, the $\mathscr{U}_j$ components in (5.1) derive from modifications to the $O(\epsilon)$ wave field by the $O(\epsilon^s\Delta)$ spanwise-periodic component of $\tilde{\mathbf{u}}$, giving rise to a spanwise-periodic variation in pseudomomentum, $\tilde{p}_1$. While in principle this can occur for all s (see Section 4.2), it is of major importance only when $s = 0$. But because the GLM-formulation provides no direct means of evaluating wave distortion a separate examination of the wave field is necessary; we do so in Section 5.3 and then determine $\tilde{p}_1$ in Section 5.4.

5.1. The primary instability initially

The primary velocity associated field Q_1 is described by (3.7) with (5.1) and (5.13). Then with the base flow U—be it Poiseuille flow or say a wind-driven boundary layer of finite extent [30]—assumed known in the absence of waves, we decompose Q_1 as $\epsilon^s Q_1 = \epsilon^s U + \epsilon^2 Q_1^P$, and seek the spanwise-independent perturbation $Q_1^P(z,t)$. The $O(\epsilon^2)$ correction to the base flow is then given by (cf. [29, Sect. 9.1])

$$\left(\frac{\partial}{\partial t} - R^{-1}\frac{\partial^2}{\partial z^2}\right)Q_1^P = -\epsilon^s \frac{1}{2} \mathrm{e}^{2\alpha c_i^w t}\left(\frac{\alpha c_i^w |\phi|^2}{U^2 + c_i^{w2}}\right)' U' + R^{-1}\left(F_1 + \epsilon^{m+2\beta} G_1\right), \tag{5.3}$$

where prime denotes d/dz and the eigenfunction $\phi(z)$ is a solution to the Orr–Sommerfeld equation

$$(U - \mathrm{i}c_i^w)(\phi'' - \alpha^2\phi) - U''\phi = \frac{1}{\mathrm{i}\alpha R}(\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi), \tag{5.4}$$

which follows by eliminating pressure from the $O(\epsilon)$ x- and z-momentum equations. Note that although the calculation of (5.3) is straightforward provided $\beta \geq \frac{1}{2}$, that is not the case for other β where computation of G_1 cannot be avoided. Indeed, in such instances it is easier to calculate the $O(\epsilon^2)$ Eulerian correction, U^P say, and deduce Q_1^P as $Q_1^P = U^P + D_1 - P_1$.

5.2. The secondary instability initially

In the presence of neutral or almost neutral waves, and on noting (5.12), both (4.1) and (4.2) take the generic form

$$\left[D^2 - l^2 - \sigma_1^v\right]\hat{u} = R^v Q_1' \hat{w}, \tag{5.5a}$$

$$\left[D^2 - l^2 - \sigma_1^v\right]\left[D^2 - l^2\right]\hat{w} = \begin{cases} -R^v l^2\left[P_1^{0\prime}\hat{u} - \epsilon^s Q_1' \hat{p}_1\right] & \text{for case (i)} \\ R^v l^2 Q_1' \hat{p}_1 & \text{for case (ii)}. \end{cases} \tag{5.5b}$$

Here $D = d/dz$ and we have introduced the vortex Reynolds number $R^v = \epsilon^{(s+2)/2} R$ for case (i) and $R^v = \epsilon^{s+1} R$ for case (ii) with the requirement that $R^v \geq O(1)$ in each case; accordingly $\sigma_1^v = R^v \sigma^v$.

With $s = 2$, (5.5) recover (for case (i)) equations given by Craik [8] to describe the initial growth of Langmuir circulations; while with $s = 0$, (5.5) recover equations given by Phillips [31] to describe an evolving secondary instability on a two-dimensional nonlinear equilibrium solution in plane Poiseuille flow.

