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Review Lecture

Water waves and their development in space and time

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[Plates 1 and 2]

Most practical predictions of water-wave propagation use linear approximations based on the concepts of 'geometric' rays and group velocity. Although this is successful, or adequate, in many instances, there are phenomena that can only be fully understood in terms of nonlinear effects.

The recent boom in soliton-related studies has shed much light on the nonlinear aspects of wave propagation in shallow water. However, for waves on deeper water some of the nonlinear effects are only now being appreciated. A few, such as the focusing pattern of steady wave fields have direct parallels in shallow water; while others, such as deep-water soliton solutions, have their own rich structure.

In deep or shallow water, wavebreaking is the most eye-catching development of a wave field. With the exception of the classical turbulent bore or hydraulic jump, our present models are still some way from giving a quantitative appreciation of important effects such as energy dissipation and momentum transfer, but causes of breaking for deep-water waves are now a little better understood.

1. Introduction

Water waves can develop in all circumstances where the surface of the water is free to move. Many different types of motion occur, but this review covers topics related to the major practical interest in water waves which arises from the ubiquity of water waves on the surface of the world's oceans, seas and large lakes. Once generated by wind, earthquakes, or ships, waves propagate substantial distances before their energy is (questionably) finally dissipated at bounding coastlines.

This review concentrates on theoretical aspects of water-wave propagation once waves have been generated. It may seem to side-step the difficulties of describing a storm-driven sea, but the growth of our understanding, only partly described here, is making good progress in that direction.

There are now numerous equations and systems of equations used to describe water waves in various circumstances but the bulk of our concepts, understanding and practical use of water-wave theory is based on the equations for irrotational inviscid motion linearized about the state of rest. Even so, as described in §2, it is often the case that these equations must be simplified further.

The full equations of motion are nonlinear and this nonlinearity is most apparent in water depths that are small compared with a wave's length, i.e. 'shallow water.' It is for shallow-water waves that we can most easily comprehend and illustrate nonlinear effects. This is explored in §3 before considering weakly nonlinear effects on shallow-water waves in §4.

Weakly nonlinear shallow-water waves should hold a major place in any balanced review. The experimental work of Russell, on the solitary wave, summarized in Russell (1845), led immediately to the theoretical works of Airy (1845) and Stokes (1847), which underlie almost all subsequent theoretical work on water waves except, surprisingly, that on the solitary wave. This was first described much later by Boussinesq (1872), but has been brought to prominence in recent years by the development of soliton solutions, initially for the Korteweg–de Vries (K.deV.) equation, which describes the solitary wave, but in this instance it was being used to model waves in a crystal lattice.

Soliton solutions for the K.deV. equation have received much recent exposure in meetings and publications (an elementary introduction is Drazin (1983), more substantial texts are Ablowitz & Segur (1981) and Dodd *et al.* (1982)). Any adequate treatment would so restrict the other topics in this review that only brief reference is given to these solitons in §4, where some continuity with adjacent sections is obtained by consideration of undular bores.

Linear theory gives a better approximation in deeper waters; Battjes & Van Heteren (1984) show local correlations of surface displacement and water velocities are described by linear theory to within observational error, even in rough seas. However, propagation distances can be hundreds or thousands times a wavelength, which gives ample time for nonlinear effects to accumulate. The simplest mathematical models in this case are nonlinear Schrödinger (n.l.S.) equations. These weakly nonlinear equations, also have soliton solutions. Some aspects of unsteady and steady wave fields are described in §§5, 6 with instabilities, focusing and waves jumps bringing in important nonlinear effects.

For steeper waves, where nonlinear effects are stronger and act more rapidly, instabilities and wave breaking figure prominently in recent theoretical and experimental work and suggest ways of interpreting the unsteady three-dimensional motions of a wind-driven sea. This work is briefly surveyed in §7.

Finally, after many previous mentions, wave breaking is discussed in §8; it is how most wave energy is dissipated, especially on beaches. This section is brief since the author has only recently reviewed breaking waves on beaches (Peregrine 1983*a*).

2. Linear theory

In most circumstances water waves are surprisingly well described by linear irrotational inviscid flow theory. The linear approximation involves linearizing the nonlinear boundary conditions and transferring them to the still-water level and is satisfactory for describing waves of sufficiently small amplitude and surface slope. However, the problems it poses are in (x, y, z, t) space whereas waves only propagate in horizontal directions, so that it is desirable to eliminate the vertical dimension to simplify the mathematics. For a high proportion of propagation problems bed slopes are also small and the simplification is then possible.

Shallow-water wave equations have always been in terms of (x, y, t) but more recently this reduction of dimension has been extended to any depth of water. In the 'mild-slope equation' due to Berkhoff (1972), each Fourier component, $\zeta(x, y, \omega)\,\mathrm{e}^{-\mathrm{i}\omega t}$, of the surface elevation satisfies

$$\nabla_1 \cdot (cc_g \nabla_1 \zeta) + \omega^2 (c_g/c)\,\zeta = 0, \tag{1}$$

in which ∇_1 is the two-dimensional vector operator (∂_x, ∂_y) and c and c_g are the magnitudes of the phase and group velocities, respectively, of plane waves with frequency ω. Comparisons with the full linear equations by Booij (1983) and by McIver & Evans (1985) give confidence in the use of the mild-slope equation for quite large slopes in suitable circumstances. See the review by Meyer (1979) for more comparisons.

Both the linearized water-wave equations and the mild-slope equation (1) are elliptic partial differential equations and are thus not amenable to efficient solution for typical wave-propagation problems extending over many wavelengths. There are also boundaries, such as beaches, where no sensible boundary conditions have been devised to account for the strongly nonlinear wave breaking that occurs. The usual method for dealing with this problem is to introduce a short wavelength or 'refraction' approximation equivalent to the geometric optics approach to light propagation. This leads to hyperbolic-type initial-value problems and integration of a transport equation along rays (see, for example, Meyer 1979).

The ray approach is satisfactory for a fair number of cases. Quantitative field studies with a good spatial distribution of observations are difficult and expensive. The early and recent studies of Munk & Traylor (1947) and Pawka (1983), respectively, in the same geographical area make an interesting contrast and illustrate the difficulties involved.

