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Berkhoff approximation in a problem on surface gravity wave propagation in a basin with bottom irregularities

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Abstract. The problem of the propagation of small-amplitude surface gravity waves in a basin of constant mean depth with one- and two-dimensional bottom roughness is solved in the framework of the Berkhoff model by a mean-field method. In both cases the solutions obtained are compared with the solutions of sets of exact linearized equations of the hydrodynamics of an incompressible fluid. The comparison of the exact and approximate mean-field attenuation coefficients has shown that the Berkhoff approximation is appropriate for the solution of this problem in the case of shallow water for an arbitrary correlation length of bottom irregularities and in the case of arbitrary depth and large-scale irregularities. An explanation is given for the limits of applicability of the Berkhoff approximation which are connected with the weak variability of the vertical structure of the wave field in shallow water and in a basin with large-scale depth fluctuations. The mean-field attenuation coefficients reach their maximum values in the region $k_0h_0 \geqslant 1$ (where k_0 is the wavenumber of the surface gravity wave in a basin of constant depth h_0). The location of these maxima is practically independent of the correlation length of the bottom irregularities. For the case of one-dimensional irregularities the effect of bottom roughness on the surface gravity wave velocity is investigated. It is shown that the surface wave in a basin with an uneven bottom may propagate more slowly, as well as faster than the wave in a basin with an even bottom, depending on the relations between the wavelength, depth and correlation length of the bottom imperfections.

1. Introduction

The mean-field method is used for waves of different physical nature (see, e.g., [1–9]) and has now been developed to describe the propagation of nonlinear waves in random media [10–13]. The propagation of surface gravity waves in a basin with a stochastically irregular bottom has been investigated in a number of papers. Usually, a set of exact linearized equations of the hydrodynamics of an incompressible fluid has been used. These equations have been solved by a Fourier transform method. A dispersion equation has been derived for the mean field of the surface gravity wave and an expression for the mean field attenuation coefficient has been obtained. Relatively simple analytical results have been obtained in the shallow-water approximation when the initial system of equations can be reduced to the simple wave equation. Together with exact equations of hydrodynamics in the theory of surface gravity waves the Berkhoff approximation is used, especially for calculations of wave fields over large distances. This approximation is obtained, conceptually, by the Galerkin procedure for some special cases of the vertical structure of the wave field [14–17].

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The substantiation of the Berkhoff approximation may be obtained in two limiting cases: shallow water with small bottom inclinations and vertical obstacles in a fluid of arbitrary but constant depth. This is why the Berkhoff model is called the refraction–diffraction model. This model does not contain a vertical coordinate: consequently, the analysis of wave propagation in a basin with varying depth is simplified considerably, especially when a numerical technique is used.

The Berkhoff model is usually used for deterministic depth profiles. Its substantiation may not be obtained by asymptotic methods for arbitrary relations between depth of the basin, wavelength and characteristic length of depth variations, and only very few numerical three-dimensional solutions, laboratory experiments and natural sea data are used for the determination of the limits of applicability of the Berkhoff approximation.

We shall evaluate herein the limits of applicability of the Berkhoff model for the case of a stochastically irregular bottom using the solution of a set of exact linearized equations of the hydrodynamics of an incompressible fluid as a benchmark case. The problem is solved by a mean-field method. The mean-field method assumes that the total wave field in a random medium may be represented as a sum of a coherent part, or a mean field, and an incoherent part, or a fluctuation field. The wavenumber for the mean field is complex, because the mean field is continually scattered by the random inhomogeneities and is converted into the incoherent field. The imaginary part of the wavenumber is the attenuation coefficient for the mean field in the random medium. The real part of the wavenumber describes the variation in the phase velocity of the wave.

2. Mean-field analysis in the framework of the Berkhoff approximation

So, we consider the propagation of small amplitude, time-harmonic surface gravity waves in a homogeneous incompressible dense fluid. The fluid has a free surface and is bounded from below by an absolutely rigid stochastically irregular bottom. Let the $z = 0$ plane of a Cartesian coordinate system coincide with the unperturbed surface of the fluid (the z-axis direction is opposite to the direction of gravitation). The depth profile is defined by the function $z = -h(x, y)$. Deviations in the level of the fluid relative to its undisturbed horizontal surface $z = 0$ due to the propagation of the wave is described in the framework of the Berkhoff approximation by the following equation (a temporal factor $\exp(-\mathrm{i}\omega t)$ is omitted):

$$cc_{\mathrm{g}}\,\Delta\eta + \nabla(cc_{\mathrm{g}})\nabla\eta + k^2 cc_{\mathrm{g}}\eta = 0. \tag{1}$$

Here

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \nabla = \frac{\partial}{\partial x}\,\boldsymbol{e}_x + \frac{\partial}{\partial y}\,\boldsymbol{e}_y$$

where $\boldsymbol{e}_{x,y}$ are the unit Cartesian vectors, and $k = k(x, y)$ is the wavenumber of the surface gravity wave, which is connected with the angular frequency by the dispersion equation

$$\omega^2 = gk\tanh kh \tag{2}$$

(g is the acceleration due to gravity),

$$c = \left[(g/k)\tanh kh\right]^{1/2} \tag{3}$$

is the phase speed, and

$$c_{\mathrm{g}} = \frac{1}{2}\left(\frac{g}{k}\tanh kh\right)^{1/2}\left(1 + \frac{kh}{\sinh kh\cosh kh}\right) \tag{4}$$

is the group speed of the wave.