5.3. Wave field modification for O(1) shear flows

Secondary velocity components of all orders affect the wave field, but because (4.1) and (4.2) require only the $O(\epsilon^{s+2}\Delta)$ component of $\tilde{\mathbf{p}}$, to wit p_1, it is evident that distortion is caused predominantly by the $O(\epsilon^{s}\Delta)$ component of $\tilde{\mathbf{u}}$; in consequence $O(\epsilon^{s+n}\Delta)$ and higher-order components of velocity may be ignored. Thus in calculating $\tilde{p}_1$, we need consider only the linear theory of wave motion in the presence of a Eulerian mean flow $\overline{U}+\tilde{u}$, assuming $\tilde{u}$ is sufficiently small [10]. Moreover, bearing in mind that the back effect of the secondary flow on the wave field plays a role only when $s=0$ in case (i) and that the analysis is easily rescaled for $s>0$ [as may be necessary for case (ii)], we shall, in this and the following section, confine attention to the case of strong shear, viz. $s=0$.

Prior to calculating p_1, however, we require the modification to the wave field by u_1. Unfortunately this cannot be determined from the GLM-equations; rather we must employ Navier Stokes and consider the wave field separately. We begin by noting that continuity is maintained provided $\mathrm{i}\alpha\mathscr{U}_1 + l\mathscr{U}_2 + \mathscr{U}_3' = 0$. Then on defining $\hat{\phi}(z) = \mathrm{i}\alpha^{-1}\mathscr{U}_3(z)$, where $\hat{\phi}(z)$ is the $O(\epsilon\Delta)$ spanwise-periodic wave field modification, we have

$$\mathscr{U}_1 - \frac{\mathrm{i}l}{\alpha}\mathscr{U}_2 = \hat{\phi}'. \tag{5.6}$$

Our intent is to construct an equation for $\hat{\phi}$ in the vein of Orr–Sommerfeld and to do so we turn to the $O(\epsilon\Delta)$ components of the Navier–Stokes equation:

$$-L\mathscr{U}_1 + \hat{u}\phi' - \hat{u}'\phi + \frac{\overline{u}'}{\mathrm{i}\alpha}\mathscr{U}_3 = -\wp, \tag{5.7a}$$

$$-\mathrm{i}\alpha L\mathscr{U}_2 = l\wp, \tag{5.7b}$$

$$-\mathrm{i}\alpha L\mathscr{U}_3 + \alpha^2\hat{u}\phi = -\wp', \tag{5.7c}$$

in which $\epsilon\Delta\rho\,\mathrm{Re}\{\mathrm{e}^{(\alpha c_i^w + \sigma^v)t}e^{\mathrm{i}\alpha x^*}\cos ly\}\wp(z)$ is the $O(\epsilon\Delta)$ pressure component and the operators are

$$L = (\mathrm{i}\alpha R)^{-1}M - (U - \mathrm{i}c_i^w) \qquad \text{and} \qquad M = D^2 - (\alpha^2 + l^2).$$

Eliminating $\wp$, $\mathscr{U}_1$, and $\mathscr{U}_2$ from (5.7) and employing (5.6) then yields

$$(U - \mathrm{i}c_i^w)M\hat{\phi} - U''\hat{\phi} + \hat{u}M\phi - \hat{u}''\phi = (\mathrm{i}\alpha R)^{-1}M^2\hat{\phi}, \tag{5.8}$$

which is devoid of the eigenvalue σ^{v} and, in the inviscid limit and for neutral waves, recovers the Rayleigh–Craik equation of [10].

5.4. *Stokes drift and pseudomomentum*

Measures of the nonlinear rectification of oscillatory disturbances are given by the Stokes drift and pseudomomentum. Thus having determined the $O(\epsilon\Delta)$ distortion to the primary wave field due to the secondary flow, we may now return to GLM and calculate the $O(\epsilon^2\Delta)$ correction to, and $O(\epsilon^2)$ components of, the pseudomomentum. In doing so it transpires that although viscosity plays a role at $O(\epsilon^2\Delta)$, it plays none at $O(\epsilon^2)$, so that $\overline{\mathbf{D}}$ and $\overline{\mathbf{P}}$ are as given by Craik [32].