Although ray methods are most often applied to steady wave fields, this treatment of each Fourier component extends to slowly varying wavefields, where group velocity is the crucial concept in describing the propagation of wave modulation in time and space (see, for example, the books of Lighthill (1978) and Whitham (1974)). This is especially well illustrated in the 'trans-oceanic' wave measurements of Barber & Ursell (1948), Barber (1949) and Munk *et al.* (1963).

Waves are most strongly refracted by depth and current variations in coastal regions. For the longest ocean waves this refraction begins at the edge of the continental shelf. In many circumstances refraction leads to considerable non-uniformities of wave energy. Ray calculations then frequently lead to crossed rays, corresponding to singularities of the approximation (see figure 1). These sometimes correspond to clear cases of focusing and caustics, but more commonly only a few adjacent rays cross. For these latter cases it is rarely useful to look for better solutions by improving the ray resolution, as in the figure, to use our knowledge of wave behaviour at caustics and their cusps (see, for example, Chao 1971; Pearcey 1946). This is because the details of the ray pattern can vary substantially for small changes of frequency and even with different representations of the bottom topography (Poole 1976). Furthermore, at the longer wave periods the

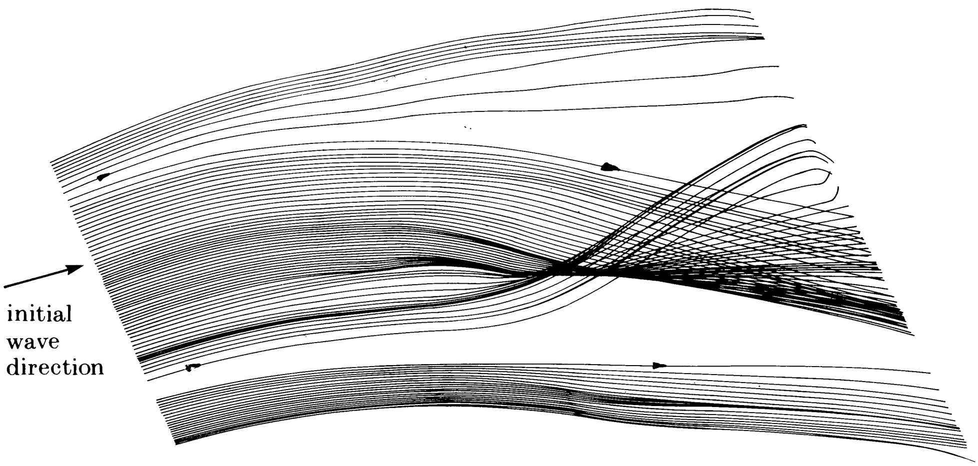

FIGURE 1. Water-wave rays showing refraction by underwater topography. (By courtesy of Hydraulic Research Ltd, Wallingford.)

wavelength becomes too long for the short-wavelength asymptotic solutions for caustics and their cusps to be of much value.

A promising way of dealing with many of these problems is to use a parabolic approximation. This is so called because the partial differential equation to be solved is parabolic. A typical derivation, for the two-dimensional wave equation, is to write the equation as

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)u = c^2\frac{\partial^2 u}{\partial y^2}\,; \tag{2}$$

assume
$$u(x,y,t) = a(x,y)\,\mathrm{e}^{\mathrm{i}k(x-ct)}, \tag{3}$$

where $a(x,y)$ is slowly varying in the sense that

$$|\nabla a| \ll ka. \tag{4}$$

Substitution of (3) in (2) gives

$$c\frac{\partial}{\partial x}\left(-2\mathrm{i}ck-c\frac{\partial}{\partial x}\right)a = c^2\frac{\partial^2 a}{\partial y^2}, \tag{5}$$

and by virtue of the assumption (4) the second x derivative is neglected to give

$$2\mathrm{i}k\frac{\partial a}{\partial x}+\frac{\partial^2 a}{\partial y^2} = 0, \tag{6}$$

the simplest parabolic wave equation or, if you prefer, a one-dimensional Schrödinger equation.

As may be seen from (3) the approximation assumes that the wave field does not vary greatly from a unidirectional wavetrain. Thus no returning reflections are permitted, though forward ones are. In a moving medium, due to currents, the amplitude function becomes $a(x,y,t)$ though only small changes in frequency can be allowed.

The parabolic approximation has proved of value for radio-wave propagation, seismic waves and in underwater acoustics (see the review by Tappert (1977) and references in Radder (1979)). The practical advantages over more accurate elliptic equations is that numerical integration can be 'marched' in the main wave-propagation direction. For water waves various derivations and equations have been used in studies to demonstrate its applications (see, for example, Radder 1979; Booij 1981; Lozano & Liu 1980; Tsay & Liu 1982; Kirby & Dalrymple 1983; Kirby 1984). A study described by Dingemans *et al.* (1984) makes comparison of computations with a parabolic equation and field measurements and concludes that it performs remarkably well for engineering purposes.

3. Strongly nonlinear effects in shallow water

When waves are long compared with the depth of water, h, nonlinear effects are much more pronounced than in deep water. Indeed, it is only for very small ratios of amplitude to depth, a/h, that nonlinear effects can be neglected. There is also contrasting behaviour in the unidirectional propagation of water waves depending on whether the waves are weakly or strongly nonlinear, i.e. $a/h \ll 1$ or $a/h = O(1)$.

Long large-amplitude waves with gentle slopes and propagating in one direction are possibly the simplest of all unsteady waves to interpret. Each part of the wave travels as if it were independent (see, for example, Lighthill 1978) and travels at the linear long-wave velocity for the local depth, $(gh)^{\frac{1}{2}}$, plus the local water velocity, u. Since the higher parts of such a wave also have a greater water velocity in the propagation direction they travel faster than lower parts of a wave. Thus, any forward-facing slope steepens until it is no longer sufficiently gentle for this to be a valid description. What happens next is frequently ill described.

It is often stated that the wave breaks. Sometimes this is clearly meant in a generalized sense that the particular mathematical description has broken down. However, what actually happens depends on the relative change of depth over the most steeply sloping part of the wave. If the change of depth is from h to $h+\Delta h$ then three types of behaviour occur.