We represent the depth of the basin by $h(x, y) = h_0 + \chi(x, y)$, where h_0 is the constant mean depth and $\chi(x, y)$ are random depth variations which are small in comparison with h_0. We assume that $\langle \chi \rangle = 0$, where the angular brackets mean an overall average. It is convenient to write down the coefficients cc_g and k^2, which appear in equation (1), in the form

$$cc_g = c_0 c_{g_0}\big[1 + u\varepsilon(x, y)\big] \qquad k^2 = k_0^2\big[1 + v\varepsilon(x, y)\big]. \tag{5}$$

Here, $\varepsilon(x, y) = \chi(x, y)/h_0$, and c_0, c_{g_0} and k_0 are the values of the velocities and propagation constant due to the mean depth h_0. These quantities are obtained from (2)–(4) when $h = h_0$. As bottom irregularities are assumed to be small, expressions for u and v can easily be found:

$$u = \frac{3k_0h_0\,(1 - k_0h_0 \tanh k_0h_0)\,\tanh k_0h_0 + k_0^2h_0^2}{(k_0h_0 + \sinh k_0h_0 \cosh k_0h_0)\big[(1 - k_0h_0 \tanh k_0h_0)\tanh k_0h_0 + k_0h_0\big]} \tag{6}$$

$$v = -\frac{2k_0h_0}{k_0h_0 + \sinh k_0h_0 \cosh k_0h_0}. \tag{7}$$

So, the equation for the amplitude of the surface gravity wave can be written down in the form

$$(1 + u\varepsilon)\Delta\eta + u\,\nabla\varepsilon\,\nabla\eta + k_0^2\eta(1 + w\varepsilon) = 0. \tag{8}$$

In (8) the notation $w = u + v$ has been introduced.

We represent the fluid level variations as the sum $\eta = \eta_0 + \eta'$, where η_0 is the coherent, or mean, field and η' is the incoherent, or fluctuation, field for which the equality $\langle \eta' \rangle = 0$ holds. Taking the overall average of equation (8) we obtain the equation for the mean field:

$$\Delta\eta_0 + k_0^2\eta_0 + u\,\big\langle \varepsilon\,\Delta\eta' \big\rangle + u\,\big\langle \nabla\varepsilon\,\nabla\eta' \big\rangle + k_0^2 w\,\big\langle \varepsilon\eta' \big\rangle = 0. \tag{9}$$

To obtain an equation for the incoherent field we use a standard procedure [1–3]. We subtract (9) from (8) and neglect small random terms of order higher than the first (the Bourret approximation [2]). The result is as follows:

$$\Delta\eta' + k_0^2\eta' + u\varepsilon\Delta\eta_0 + u\nabla\varepsilon\nabla\eta_0 = 0. \tag{10}$$

We write down the quantities η_0 and η' in the form of Fourier integrals of the type

$$\eta_0(\boldsymbol{r}) = \iint\limits_{-\infty}^{+\infty} \tilde{\eta}_0(\boldsymbol{\varkappa})\mathrm{e}^{\mathrm{i}\boldsymbol{\varkappa}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{\varkappa} \tag{11}$$

$$\eta'(\boldsymbol{r}) = \iint\limits_{-\infty}^{+\infty} \tilde{\eta}'(\boldsymbol{\varkappa})\mathrm{e}^{\mathrm{i}\boldsymbol{\varkappa}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{\varkappa} \tag{12}$$

where $\tilde{\eta}(\boldsymbol{\varkappa})$ and $\tilde{\eta}'(\boldsymbol{\varkappa})$ are the Fourier transform of the mean field and the fluctuation field, respectively, and $\boldsymbol{r} = (x, y)$ and $\boldsymbol{\varkappa} = (\varkappa_x, \varkappa_y)$ are the two-dimensional vectors. It follows from equations (10)–(12) that the Fourier transform of the incoherent field is given by the expression

$$\tilde{\eta}'(\boldsymbol{\varkappa}) = \frac{1}{\varkappa^2 + k_0^2}\iint\limits_{-\infty}^{+\infty} (k_0^2 w - u\boldsymbol{\varkappa}\cdot\boldsymbol{\xi})\,\tilde{\varepsilon}(\boldsymbol{\varkappa} - \boldsymbol{\xi})\,\tilde{\eta}_0(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \tag{13}$$

where

$$\tilde{\varepsilon}(\boldsymbol{q}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} \varepsilon(\boldsymbol{r})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\, \mathrm{d}\boldsymbol{r} \tag{14}$$

is the spectrum of the bottom irregularities. Making use of equations (13), (9) we obtain an equation for the spectrum of the mean field:

$$(q^2 - k_0^2)\tilde{\eta}_0(\boldsymbol{q}) = \iiiint_{-\infty}^{+\infty} \frac{(k_0^2 w - u\boldsymbol{\varkappa}\cdot\boldsymbol{\xi})(k_0^2 w - u\boldsymbol{q}\cdot\boldsymbol{\varkappa})}{\varkappa^2 - k_0^2}\, \tilde{\eta}_0(\boldsymbol{\xi})$$

$$\times\left\langle \tilde{\varepsilon}\,(\boldsymbol{q}-\boldsymbol{\varkappa})\,\tilde{\varepsilon}\,(\boldsymbol{\varkappa}-\boldsymbol{\xi})\right\rangle \mathrm{d}\boldsymbol{\xi}\; \mathrm{d}\boldsymbol{\varkappa}. \tag{15}$$

At this stage we need to introduce the statistical properties of the depth fluctuations. First, we consider the case when the depth fluctuations are statistically homogeneous and isotropic, so that

$$\langle \varepsilon(\boldsymbol{r})\,\varepsilon(\boldsymbol{r}')\rangle = \langle \varepsilon^2\rangle\, \Gamma\big(|\boldsymbol{r}-\boldsymbol{r}'|\big). \tag{16}$$

where Γ is the correlation coefficient of the bottom irregularities and $\langle\varepsilon^2\rangle$ is their constant dispersion. For the Fourier spectrum of the function $\eta(\boldsymbol{r})$ under the condition (16) the relationship

$$\left\langle \tilde{\varepsilon}\,(\boldsymbol{q}-\boldsymbol{\varkappa})\,\tilde{\varepsilon}\,(\boldsymbol{\varkappa}-\boldsymbol{\xi})\right\rangle = \langle\varepsilon^2\rangle\, T\big(|\boldsymbol{q}-\boldsymbol{\varkappa}|\big)\, \delta(\boldsymbol{q}-\boldsymbol{\xi}) \tag{17}$$

holds. Here, T is the two-dimensional spatial spectrum of the correlation coefficient Γ. The quantities $T(\boldsymbol{k})$ and $\Gamma(\boldsymbol{r})$ are connected by means of the reciprocal relations

$$\Gamma(\boldsymbol{r}) = \iint_{-\infty}^{+\infty} T(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, \mathrm{d}\boldsymbol{k} \tag{18}$$

$$T(\boldsymbol{k}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} \Gamma(\boldsymbol{r})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, \mathrm{d}\boldsymbol{r}. \tag{19}$$