$\overline{\mathbf{d}}$ and $\overline{\mathbf{p}}$ follow from (2.6) and (2.12) after first obtaining the particle displacements. To wit, on noting that $\overline{D}^{\mathrm{L}}\xi_j = d\xi_j/dt$ and employing (2.3) and (2.13), we see that $\xi_j(\mathbf{x},t)$ is given by integration of

$$\frac{d\xi_j}{dt} = \breve{u}_j + \xi_k \bar{u}_{j,k} \tag{5.9}$$

along mean trajectories

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}^{\mathrm{L}}(\mathbf{x},t). \tag{5.10}$$

Of course the displacement field must conform with [7] postulate (viii) that ξ_j be zero at some time t_0 and position $\mathbf{x}_0$. But if ξ_j takes the form $f(z)e^{\mathrm{i}\alpha x^*}e^{\sigma^w t}$, then if $t=t_0$ is finite so too is the wave amplitude; and this means that different particles in an averaged ensemble are located on different streamlines, causing $\overline{\mathbf{d}}$ and $\overline{\mathbf{p}}$ to oscillate. To circumvent such behavior we follow [32] and allow $t_0 \to -\infty$, at which point the wave amplitude is essentially zero and the particles to be averaged are equally spaced along the same streamline. One further point: when integrating (5.9) along mean trajectories (5.10) to determine ξ_j, we shall assume ξ_j is small compared with the radius of curvature of $\bar{u}_j$, and thus treat $\bar{u}_j$ constant with respect to time, i.e., we assume $x^* = x_0 + Ut$.

So on writing

$$\xi_j = \epsilon\,\mathrm{Re}\left\{\Xi_j^0 \mathrm{e}^{\alpha c_i^w t + \mathrm{i}\alpha x^*}\right\} + \epsilon\Delta\,\mathrm{Re}\left\{\hat{\xi}_j \mathrm{e}^{\alpha c_i^w t + \mathrm{i}\alpha x^*}\,\mathrm{e}^{\sigma^v t}\cos ly\right\} \qquad (j=1,3), \tag{5.11}$$

where ξ_j is displacement from the *average* position, then Craik [32] finds at $O(\epsilon)$ that

$$\Xi_1^0 = \left(\frac{\phi}{\mathrm{i}\,\alpha(U - \mathrm{i}c_i^w)} \right)', \qquad \Xi_2^0 = 0, \qquad \Xi_3^0 = \frac{-\phi}{U - \mathrm{i}c_i^w},$$

while we find at $O(\epsilon\Delta)$ that

$$\hat{\xi}_1 = \left(\frac{-\phi\hat{u}}{\mathrm{i}\,\alpha(U - \mathrm{i}c_i^w)^2} \right)' - \frac{U'\mathscr{U}_3 + \mathrm{i}\,\alpha(U - \mathrm{i}c_i^w)\mathscr{U}_1}{\alpha^2(U - \mathrm{i}c_i^w)^2},$$

$$\hat{\xi}_3 = \frac{\phi\hat{u}}{(U - \mathrm{i}c_i^w)^2} + \frac{\mathscr{U}_3}{\mathrm{i}\,\alpha(U - \mathrm{i}c_i^w)}.$$

Note that although $\hat{\xi}_2$ is nonzero at this order, it is not required to determine $\hat{p}_1$.

Likewise, writing the Stokes drift and pseudomomentum as (cf. Section 3.2)

$$\begin{aligned} \left(\bar{d}_j, \bar{p}_j\right) = {} & \epsilon^2 \operatorname{Re}\left\{\left(D_j^0, P_j^0\right) \mathrm{e}^{2\alpha c_i^? t}\right\} \\ & + \epsilon^{3-\delta_{j1}} \Delta \operatorname{Re}\left\{\left(\hat{d}_j, \hat{p}_j\right) \mathrm{e}^{2\alpha c_i^w t}\, \mathrm{e}^{\sigma^v t} \cos ly\right\} \qquad (j = 1,3); \end{aligned} \tag{5.12}$$