The most strongly nonlinear case is when $\Delta h/h \geqslant 1$. The wave then develops into a breaking wave, and this is illustrated with an accurate solution for $\Delta h/h = 1.25$ in figure 2. It may be seen that breaking occurs sooner than the simple theory

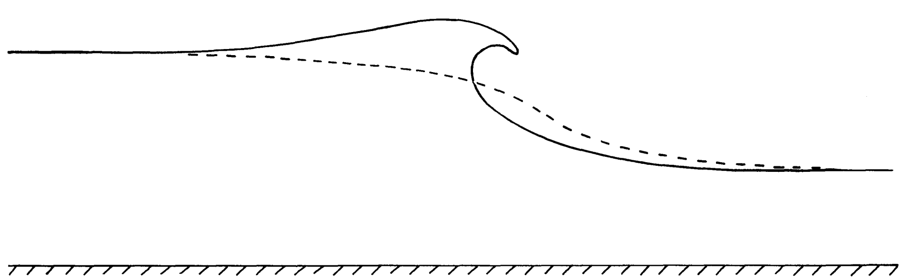

Figure 2. A propagating wave of elevation, $\Delta h = 1.25h$, after nonlinear shallow-water steepening. The full line corresponds to an accurate potential flow numerical solution, where integration was begun at a point where the shallow-water theory was giving an adequate description. The broken line is the corresponding shallow-water approximation.

predicts: in this case, a time $4(h/g)^{\frac{1}{2}}$ earlier. This is $O(1)$ in terms of the natural time scale of the motion. However, for the long scales relevant to the shallow-water wave theory it is short enough to be unimportant in many applications and for this case the description of wave steepening leading to breaking is appropriate. For other cases see the next section.

In this context of large-amplitude long-wave theory, the length and time scales of breaking, and also the subsequent development of the wave to a quasi-steady turbulent bore, are short; so it is possible to neglect consideration of all short-scale details and describe the bore as a discontinuity in the mathematical solution across which momentum flow and mass are conserved but energy is dissipated, as in the classical hydraulic jump.

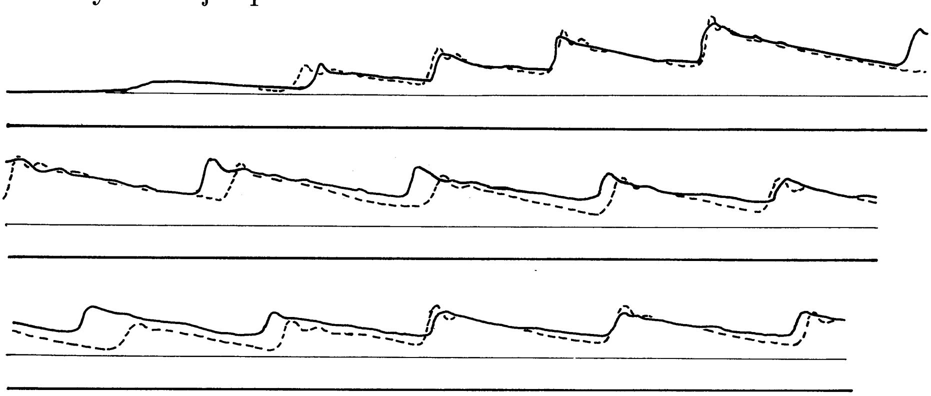

FIGURE 3. Measurements in a laboratory wave flume (by courtesy of J. B. Hansen and I. A. Svendsen, Technical University of Denmark) and inviscid shallow-water numerical solution (Packwood 1980) forced by wave-gauge readings from a point seven times further offshore. The full line is from experiment and the broken line is from numerical solution.

An example of such modelling is shown in figure 3. Breaking waves occur on most beaches, and although long-wave theory rarely applies in the breaker zone, the subsequent development of the wave through the surf zone on a gently sloping beach leads to each wave front forming a bore. The figure shows bores arriving on a laboratory beach as recorded by a wave gauge near the shoreline. The motion started from rest. The numerical model (Packwood 1980) was forced by using surface displacement measurements from a wave gauge placed at a point where bores had formed, about 7 times as far from the shoreline. The numerical model is successful, in describing both individual waves and the increase in mean water level that occurs as waves are incident on a beach. Later, reflections from the numerical seaward boundary occur and reduce the accuracy. These arise since no measurement of velocity was available to give the correct non-reflecting boundary conditions. The model also describes wave run-up well.

Recent progress has been made in modelling the structure of such turbulent bores (Svendsen & Madsen 1984) by using a simple description of the turbulent region and guided by the descriptive account of Peregrine & Svedsen (1978).

4. Weakly nonlinear effects in shallow water

The wave steepening described above also occurs for weakly nonlinear waves. However, for these waves the nonlinear steepening effect can be balanced by the pressure gradients that accelerate the surface water up the slope (Peregrine 1966), an effect usually described in a more abstract way as frequency dispersion. An exact balance between these effects gives the solitary wave, an isolated wave of elevation.

The simplest, and most useful, approximation for unidirectional waves in this case is the K.deV. equation

$$\zeta_t + \zeta_x + \tfrac{3}{2}\zeta\zeta_x + \tfrac{1}{6}\zeta_{xxx} = 0. \tag{7}$$

Its 'soliton' solution

$$\zeta = a \operatorname{sech}^2 \tfrac{1}{2}(3a)^{\frac{1}{2}}(x - ct), \tag{8}$$

where

$$c = 1 + \tfrac{1}{2}a \tag{9}$$

is a first approximation to the solitary wave.