After substitution of (17) in (15) we obtain the dispersion equation for surface gravity waves in the basin with two-dimensional bottom irregularities:

$$q^2 = k_0^2 + \langle\varepsilon^2\rangle \iint_{-\infty}^{+\infty} \frac{(k_0^2 w - u\boldsymbol{\varkappa}\cdot\boldsymbol{q})^2}{\varkappa^2 - k_0^2}\, T\big(|\boldsymbol{q}-\boldsymbol{\varkappa}|\big)\, \mathrm{d}\boldsymbol{\varkappa}. \tag{20}$$

In order to simplify this equation we neglect terms which contain a small value $\langle\varepsilon^2\rangle$ to the power two and replace the effective propagation constant q on the right-hand side of (20) by the propagation constant k_0, corresponding to the constant mean depth of the basin. After that the expression for the quantity q can be written in the form

$$q \simeq k_0 + \frac{\langle\varepsilon^2\rangle}{2k_0} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\infty} \frac{k_0^2\,(k_0 w - u\varkappa\cos\varphi)^2}{\varkappa^2 - k_0^2}\, T\left(\sqrt{k_0^2 + \varkappa^2 - 2k_0\varkappa\cos\varphi}\right)\, \varkappa\, \mathrm{d}\varkappa. \tag{21}$$

In equation (21) we have used a polar coordinate system in $\varkappa$ space.

The integral which appears in expression (21) for the effective propagation constant for the surface wave is complex. Its real part, which results from the Cauchy principal value integration, describes wave-velocity variations due to the bottom roughness. The imaginary part of the integral (21), which is proportional to the semiresidue in a pole $\varkappa = k_0$, defines the attenuation coefficient for the mean field of the surface wave. Our prime interest is in investigation of this coefficient. Carrying out the corresponding calculations in the framework of the Berkhoff approximation we obtain the following expression for the mean-field attenuation coefficient:

$$\operatorname{Im} q = \gamma_{\mathrm{B}} = \frac{\pi\langle\varepsilon^2\rangle k_0^3}{4} \int_0^{2\pi} \left(2u \sin^2\frac{\varphi}{2} + v\right)^2 T\left(2k_0 \sin\frac{\varphi}{2}\right) \, \mathrm{d}\varphi. \qquad (22)$$

Equation (22) will be analysed in sections 4 and 5.

3. Mean-field analysis in the framework of the potential model

We shall evaluate the limits of applicability of the Berkhoff approximation to the solution of the problem on surface wave scattering by comparing the approximate attenuation coefficient γ_{B} with the attenuation coefficient γ obtained from the solution of an exact linearized set of equations of hydrodynamics [7]. This set of equations, written in terms of the velocity potential φ, takes the form

$$\Delta\varphi + \frac{\partial^2\varphi}{\partial z^2} = 0 \qquad -h(x, y) < z < 0 \qquad (23)$$

$$\frac{\partial\varphi}{\partial z} = \frac{\omega^2}{g}\varphi \qquad z = 0 \qquad (24)$$

$$\frac{\partial\varphi}{\partial z} = -\nabla h\,\nabla\varphi \qquad z = -h(x, y). \qquad (25)$$

The boundary-value problem (23)–(25) has been solved in [7]. We only note herein that for its solution it is necessary to transport the boundary condition (25) from the surface $z = -h(x, y)$ to the plane $z = -h_0$ and rewrite it in the form

$$\frac{\partial\varphi}{\partial z} = -\nabla\left[\chi(x, y)\,\nabla\varphi\right] \qquad z = -h_0. \qquad (26)$$

After that, a Fourier transform method must be used. The solution of equations (23)–(25) yields the following dispersion equation for the surface wave:

$$\frac{\omega^2 - gk\tanh kh_0}{gk - \omega^2\tanh kh_0} = I \qquad (27)$$

where

$$I = \langle\chi^2\rangle \iint_{-\infty}^{+\infty} T\left(|\boldsymbol{k} - \boldsymbol{\xi}|\right) \frac{(\boldsymbol{k}\boldsymbol{\xi})^2}{k\xi} \frac{g\xi - \omega^2\tanh\xi h_0}{\omega^2 - g\xi\tanh\xi h_0} \, \mathrm{d}\boldsymbol{\xi}. \qquad (28)$$

It is convenient to represent the solution of (27) in the form $k = k_0 + \varkappa$, where k_0 is the solution of the dispersion equation (2) subject to the constant mean depth h_0 and $\varkappa$ is small in comparison with k_0. Upon solving (27) by expanding it in powers of $\varkappa$ we obtain the following dispersion equation for the mean field of a surface gravity wave:

$$\varkappa = \delta k + \mathrm{i}\gamma = -\frac{2k_0}{2k_0h_0 + \sinh 2k_0h_0} I. \qquad (29)$$

The mean-field attenuation coefficient γ is defined by the semiresidue in the pole $\varkappa = k_0$ of the integrand in (29) and is described by the expression [7]:

$$\gamma = \frac{\pi \langle \varepsilon^2 \rangle k_0^3}{4} \left[\frac{4(2k_0 h_0)^2}{(2k_0 h_0 + \sinh 2k_0 h_0)^2} \right] \int_0^{2\pi} T \left(2k_0 \sin \frac{\varphi}{2} \right) \cos^2 \varphi \, \mathrm{d}\varphi. \qquad (30)$$