then at $O(\epsilon^2)$ and on assuming a streamwise average [32],

$$D_3^0 = \frac{1}{2} \left(\frac{\alpha c_i^w |\phi|^2}{U^2 + c_i^{w2}} \right)' \tag{5.13}$$

$$P_1^0 = -\frac{1}{2} U \left\{ \left| \left(\frac{\phi}{U - \mathrm{i}c_i^w} \right)' \right|^2 + \alpha^2 \left| \frac{\phi}{U - \mathrm{i}c_i^w} \right|^2 \right\}. \tag{5.14}$$

In consequence the Jacobian (2.11) becomes

$$J = 1 - \frac{\epsilon^2}{4} \left(\frac{|\phi|^2}{U^2 + c_i^{w2}} \right)'' \mathrm{e}^{2\alpha c_i^w t}, \tag{5.15}$$

indicating that provided ϕ and U are analytic, difficulties with the mapping (2.1) are to be expected only if $U^2 + c_i^{w2} \le O(\epsilon^2)$.

Looking now to the $O(\epsilon^2\Delta)$ correction to pseudomomentum, we find after some lengthy algebra that

$$\hat{p}_1 = \mathscr{A}_1(z)\hat{u}(z) + \mathscr{A}_2(z)\hat{u}'(z) + \mathrm{Re}\left\{\mathscr{A}_3(z)\hat{\phi}(z) + \mathscr{A}_4(z)\mathscr{U}_1\right\}, \quad (5.16)$$

where $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, and $\mathscr{A}_4$ are functions that are independent of σ^v, specifically

$$\mathscr{A}_1 = \frac{1}{2}\left\{\frac{U^2 - c_i^{w2}}{U^2 + c_i^{w2}}\left[\alpha^2\left|\frac{\phi}{U - \mathrm{i}c_i^w}\right|^2 + \left|\left(\frac{\phi}{U - \mathrm{i}c_i^w}\right)'\right|^2\right]\right.$$

$$\left. - UU'\left[\frac{\phi}{(U - \mathrm{i}c_i^w)^3}\left(\frac{\phi^*}{U + \mathrm{i}c_i^w}\right)' + \frac{\phi^*}{(U + \mathrm{i}c_i^w)^3}\left(\frac{\phi}{U - \mathrm{i}c_i^w}\right)'\right]\right\},$$

$$\mathscr{A}_2 = \frac{1}{4}\left[\frac{(U + \mathrm{i}c_i^w)\phi}{(U - \mathrm{i}c_i^w)^2}\left(\frac{\phi^*}{U + \mathrm{i}c_i^w}\right)' + \frac{(U - \mathrm{i}c_i^w)\phi^*}{(U + \mathrm{i}c_i^w)^2}\left(\frac{\phi}{U - \mathrm{i}c_i^w}\right)' + \left(\left|\frac{\phi}{U - \mathrm{i}c_i^w}\right|^2\right)'\right],$$

$$\mathscr{A}_3 = \frac{1}{U - \mathrm{i}c_i^w}\left[\frac{UU'}{U - \mathrm{i}c_i^w}\left(\frac{\phi^*}{U + \mathrm{i}c_i^w}\right)' - \frac{\alpha^2 U\phi^*}{U + \mathrm{i}c_i^w}\right],$$

$$\mathscr{A}_4 = -\frac{U}{U - \mathrm{i}c_i^w}\left(\frac{\phi^*}{U + \mathrm{i}c_i^w}\right)'.$$

Also required in (5.16) is $\mathscr{U}_1$, which follows from (5.6) and (5.7), as

$$L\mathscr{U}_1 = \frac{\alpha^2}{\alpha^2 + l^2}L\hat{\phi}' + \frac{l^2}{\alpha^2 + l^2}\left(\hat{u}\phi' - \hat{u}'\phi - U'\hat{\phi}\right). \quad (5.17)$$

Note that in contrast to its inviscid counterpart, which is algebraic, (5.17) is an ordinary differential equation. For that reason our $\mathscr{A}_i$'s $(i = 1, 4)$ should not be confused with Craik's [10] $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$, although of course (5.16) recovers Craik's form in the appropriate limit.