The significance of the name 'soliton' for solution (8) is that with the use of the inverse-scattering transform (see books cited in §1) it can be shown that solitons emerge from a wide range of initial conditions. Indeed observation of waves in shallow water soon reveals numerous near-solitary-wave motions. Solitons also interact nonlinearly with other waves and can be identified once more unchanged after the interaction, except for a spatial shift. Rather than summarize readily available texts, I include one example of K.deV. soliton theory in figure 4, which

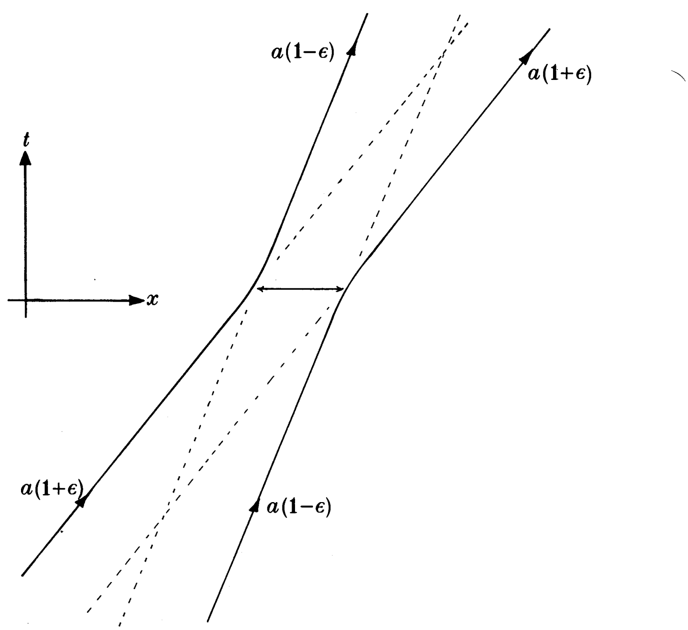

Figure 4. The trajectory of the crests of two nearly equal solitons, amplitudes $a(1+\epsilon)$ and $a(1-\epsilon)$ initially; a sketch from the explicit solution, (see, for example, Whitham 1974). The minimum separation distance is $2h(h/3a)^{\frac{1}{2}} \ln(2/\epsilon) + O(h)$, where h is the water depth, and at that time the rate of change of soliton amplitude is $\frac{1}{2}\epsilon^2(a/h)^{\frac{5}{2}}(3gh)^{\frac{1}{2}}$. See Stiassnie & Peregrine (1980) for an application to waves on beaches.

is explained in the figure caption. Note also the value for three-dimensional wave studies of Miles's (1977*a,b*) full solution, at this approximation, for obliquely interacting solitons.

However, one important aspect of weakly nonlinear water waves is not readily described by the inverse-scattering transform. That is the events that occur when a very long wave steepens at its front, as described in §3. For a weakly nonlinear wave, i.e. $\Delta h < 0.3h$, no breaking occurs; the interaction of nonlinear and frequency dispersive effects lead to the formation of an ever growing train of waves: an undular bore. An undular bore is partly shown in figure 5, it is the weakly nonlinear counterpart of figure 2.

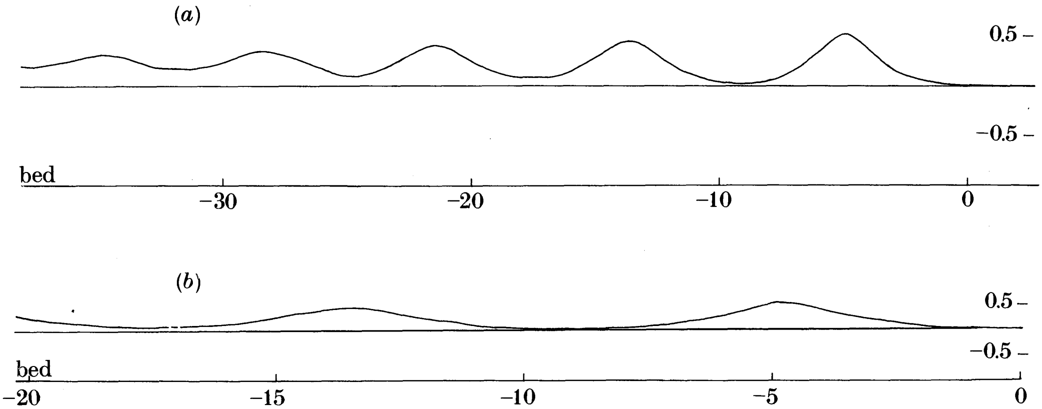

FIGURE 5. The leading waves of an undular bore arising from a wave of elevation $\Delta h = 0.25h$ at a time about $120(h/g)^{\frac{1}{2}}$ after shallow-water theory becomes inaccurate. Computation made by using an accurate irrotational flow model as in Dold & Peregrine (1984): (*a*) vertical exaggeration 4:1; (*b*) natural scale.

The leading waves of an undular bore draw ahead until they are like solitary waves, while the smaller amplitude trailing waves illustrate the dispersion of linear wave theory. The asymptotic state of the K.deV. undular bore is given by Gurevich & Pitaevskii (1974) and computations of its development are given by Peregrine (1966) and Fornberg & Whitham (1978).

When waves approach sufficiently gently sloping beaches instead of breaking, undular bores start to form and 'multiple crests' appear. These can be interpreted either as the long-wave crest developing an undular bore at its head or as the crest splitting into a set of solitons.

Study of undular bores has been rather neglected in the boom in nonlinear mathematics of recent years. This is a little strange, since although many equations and systems of equations have solitary-wave solutions where nonlinear and dispersive effects balance, with corresponding undular bore behaviour, they do not so often have soliton properties. A different undular bore example appears in §6 of this paper.

For increasing long-wave amplitudes undulations develop and lead to breaking after some delay, i.e. in the range $0.3h < \Delta h < h$. When a final steady state occurs it may include both breaking and waves. For example, see figure 6, plate 1, which

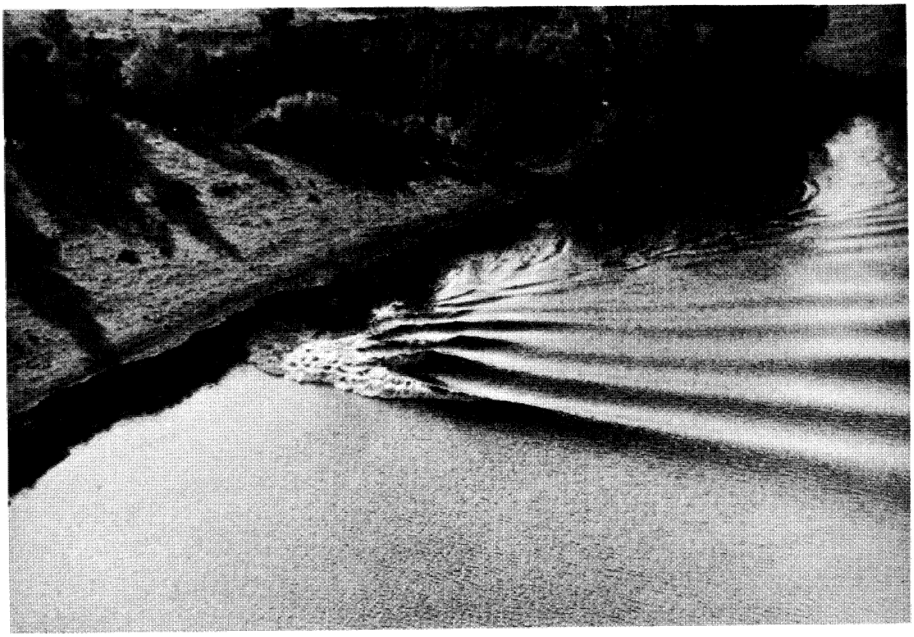

FIGURE 6. The tidal bore in the River Severn, which shows effects of depth variation across the river. (Photograph by courtesy of P. Dash.)