4. Comparison of the approximate and the exact attenuation coefficients

In this section we analyse the approximate expression (22) and the exact expression (30) for the mean-field attenuation coefficient of the surface gravity wave. In the shallow-water case, when $k_0 h_0 \ll 1$, it follows from (6), (7) that $u = -v = 1$. With these equation (22) takes the form

$$\gamma_{\mathrm{B}} \simeq \frac{\pi \langle \varepsilon^2 \rangle k_0^3}{4} \int_0^{\infty} T \left(2k_0 \sin \frac{\varphi}{2} \right) \cos^2 \varphi \, \mathrm{d}\varphi. \qquad (31)$$

The same expression is derived for the exact value of the mean-field attenuation coefficient from (30) under the condition $k_0 h_0 \ll 1$. The exact (γ) and approximate (γ_{B}) attenuation coefficients coincide for arbitrary scales of bottom irregularities, because both the Berkhoff and potential models in the case $k_0 h_0 \ll 1$ are reduced to the shallow-water equation

$$\nabla \eta + k_0^2 \eta = -\nabla [\varepsilon(x, y) \nabla \eta] \qquad (32)$$

the solution of which also leads to the result (31) [17].

In the case of large-scale bottom irregularities, when $k_0 L \gg 1$ (L is their correlation length) the spectrum $T(k)$ of the correlation coefficient decreases rapidly while its argument grows, so a considerable contribution to integrals (22) and (30) is given by the region of small values of the integration variable φ. Assuming $\cos \varphi \approx 1$, from (22), (30) we obtain the following expression for γ and γ_{B}:

$$\gamma \approx \gamma_{\mathrm{B}} \approx \frac{\pi \langle \varepsilon^2 \rangle k_0^3}{4} \left[\frac{4(2k_0 h_0)^2}{(2k_0 h_0 + \sinh 2k_0 h_0)^2} \right] \int_0^{2\pi} T \left(2k_0 \sin \frac{\varphi}{2} \right) \mathrm{d}\varphi. \qquad (33)$$

Hence, in the case of large-scale bottom irregularities the Berkhoff approximation is valid for arbitrary basin depth.

In the limiting case of deep water, $k_0 h_0 \gg 1$, it follows from (6), (7) that

$$u = -8k_0^2 h_0^2 \mathrm{e}^{-2k_0 h_0} \qquad v = -8k_0 h_0 \mathrm{e}^{-2k_0 h_0}. \qquad (34)$$

In this situation the surface wave attenuation coefficient calculated in the framework of the Berkhoff approximation is described by the expression

$$\gamma_{\mathrm{B}} \approx 16\pi \langle \varepsilon^2 \rangle k_0^3 (k_0 h_0)^2 \, \mathrm{e}^{-4k_0 h_0} \int_0^{2\pi} \left(2k_0 h_0 \sin^2 \frac{\varphi}{2} + 1 \right)^2 T \left(2k_0 \sin \frac{\varphi}{2} \right) \mathrm{d}\varphi. \qquad (35)$$

The attenuation coefficient calculated for the case $k_0 h_0 \gg 1$ in the framework of the potential model has the form

$$\gamma \approx 16\pi \langle \varepsilon^2 \rangle k_0^3 (k_0 h_0)^2 \, \mathrm{e}^{-4k_0 h_0} \int_0^{2\pi} T \left(2k_0 \sin \frac{\varphi}{2} \right) \cos^2 \varphi \, \mathrm{d}\varphi. \qquad (36)$$

Analysis of equations (35) and (36) shows that there is a considerable discrepancy in the values of the integrals appearing in these expressions in the cases of small- and medium-scale bottom imperfections; where this occurs the Berkhoff approximation is not applicable.

For arbitrary basin depths and correlation lengths of bottom irregularities it is necessary to carry out numerical investigations of the mean-field attenuation coefficients of surface gravity waves. To do this we represent the quantities γ and γ_B in the form

$$\gamma = \frac{\langle \varepsilon^2 \rangle}{L}\,\tilde{\gamma} \qquad \gamma_B = \frac{\langle \varepsilon^2 \rangle}{L}\,\tilde{\gamma}_B \tag{37}$$

where $\tilde{\gamma}$ and $\tilde{\gamma}_B$ are numerical coefficients whose forms are defined by an actual correlation coefficient of bottom roughness. For example, in the case of an exponential correlation coefficient, $\Gamma(r) = \exp(-r/L)$, for which

$$T(k) = \frac{L^2}{2\pi\,(1 + L^2 k^2)^{3/2}} \tag{38}$$

the quantities $\tilde{\gamma}$ and $\tilde{\gamma}_B$ are described by the following expressions:

$$\tilde{\gamma} = \frac{1}{8}\,(k_0 L)^3 \left[\frac{4(2k_0h_0)^2}{\left(2k_0h_0 + \mathrm{sh}^2 k_0h_0\right)^2} \right] \int_0^{2\pi} \frac{\cos^2\varphi \, \mathrm{d}\varphi}{\left[1 + \left(2k_0 L \sin\frac{1}{2}\varphi\right)^2\right]^{3/2}} \tag{39}$$

$$\tilde{\gamma}_B = \frac{1}{8}\,(k_0 h)^3 \int_0^{2\pi} \frac{\left(2u \sin^2 \frac{1}{2}\varphi - v\right)^2 \mathrm{d}\varphi}{\left[1 + \left(2k_0 L \sin\frac{1}{2}\varphi\right)^2\right]^{3/2}}. \tag{40}$$

The results of numerical calculations from equations (39), (40) are shown in figure 1. Here we have used the ratio of the correlation length of the bottom irregularities to the depth of the basin, $\mu = L/h_0$, as a parameter. The quantities $\tilde{\gamma}$ and $\tilde{\gamma}_B$ reach their maximum values in the region $k_0h_0 \geqslant 1$, and the locations of these maxima are practically independent of the parameter μ. In shallow-water cases, $k_0h_0 \ll 1$, the curves for $\tilde{\gamma}$ and $\tilde{\gamma}_B$ merge. They also merge when $k_0L \gg 1$.