Thus given U for the problem at hand, ϕ and the primary flow field are determined by (5.3) and (5.4), while the eigenvalue problem for σ_1^v is completely specified by appropriate boundary conditions and the coupled system (5.5), (5.8), (5.16), and (5.17) together with (5.14) and, if needed, (5.13). Finally, although the viscous eigenvalue problem is significantly more complicated than its inviscid counterpart, it is nevertheless suitable for numerical treatment by methods similar to those employed by Phillips and co-workers [12, 14]. Some preliminary results are reported by Phillips [26].

6. Recapitulation and concluding remarks

This article deals with growing finite-amplitude waves and their nonlinear interaction with unidirectional viscous shear flows. The shear may take any of a range of strengths and the wave field is initially independent of the spanwise coordinate. The analysis largely employs the generalized Lagrangian-mean formulation of Andrews and McIntyre [7] and is in part a generalization of a theory developed by Craik [10], who dealt with secondary instabilities owing to neutral waves in inviscid shear flows.

Once the waves are other than infinitesimal, the primary instability acts as a catalyst for, and is thus subject to, secondary instabilities. These instabilities manifest as structures of longitudinal vortex form and arise through the stretching and rotation of mean vorticity. Such events are concealed by the Eulerian equations of mean motion, but not by GLM, which describes mean vorticity kinematics in a manner akin to the description of instantaneous vorticity kinematics.

Two instability mechanisms to longitudinal vortex form were observed, both inviscid: The first has as its basis the Craik−Leibovich type 2 mechanism, which was originally conceived as an explanation for Langmuir circulations. Unfortunately the physical basis for the mechanism in strong $O(1)$ shear lacks the clarity of its forebear at $O(\epsilon^2)$, in part because the Stokes drift, which has clear physical interpretation, is replaced by the physically nebulous pseudomomentum. The kinematics are further confused by wave distortion which, while negligible in $O(\epsilon^2)$ shear, plays an increasingly important role as the level of shear increases. But the mechanism remains inviscid and wave catalyzed and has a growth rate very much higher than the diffusive growth rate of the primary instability. Finally, while the instability is centrifugal in $O(\epsilon^2)$ shear that is not the case in $O(1)$ shear.

The distinctive feature of the second instability mechanism is that the wave and primary flow field distort in concert at all levels of shear. This mechanism would occur in regions of the flow where the vertical gradient of the streamwise component of pseudomomentum is zero (or almost zero) and in consequence requires a unique relationship between the wave field and the mean flow field. The equations describing the instability bear similarity to those describing the Benney−Lin [28] and CL1 [33] instabilities, but these mechanisms are fundamentally different. Indeed this apparently new and still unproven mechanism assumes a primary wave field that is initially independent of the spanwise coordinate, while Benney−Lin and CL1 require a wave field with spanwise structure. Moreover Benney−Lin and CL1 grow algebraically in time, while this mechanism can initially grow exponentially fast.

GLM does not directly account for wave distortion and to determine it requires a separate examination of the wave field. This examination was carried out for monochromatic waves and indicates that distortion is described by two ordinary differential equations, one of fourth order and the other of second; this pair replace Craik's [10] second-order Rayleigh–Craik equation and its algebraic companion. Of course as Craik notes, viscosity plays a role only at rigid boundaries and critical layers, thereby simplifying the analysis over much of the domain. Nevertheless, the calculation of wave-mean interactions that contain regions where viscous effects are important, will require significant computational effort. The methods used, however, need not be greatly different from those employed by Phillips and co-workers [12, 14].

Finally, the ostensibly formidable viscous contribution to GLM turns out to be difficult to calculate only in the case of very weak shear in the presence of rotational waves.

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