FIGURE 9. Waves in a wave-modelling basin. Waves are being reflected off the edge of a deeper channel. The pattern is explained in the text. (Photograph by courtesy of Hydraulics Research Ltd, Wallingford.)

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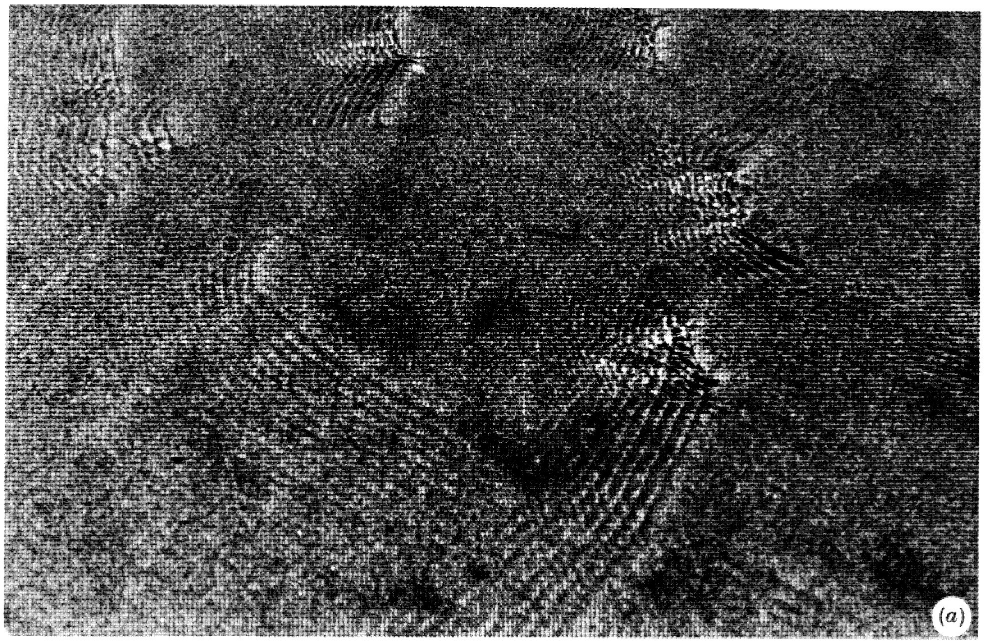

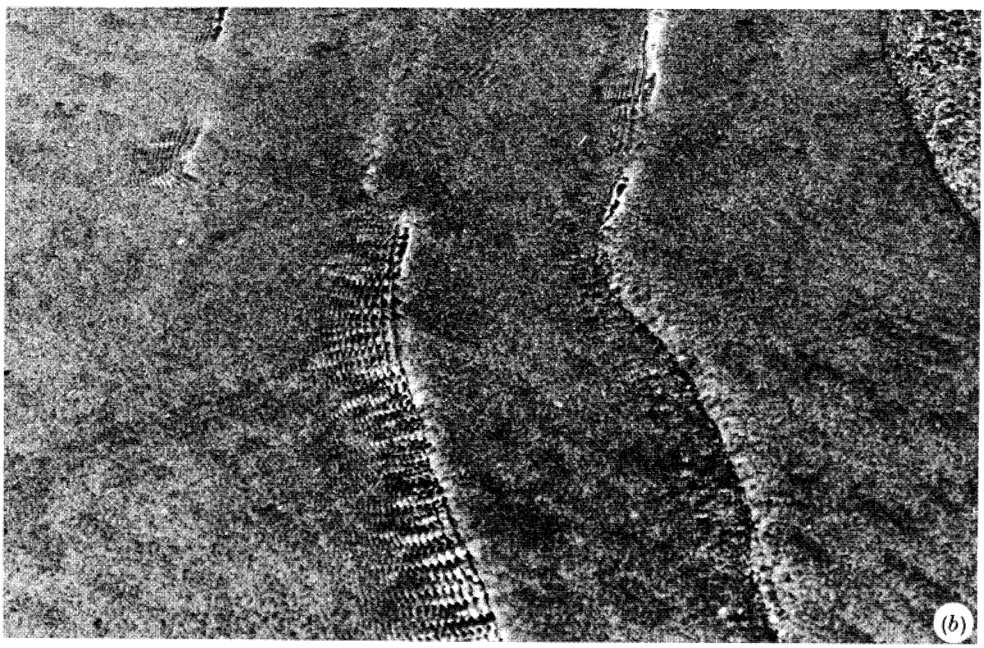

FIGURE 7. Capillary waves generated by breaking at the crests of gravity water waves. (*a*) Capillary waves of low steepness without significant nonlinear effects. (*b*) Steep capillary waves showing self-focusing.

shows a bore in a river where there is transverse variation of depth, h, giving rise to variations across the bore that correspond roughly with this range.

5. Nonlinear effects in deeper water: unsteady waves

For many years the study of nonlinear effects in deep, or moderate-depth, water was entirely concerned with extensions of Stokes (1847, 1880) work on uniform plane wave trains and the limiting wave with its crest angle of 120°. Such analysis is almost complete after the work of Cokelet (1977) and Longuet-Higgins & Fox (1977, 1978).

A transformation of our views was wrought by Benjamin and Fier's discovery that Stokes waves are unstable to long modulations. The instability was explained (Benjamin 1967) by using the ideas of resonant wave interactions that had been developed in the previous few years (see Phillips 1977) and also by examining equations for the modulation of wave trains (Lighthill 1965; Whitham 1967).