The explanation of these facts is as follows. Within the framework of the Berkhoff model the vertical structure of the wave field is assumed to be constant. In the case of shallow water, as well as in the case where the correlation length of the bottom irregularities is large, the vertical structure of the wave field being calculated from the more explicit potential model also practically does not vary. That is why expressions (22) and (30) coincide for shallow water (see equation (31)) and for large-scale irregularities (see equation (33)).

5. Surface waves scattering due to one-dimensional depth fluctuations

Let us now turn to consideration of the effect of one-dimensional depth fluctuations on the propagation of surface gravity waves. Let the surface of the bottom be defined by the function $z = -h(x)$. For simplicity, we consider a two-dimensional problem and assume that the surface wave propagates along the x axis. We also assume that the depth fluctuations are statistically homogeneous; hence the following equality holds:

$$\left\langle \varepsilon(x)\,\varepsilon(x') \right\rangle = \langle \varepsilon^2 \rangle\, \Gamma_1\left(|x - x'|\right). \tag{41}$$

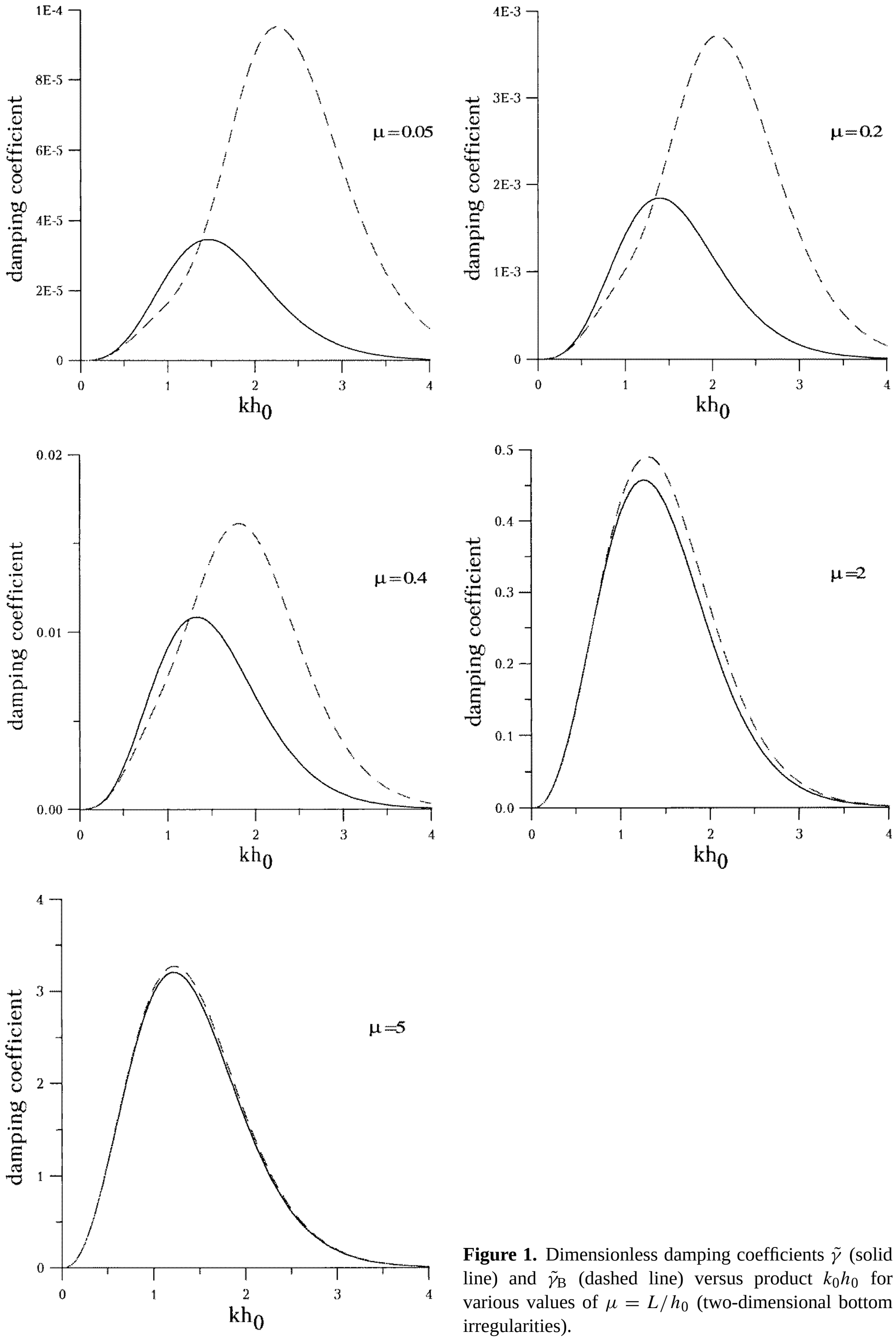

Figure 1. Dimensionless damping coefficients $\tilde{\gamma}$ (solid line) and $\tilde{\gamma}_B$ (dashed line) versus product k_0h_0 for various values of $\mu = L/h_0$ (two-dimensional bottom irregularities).