This latter approach has shown that Benjamin–Feir instability is a property of a wide class of dispersive wave systems. Zakharov (1968) derived the deep-water water-wave modulation equations in the form

$$2\mathrm{i}A_t + A_{xx} + |A|^2 A = 0, \tag{10}$$

where $A(x,t)$ is the complex representation of a first-order amplitude in a frame of reference moving with the linear group velocity c_g. Equation (10) is a nonlinear extension of the parabolic wave equation; it is known as the nonlinear Schrödinger equation, or cubic nonlinear Schrödinger equation, n.l.S. equation or n.l.S.+ equation (see below), and it arises in numerous circumstances where dispersive waves are weakly nonlinear; for another, recent, water-wave example see Larraza & Putterman (1984) and Miles (1984).

The Benjamin–Feir instability leads to the growth of modulations that form into nearly separate wave groups. The n.l.S.+ equation (10) also has a solution for a wave group:

$$A = A_0 \operatorname{sech}(A_0 x/2^{\frac{1}{2}}) \exp(\tfrac{1}{4}\mathrm{i}A_0^2 t). \tag{11}$$

The phase variation corresponds to one half the finite-amplitude correction to the frequency of a plane wave. This 'envelope-soliton' solution (11) was shown to have soliton properties by Zakharov & Shabat (1972). These two properties of water-wave modulations has stimulated considerable interest in wave groups occurring in the ocean. As yet there is no clear statistical indication of nonlinearly bound wave groups despite the well known, among surfers, tendency for swell to arrive in 'sets', usually of 5 to 7 waves but occasionally just the 'freak' single wave.

Theoretical grounds for expecting well-defined groups of waves are not strong. The mathematical solutions are only two-dimensional and three-dimensional instabilities exist (see below). The envelope solitons do interact with each other even though they retain their identity; they can also be formed as bound 'multi-solitons', which have unsteady modulations (Peregrine 1983*b*; Mollenauer 1985). Even so, I suspect that three-dimensional groups do exist; I have seen what appeared to be the same short, steep wave-group at points more than 6 km apart on the River Severn (not the tidal bore, but following some minutes behind it).

A surprising feature of the wave groups arising from Benjamin–Feir instability was found by Lake *et al.* (1977). The groups they generated continued to develop and eventually reformed into a uniform wave train. This 'long-time recurrence' of a plane wave train is modelled by numerical solutions of the n.l.S+ equation; and also, for recurrence of a single group in an otherwise uniform wave train, by an analytic solution for a soliton plus a wave train given by Ma (1979). The period of recurrence for the latter arises from the differing frequencies of the plane wave and soliton (see Peregrine 1983*b*).

A further, important, detail of the experiments by Lake *et al.* (1977), which is not modelled by the weakly nonlinear approximation of the n.l.S.+ equation, is that for sufficiently steep initial waves the second uniform wave train has a longer period than the initial one. This is likely to be associated with strongly nonlinear effects at the time of maximum modulation. It is not at present clear whether or not this frequency change necessarily involves some wave breaking, or equivalent energy transfer to capillary-waves at wave crests. Su & Green's (1984) experiments indicate that the three-dimensional instabilities of very steep waves may be a necessary part of the frequency change. Whatever the mechanism, it has a possible connection with the decrease of frequency with duration and fetch of wind-driven waves (Lake & Yuen 1978).

6. Wave focusing and steady wave patterns

The Benjamin–Feir instability and the growth of wave modulations is a form of focusing in (x, t) space. The n.l.S.+ equation (10) is said to describe 'self-focusing waves'. The limiting Ma soliton of soliton amplitude approaching 0 gives a single self-focusing event. In its simplest form (Peregrine 1983*b*)

$$A = \mathrm{e}^{\mathrm{i}t}[1-(1+2\mathrm{i}t)/(x^2+t^2)]. \tag{12}$$

Nonlinear Schrödinger equations also describe steady wave fields in two spatial dimensions. In this case t is the spatial coordinate in the main wave propagation direction. For gravity water waves it is the n.l.S.− equation

$$2\mathrm{i}A_t+A_{xx}-|A|^2A = 0 \tag{13}$$

that is appropriate, while the n.l.S.+ equation (11) is appropriate for capillary waves.

However, time modulation of capillary waves is described by the n.l.S.− equation (13). The self-focusing (n.l.S.+) behaviour of nonlinear capillary wave patterns is readily seen. Figure 7*a*, plate 2, shows capillary waves being generated at the crest of a small weakly breaking gravity wave. The capillary waves have almost parallel crests. Figure 7*b* shows a stronger more rapidly moving breaker also generating capillary waves. These capillary waves are much steeper and their self-focusing is easily seen. Self-focusing of capillary waves is also reported by Banerjee *et al.* (1983).

In contrast to the n.l.S.+ equation the n.l.S.− equation (13) has solutions showing defocusing. For example, an experimental comparison with the solutions of the n.l.S.− equation and linear theory is given by Stamnes *et al.* (1983), who used a Fresnel lens of concrete wedges to focus water waves. Whereas in linear theory a typical wave focus may be interpreted as a cusp where two caustics join, Peregrine (1983*c*) gives a nonlinear description in terms of jumps between wave trains of different properties.

Wave jumps were first discussed by Whitham (1965). The first realistic examples were given by Ostrovskii (1968), Ostrovskii & Shira (1976) and Miles (1977*c*) for solitary waves and by Yue & Mei (1980) for dispersive water waves. For the latter the wave jump occurred where waves met and were reflected from a semi-infinite wall at glancing incidence. We shall examine more closely a similar case of a reflected wave field.

Since the n.l.S. – equation has a dispersive nature the wave jumps take on the character of undular bores. This is accentuated by a close analogy to Boussinesq's equations, the bidirectional equivalent of the K.deV. equation. Thus the largest modulation becomes like the soliton solution of (13), which is

$$A^2 = A_0^2(1-\sin^2 B\,\mathrm{sech}^2\,\theta)\,\mathrm{e}^{2\mathrm{i}S}, \tag{14}$$

where A_0 is the amplitude of a uniform wave on which the soliton propagates with a modulation down to $A_0 \cos B$ (B is a constant). The amplitude-modulation phase function

$$\theta = (2^{-\frac{1}{2}}A_0 \sin B)\,(x-ct), \tag{15}$$

where
$$c = \pm 2^{-\frac{1}{2}}A_0 \cos B, \tag{16}$$

and the wave phase

$$S = -\tfrac{1}{2}A_0^2 t + \arctan\,(\tan B \tanh \theta). \tag{17}$$

This is termed a 'dark' soliton since it corresponds to a decrease of wave intensity below the mean level (and was first observed on beams of light). See figure 8 for solitons forming in an undular wave jump.

As may be seen from (16) the deeper the soliton is modulated below A_0 the slower it propagates. The limiting case of modulation down to zero is $B = \frac{1}{2}\pi$, with

$$A = A_0 \tanh 2^{-\frac{1}{2}}x\,\exp(-\tfrac{1}{2}\mathrm{i}A_0^2 t), \tag{18}$$

and a velocity of zero. The wave phase changes across the soliton by $2B$, for example by π for the limiting soliton. The soliton behaviour of these solutions is proven in Zakharov & Shabat (1973). The gravity-water-wave case of the limiting soliton is given to greater accuracy by Roberts & Peregrine (1983). In a wave pattern in the (x, t) plane, a dark soliton appears as a line of lowered amplitude at an angle to the direction of propagation. (If unsteady two-dimensional wave patterns are considered such a line can also move.)

In a wave jump, which develops like an undular bore, the slowest wave becomes more and more like a soliton, increasing in depth of modulation with time. Unlike the K.deV. solitons there is a limit to soliton amplitude (solution (18)). However, in practice this does not act as a limiting modulation but as a transition between solitons with positive and negative velocities. Figure 8 demonstrates this point. Figure 9 shows a realization of a wave jump.

At first sight figure 9 appears to show hexagonal waves. This striking impression

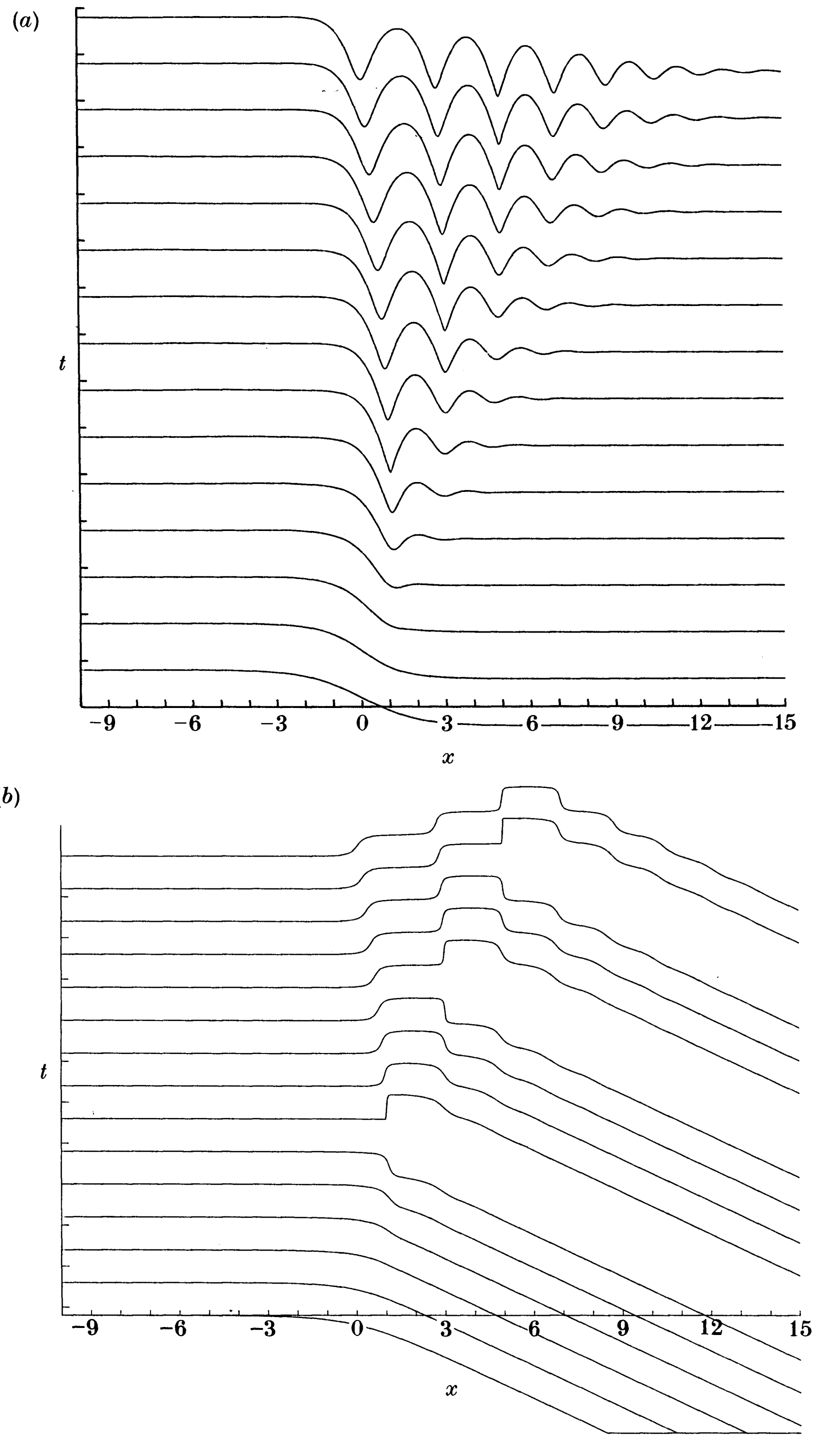

FIGURE 8. Undular bore of the n.l.S. equation (13): (*a*) amplitude of modulation; (*b*) phase of modulation, the direction of phase change for a soliton is related to its direction of propagation.

is misleading. The wave pattern arises in a model basin where waves incident from the left meet a deeper channel at glancing incidence and are reflected. There is a steeper wave train propagating along the edge of the channel, the crests are relatively short. The hexagonal pattern is the 'undular bore' region of the wave jump between this higher wave and the incident wave. Only those parts of the hexagons that are almost parallel with the incident wave crests correspond to wave crests. The oblique lines are the slopes up from the troughs to the near-zero level between the crests. Examination of the photograph reveals the change of wave phase across the soliton-like minimum of amplitude. As the first such minimum is followed in the propagation direction there is a progressive change in this phase shift. The lowest modulation level corresponds to the phase shift π at the point level with the central tripod-supported measuring instrument.