Here, Γ_1 is the correlation coefficient of the one-dimensional irregularities. Its spatial Fourier spectrum is given by the expression

$$T_1(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Gamma_1(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x. \tag{42}$$

It can be shown that in the case of one-dimensional bottom imperfections equation (17) must be replaced by the formula

$$\left\langle \tilde{\varepsilon}\,(\boldsymbol{q}-\boldsymbol{\varkappa})\, \tilde{\varepsilon}\,(\boldsymbol{\varkappa}-\boldsymbol{\xi})\right\rangle = \langle \varepsilon^2 \rangle\, T_1\big(|q_x - \varkappa_x\,|\big)\, \delta(\boldsymbol{q}-\boldsymbol{\xi})\delta(q_y - \varkappa_y)\, \delta(\varkappa_y - \xi_y). \tag{43}$$

Substitution of equation (43) in (15) leads to the dispersion equation for surface waves propagating in the basin with one-dimensional depth fluctuations. The solution of this equation is

$$q \simeq k_0 + \frac{\langle \varepsilon^2 \rangle}{2k_0} \int_0^{\infty} \frac{k_0^2\,(k_0 w - u\varkappa)^2}{\varkappa^2 - k_0^2}\, T_1\big(|k_0 - \varkappa|\big)\, \mathrm{d}\varkappa. \tag{44}$$

The imaginary part of the integral, which appears in (44), is defined by semi-residues in the poles $\varkappa = \pm k_0$. After some algebra we obtain the expression for the mean-field attenuation coefficient due to surface wave scattering by an uneven bottom:

$$\gamma_{\mathrm{B}}^{(1)} = \frac{\pi\,\langle \varepsilon^2 \rangle\, k_0^2}{4}\, \big[v^2 T(0) + (4u^2 + 4\,u\,v + v^2) T(2k_0)\big]\,. \tag{45}$$

The solution of the boundary-value problem (23)–(25) for the case of one-dimensional depth fluctuations leads to a dispersion equation of the type (27) in which

$$I = I_1 = \left\langle \chi^2 \right\rangle k \int_{-\infty}^{+\infty} T_1\big(|k-\xi|\big)\, \frac{g\xi - \omega^2 \tanh \xi h_0}{\omega^2 - g\xi\, \tanh \xi h_0}\, \xi\, \mathrm{d}\xi. \tag{46}$$

It follows from the solution of the dispersion equation that the 'exact' value of the mean-field attenuation coefficient is given by the expression

$$\gamma^{(1)} = \pi\,\langle \varepsilon^2 \rangle \left(\frac{2k_0^2 h_0}{2k_0 h_0 + \sinh 2k_0 h_0} \right)^2 \big[T_1(0) + T_1(2k_0)\big]. \tag{47}$$

For some given correlation coefficients of bottom roughness relatively simple analytical expressions for quantities $\gamma_{\mathrm{B}}^{(1)}$ and $\gamma^{(1)}$ can be derived from equations (45), (47). For example, in the case of an exponential correlation coefficient, $\Gamma_1(x) = \exp(-x/L)$, for which the corresponding spectrum is

$$T_1(k) = \frac{L}{\pi(1 + L^2 k^2)} \tag{48}$$

the expressions for the 'exact' and the approximate mean-field attenuation coefficients of the surface wave take the form

$$\gamma^{(1)} = \frac{8\langle \varepsilon^2 \rangle k_0^4 h_0^2 L (2k_0^2 L^2 + 1)}{(4k_0^2 L^2 + 1)(2k_0 h_0 + \sinh 2k_0 h_0)^2} \tag{49}$$

$$\gamma_{\mathrm{B}}^{(1)} = \frac{\langle \varepsilon^2 \rangle}{4}\, k_0^2\, L \left[v^2 + \frac{4u^2 + 4\,u\,v + v^2}{4k_0^2\, L^2 + 1} \right]. \tag{50}$$

As in the case of two-dimensional bottom irregularities the 'exact' and approximate mean-field attenuation coefficients coincide when the depth of the basin is small in comparison with the wavelength. For the exponential correlation coefficient the quantities $\gamma_B^{(1)}$ and $\gamma^{(1)}$ (provided $k_0 h_0 \ll 1$) are described by the formula

$$\gamma_B^{(1)} \approx \gamma^{(1)} \approx \frac{\langle \varepsilon^2 \rangle}{2} k_0^2 L \frac{1 + 2 k_0^2 L^2}{4 k_0^2 L^2 + 1}. \tag{51}$$

If the depth fluctuations are large scale, $k_0 L \gg 1$, then $T(2k_0) \ll T(0)$ and

$$\gamma_B^{(1)} \approx \gamma^{(1)} \approx \pi \langle \varepsilon^2 \rangle \left(\frac{2k_0^2 h_0}{2k_0 h_0 + \sinh 2k_0 h_0} \right)^2 T_1(0). \tag{52}$$

In the case of an exponential correlation coefficient it follows from equation (52) that

$$\gamma_B^{(1)} \approx \gamma^{(1)} \approx \langle \varepsilon^2 \rangle L \left(\frac{2k_0^2 h_0}{2k_0 h_0 + \sinh 2k_0 h_0} \right)^2. \tag{53}$$

For the deep water case in the framework of the Berkhoff model we obtain the following expression for the mean-field attenuation coefficient:

$$\gamma_B^{(1)} = 64 \langle \varepsilon^2 \rangle k_0^4 h_0^2 L \, \mathrm{e}^{-4k_0 h_0} \frac{k_0^2 h_0^2 + k_0^2 L^2}{4 k_0^2 L^2 + 1}. \tag{54}$$

Within the framework of the potential model the mean-field attenuation coefficient is described by the formula

$$\gamma^{(1)} = 32 \langle \varepsilon^2 \rangle \mathrm{e}^{-4k_0 h_0} \frac{k_0^4 L h_0^2 (2 k_0^2 L^2 + 1)}{4 k_0^2 L^2 + 1}. \tag{55}$$

Expressions (54) and (55) coincide only in the case of large-scale bottom irregularities (provided the inequality $k_0 L \gg k_0 h_0 \gg 1$ holds), and they diverge considerably for small-scale imperfections. It is necessary, however, to keep in mind that in the case of deep water, as expected, the effect of bottom roughness on the propagation of surface waves is exponentially small.