Dark-soliton-like modulations may be seen in ship waves, a topic, discussed by Hui & Hamilton (1979), where more study is required. Very similar modulations appear in experiments by Su (1982), which he terms 'skew waves' because they are also similar to higher order skew-symmetric wave solutions due to Saffman & Yuen (1980), (see, for example, Su 1982, figure 4). Dark solitons may be relevant but do not figure in the discussion.

Saffman & Yuen's (1980) skew waves and other solutions were found by finding bifurcations from the plane wave solutions of the Zakharov equation (Zakharov 1968) with minor corrections by Crawford *et al.* (1980) and extension to finite water depth and higher order by Stiassnie & Shemer (1984)). This equation is of the same order in wave steepness as the n.l.S. equations, but more Fourier components are included so that it is not restricted to long modulations. In discussing the similarity between the experimental waves and the skew-symmetric waves found by Saffman & Yuen (1980), Meiron *et al.* (1982) note that for the examples calculated there is a reduction of momentum and energy densities as the degree of three dimensionality increases. Thus the three-dimensional waves may be favoured compared with plane waves.

The examples mentioned above are only a few examples from a wide range of steady or periodic wave patterns, or both. The work of Bryant (1979, 1982, 1983, 1984) and Roberts (1983) is interesting. They show how high-order interactions can be significant (see, for example, figure 6 in Bryant (1984) and the whole family of resonances for crossing wave trains tables 1 and 2 of Roberts (1983)). High-order interactions are likely to require very long time scales to accumulate much effect, so their practical importance may be small unless they become especially important at high wave steepness.

7. Steep unsteady waves

For waves of moderate steepness, for example $ak = 0.15$ (a = amplitude, k = radian wavenumber), it takes many wavelengths or wave periods for the Benjamin–Feir instability to develop. The steepest theoretical steady-wave solution has $ak = 0.44$ for deep water (less in water of finite depth). Waves steeper than $ak = 0.3$ are affected by a more rapidly growing instability. First identified by Longuet-Higgins (1978*b*) for two-dimensional flow, it is shown by McLean *et al.*

(1981) and McLean (1982) to have its fastest growth rate for three-dimensional disturbances with a wavelength of twice the primary wave in the wave's propagation direction. McLean *et al.* describe this instability as 'type II'. Their type I instability generalizes Benjamin–Feir instability for which the fastest growing disturbances are the two-dimensional ones already mentioned. All these results involve using numerical solutions with many terms of Fourier series.

Experiments on steep waves (Su *et al.* 1982; Su 1982; Melville 1982) all show the growth leading to wave breaking of alternate crests, or less often every third crest, with a periodicity along the crests that is consistent with the type II instability. Su & Green's (1984) experiments show that type II instability, or some closely related disturbance, acts to cause wave breaking among steep waves in modulations arising from the Benjamin–Feir instability. Su & Green further suggest that the period lengthening observed by Lake *et al.* (1977) may be a strictly three-dimensional phenomenon.

The tale of instabilities is not yet finished. Tanaka (1983) found a two-dimensional instability that occurs for waves with a steepness greater than that of the waves with maximum energy density. This is at a lower steepness than Longuet-Higgins (1978*a*) suggested in an earlier study. Tanaka (1985) finds that the eigenvector for the marginally stable disturbance is linearly dependent on the eigenvector for the trivial phase-shift mode. This clarifies apparent contradictions with previous results. It is confirmed analytically by Saffman (1985) (see also Longuet-Higgins 1984).

Deep-water waves for which the above instability is computed rarely break in a two-dimensional manner, yet waves in shallow water, as on beaches, usually do. Occasionally as much as one kilometre of wave crest overturns at once. Thus similar computations for shallow-water waves, such as the solitary wave, are awaited with interest since they may have more practical relevance.

8. Wave breaking

Steep waves generated in experiments break through instabilities. Waves in nature are usually generated by the wind and are breaking as they are generated. Banner & Melville's (1976) experiments demonstrate the importance of such breaking. Wave breaking also occurs occasionally as waves propagate freely and, more often, when propagation conditions change due to varying water depth or currents. Most waves break when they meet a beach, but since this is the topic of a recent review (Peregrine 1983*a*), a few other aspects of wave breaking are treated here.

Our understanding of the breaking process is mainly limited to three areas.

(i) General statements about the causes (for example, waves too steep, instabilities, shoaling water, etc).

(ii) Measurements in the field and the laboratory. This is a difficult environment for measurements. The most interesting, large ocean waves are such that it is difficult to ensure equipment survives their impact. The interpretation of measurements is also difficult in the two-phase flows that result.

(iii) Two-dimensional numerical modelling.

Numerical modelling of wave overturning, first successfully implemented by Longuet-Higgins & Cokelet (1976), has now developed, via work by Baker *et al.* (1982), to a more efficient second-generation program (Dold & Peregrine (1984). This program was used to create figures 2 and 5.

These numerical methods have been used to interpret the motion, showing, for example regions of high water-accelerations under the overturning jet (Peregrine *et al.* 1980; New *et al.* 1985), which hindsight shows to be an obvious feature of the motion. As well as being able to provide data for engineers, these methods also provide accurate easily observed, numerical 'wave tanks' for testing theoretical developments, for example Longuet-Higgins & Cokelet's (1978) study of instabilities. More surprisingly, numerical solutions have stimulated analytical speculation leading to mathematical solutions that appear to have some relevance to wave overturning (New 1983; Greenhow 1983).

The subsequent splash and other motions that sooner or later degenerate into turbulence have been little studied except in the context of waves on beaches. See Peregrine (1983*a*) for more discussion.

Concluding remarks

The range of topics covered has been necessarily limited and shows a bias to my own interests. However, other views can be readily obtained from the book by Mei (1983) and the substantial review paper by Yuen & Lake (1982).

The interaction of experiment and observation with theory has been an important element of the development of this area of study from the days of Airy (1845), Russell (1845) and Stokes (1847) to the present day. It is a very satisfying aspect of the subject that simple visual observation still has a part to play in this process.

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