The results of numerical investigations of the dimensionless coefficients $\tilde{\gamma}^{(1)}$ and $\tilde{\gamma}_B^{(1)}$, which have been introduced similarly to equations (37), are presented in figure 2 for different values of μ. It can be seen that in the vicinity of extrema of quantities $\tilde{\gamma}^{(1)}$ and $\tilde{\gamma}_B^{(1)}$ the Berkhoff approximation only produces correct results when the bottom irregularities are relatively smooth. This fact is explained by the above-mentioned effect of the conservation of the vertical structure of the wave field. In the case of small-scale irregularities calculations based on the Berkhoff model lead to significant errors. So, in those cases where the vertical structure of the wave field varies, the Berkhoff approximation is of limited use.

6. Effects of bottom roughness on the propagation velocity of surface waves

Together with mean-field attenuation bottom irregularities cause propagation velocity variations of surface waves. Alternation of the real part of the effective propagation constant is obtained using Cauchy principal value integration in (21), (28) for the case of two-dimensional imperfections and in (44), (46) for the case of one-dimensional imperfections. However, such calculations are rather complicated. It is worth noting that for waves of different physical nature the approach, which uses the Fourier transform method for a solution of a set of equations for a mean field and a fluctuation field, makes it possible to obtain the mean-field attenuation coefficient relatively easily. As this takes place, in order to obtain the real part of the propagation constant it is necessary to calculate the principal values of the integrals appearing in the dispersion equation. Such calculations usually

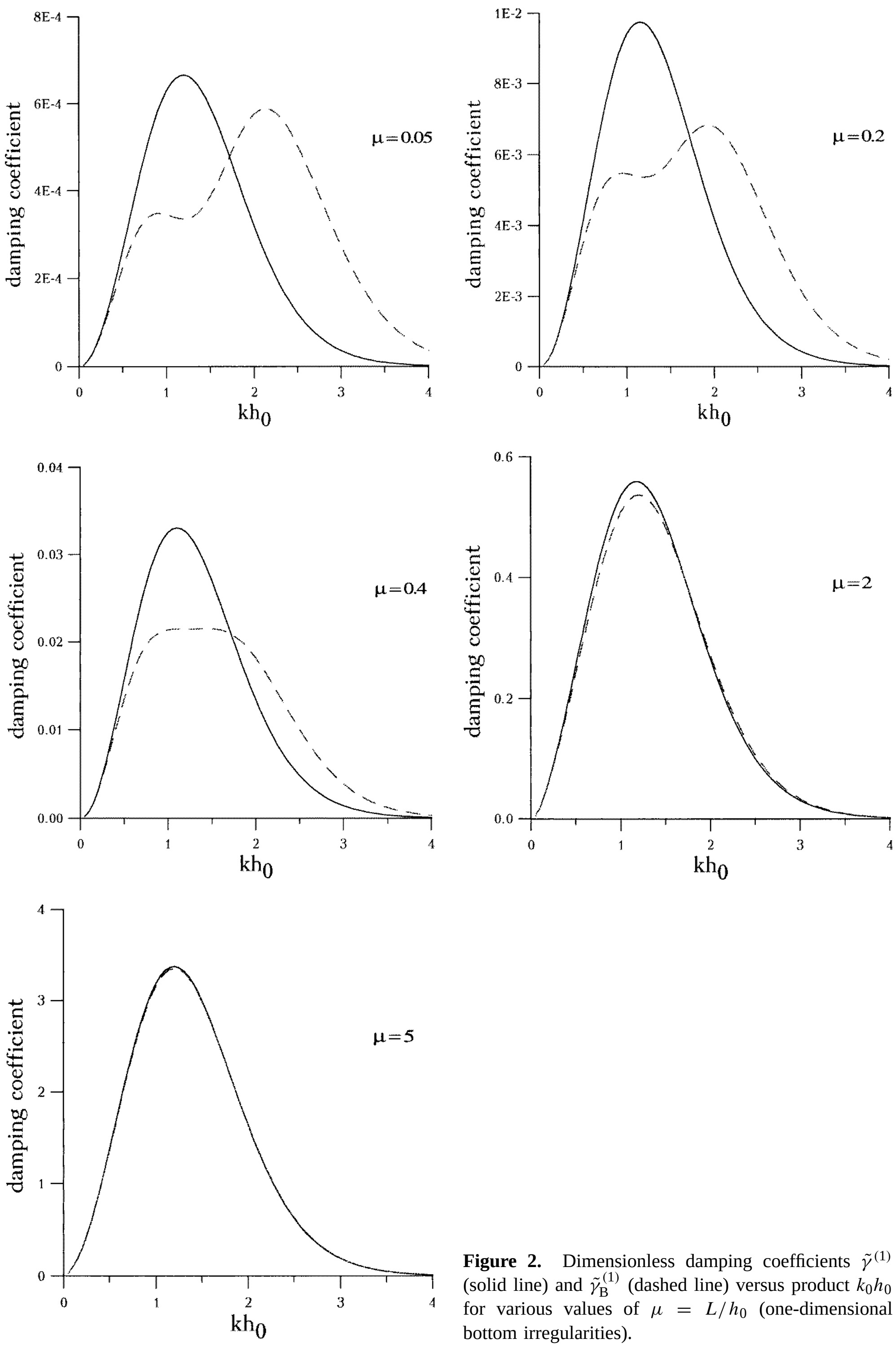

Figure 2. Dimensionless damping coefficients $\tilde{\gamma}^{(1)}$ (solid line) and $\tilde{\gamma}_{\mathrm{B}}^{(1)}$ (dashed line) versus product $k_0 h_0$ for various values of $\mu = L/h_0$ (one-dimensional bottom irregularities).

involve difficulties. The approach which applies the Green function method (the fluctuation field is expressed in terms of the mean field via a Green function and is inserted in the mean-field equation, which therefore becomes an integro-differential one) requires a larger amount of calculation. However, this approach makes it possible to obtain an expression for the real part of the effective propagation constant and the mean-field attenuation coefficient relatively easily. We shall herein apply the Green function method to investigate variations of the surface wave propagation velocity in a basin with one-dimensional depth fluctuations in the framework of the Berkhoff model. We omit description of the rather cumbersome calculations and give only the final expression for the effective propagation constant (this expression amounts to the solution of (44)):

$$q \simeq k_0 + \delta k_{\mathrm{B}} + \mathrm{i}\gamma_{\mathrm{B}}^{(1)}. \tag{56}$$

Here

$$\delta k_{\mathrm{B}} = \frac{1}{2}\langle\varepsilon^2\rangle k_0\, u^2 - \langle\varepsilon^2\rangle k_0^2 \left(u^2 + u\,v + \frac{v^2}{4}\right)\int\limits_0^\infty \Gamma(z)\ \sin 2k_0 z\ \mathrm{d}z \tag{57}$$

$$\gamma_{\mathrm{B}}^{(1)} = \frac{\langle\varepsilon^2\rangle}{4}\,k_0^2\left[v^2\int\limits_0^\infty \Gamma(z)\ \mathrm{d}z + (4u^2 + 4\,u\,v + v^2)\int_0^\infty \Gamma(z)\ \cos 2k_0 z\ \mathrm{d}z\right]. \tag{58}$$

It is evident that as the quantities $\Gamma_1(x)$ and $T_1(k)$ are connected by the relationship (42), equations (45) and (58) coincide.

Let us analyse equation (57). In the case of shallow water and an exponential correlation coefficient the expression for δk_{B} takes the form

$$\delta k_{\mathrm{B}} = k_0\,\langle\varepsilon^2\rangle\,\sigma \tag{59}$$

where

$$\sigma = \frac{3k_0^2\,L^2 + 1}{2(4k_0^2 L^2 + 1)}. \tag{60}$$

The quantity σ is always positive and decreases monotonically from $\sigma = 0.5$ (this value corresponds to the case of small-scale irregularities, $k_0 L \ll 1$) to $\sigma = 0.375$ in the opposite limiting case of large-scale imperfections, $k_0 L \gg 1$. This fact means that in the shallow-water case the surface waves propagate in a basin with a rough bottom on average more slowly than in a basin with an even bottom.

The same result follows from the shallow-water equations without use of the Berkhoff approximation. This result is important in explaining the increase in the travel time of a tsunami compared with the calculated travel time. The accuracy of tsunami travel time calculations and measurements are discussed in [18, 19].

In the case of deep water and an exponential correlation coefficient the quantity σ is described by the expression

$$\sigma \approx\ 32k_0^2 h_0^2 \mathrm{e}^{-4k_0 h_0}\,\frac{k_0^2 h_0^2 - 4k_0^3 h_0 L^2 - k_0^2 L^2}{4k_0^2 L^2 + 1}. \tag{61}$$

If the bottom irregularities are large, $k_0 L \gg k_0 h_0 \gg 1$, then the quantity σ is negative:

$$\sigma \approx -32\,k_0^3\,h_0^3\,\mathrm{e}^{-4k_0 h_0} \tag{62}$$

and the surface wave propagates on average faster than in a basin with an even bottom.

The results of calculations of the quantities $\sigma_{\mathrm{ph}} = c/c_0 - 1$ and $\sigma_{\mathrm{g}} = c_{\mathrm{g}}/c_0 g - 1$, which characterize the variations in the phase and group velocities of a surface gravity wave are

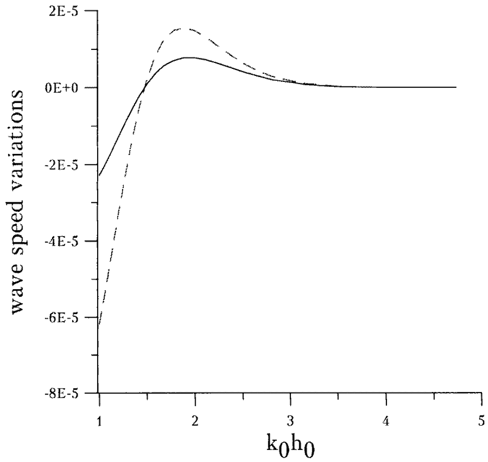

Figure 3. Wave speed variations due to bottom roughness for $\mu = 5$ and $\langle\varepsilon^2\rangle = 10^{-3}$. The solid line is $\sigma_{\rm ph}$, and the dashed line is $\sigma_{\rm g}$.

shown in figure 3 for $\langle\varepsilon^2\rangle = 10^{-3}$ and $\mu = L/h_0 = 5$. It can be seen that the effect of bottom roughness on surface wave velocity is very modest. The velocity variations are of the order of 10^{-5}, and, therefore, travel time variations do not exceed 1 or 2 s even for transoceanic paths. Thus we shall not discuss this effect in detail.

7. Conclusion

Results of analytical and numerical investigations of surface waves propagation in a basin with one- or two-dimensional random bottom irregularities have been presented. Two different approaches to the solution of the problem have been used. One of them is based on the refraction–diffraction Berkhoff model. The other uses a linearized set of equations of the hydrodynamics of an incompressible fluid, and, consequently, is more precise. Conditions have been established under which the Berkhoff approximation is applicable by comparing the exact and approximate solutions. It has been ascertained that the Berkhoff approximation is valid in situations where the vertical structure of the wave field practically does not vary. These are the cases of shallow water and arbitrary correlation length of bottom irregularities and that of large-scale imperfections and arbitrary depth of the basin. It has been shown that the mean-field attenuation coefficient of surface waves reaches its maximum value in the depth range $k_0h_0 \geqslant 1$. The location of this maximum is practically independent of the correlation length of bottom roughness. Variations in the propagation velocity of surface waves have been investigated. It has been shown that in shallow water surface waves propagate in a basin with a rough bottom on average more slowly than in a basin with an even bottom. In deep water with large-scale bottom irregularities surface waves propagate on average faster than in a basin with a perfect bottom. However, for small bottom irregularities the effect of surface wave velocity variations is very modest.

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