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The momentum of light in a refracting medium
II. Generalization. Application to oblique reflexion

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The expression for the momentum of a light wave in a refractive medium previously derived for the case of a sufficiently wide and short light pulse, is shown to be valid for a pulse of any shape, as long as its linear dimensions are large compared to the wavelength, and dispersion may be neglected. The 'prompt' momentum transfer to a mirror immersed in a refractive fluid by the reflexion of a light pulse incident obliquely, with the electric vector in the plane of incidence, (4.7), is shown to differ appreciably from that which would follow from Minkowski's theory. The momentum deposited in the medium is also determined, and its subsequent fate discussed. If the mirror is large enough, it also receives an amount of momentum equal to the 'deposited' momentum, and then the total received by the mirror equals the momentum change of the light.

1. Introduction

A recent paper by the author (Peierls 1976, referred to as Paper I)† discussed the momentum of a light wave in a refractive medium. The result, equation (2.12), was derived assuming a wide and short light pulse. It is different both from Minkowski's and from Abraham's expression. In the present paper it will be shown that the

† The author regrets that a number of errors in the equations of I were left uncorrected: equation (3.11) should have a semicolon after the $\frac{1}{5}$, to read

$$\frac{\partial^2\chi}{\partial z^2} = \frac{1}{5}; \quad \frac{\partial^2\chi}{\partial x\,\partial z} = \frac{1}{5}ik.$$

(3.22) similarly should have a semicolon after a_1, and in the last term a_2 should be replaced by a_1:

$$\frac{(\epsilon-\epsilon_0)^2}{\epsilon_0}\tau = -\tfrac{1}{2}a_1; \quad \frac{(\epsilon-\epsilon_0)^2}{\epsilon_0}\sigma = \frac{(\epsilon-\epsilon_0)^2}{3\epsilon_0} + \tfrac{1}{3}a_1.$$

In the line above equation (2.5), $(\epsilon_0-\epsilon_0)$ should read $(\epsilon-\epsilon_0)$. In the last integral of (3.9), the lower limit should be r, not 1.

The last term inside the square bracket of (5.17) should be -1, instead of -2. In (5.18) the second term in the bracket should read $-2n_1^4$, instead of $+1$, making (5.18) come to

$$\Delta G = \frac{2\sigma\mathscr{E}}{c}\frac{n_1(n_1-n_2)}{n_1+n_2}(2-n_1^2-n_2^2).$$

I am grateful to Dr Kenneth Young, of the Chinese University of Hong Kong, for pointing out the last two errors.

restriction is unnecessary, and that the formula for the momentum is independent of the shape of the pulse.

Paper I also considered a number of relevant experiments. These do not give the momentum of the light directly, because of the presence of acoustic pulses, which affect the momentum balance. In many of these experiments the correct answer is obtained by considering the balance of pseudo-momentum which, according to a theorem by Gordon (1973), is conserved in a fairly wide range of circumstances. One therefore obtains, in many circumstances, the same result as if there were no acoustic pulses, and the momentum of the light was given by Minkowski's expression. The present paper will give a simple example in which the experiment should give a different answer from the simple Minkowski result, and in which it might even be possible to test the assumptions made about the response of the atoms in the medium to electric fields.

The experiment which will be discussed consists in measuring the impulse given to a mirror by a light pulse reflected from it, as in the experiment of Jones & Richards (1954). Their work gave the 'simple' Minkowski answer, and this was explained by Gordon in terms of pseudo-momentum conservation for reflexion at normal incidence. However, Gordon's theorem requires that the electric vector vanish at the reflector surface. This is true in the case of normal incidence, but not for oblique incidence. It follows therefore that the same experiment, done at oblique incidence, might give an answer different from the 'simple' Minkowski result.

We shall calculate the momentum imparted to the mirror by a light pulse of finite width and length (but both large compared to the wavelength). The calculation will be done for the limiting case of an infinitely extended plane mirror. This is, of course, unrealistic, but the calculation of the edge effects for a finite mirror, particularly the hydrodynamic part of the problem, would be very difficult.

The force on the mirror consists of two parts: an instantaneous force acting during the reflexion of the light, and a delayed force due to the acoustic pulses which eventually reach the mirror. We shall give results for both parts. The instantaneous force will probably apply also to the case of a finite mirror, provided only that the mirror is wider than the light pulse. However, some of the acoustic pulses are likely to miss a small mirror, so that the expected total impulse is then somewhere between the results including and excluding the delayed part.

2. Mechanical momentum for general shape of pulse

We start from equation (2.8) of paper I for the force density. By using Maxwell's equations this can be put in the form

$$\boldsymbol{f} = \epsilon_0 \tau (n^2-1)^2 (\boldsymbol{E}.\nabla)\,\boldsymbol{E} + \tfrac{1}{2}\epsilon_0[(n^2-1)+\sigma(n-1)^2]\,\nabla E^2 + \epsilon_0(n^2-1)\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B}). \quad (2.1)$$

Here σ and τ are numerical coefficients whose significance is discussed in paper I.

That discussion envisaged a medium of the Clausius–Mosotti type, i.e. a medium in which an electric field induces dipoles whose linear extension is small compared to their distances from each other. It also assumed that the atomic polarizability is independent of the density. For such a medium it was shown in paper I that σ and τ are both equal to $\frac{1}{3}$.

This, of course, is not the most general situation for real media. If the linear dimensions of the atomic or molecular structure carrying the dipole moments are comparable with their spacing the argument for the particular values of σ and τ no longer holds, but we can still justify an expression of the form (2.1). However, σ and τ may now depend on the medium, and for a particular medium, may also vary upon compression. There is then no special advantage in a definition of the coefficients which includes an explicit dependence on the refractive index n, if the coefficients themselves can vary with n. An alternative form for (2.1) can always be written in terms of the coefficients a_1 and a_2 introduced by Landau & Lifshitz (1960) in terms of the variation of the dielectric tensor with strain:

$$\boldsymbol{f} = -\tfrac{1}{2}a_1(\boldsymbol{E}.\boldsymbol{\nabla})\,\boldsymbol{E} - \tfrac{1}{2}[\epsilon_0(n^2-1)+a_2]\,\boldsymbol{\nabla}E^2 + \epsilon_0(n^2-1)\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B}). \tag{2.1a}$$

The relation between the two sets of coefficients is then

$$a_1 = -2\tau(n^2-1)^2\epsilon_0, \quad a_2 = -(n^2-1)\,\epsilon_0 - \sigma(n^2-1)^2\,\epsilon_0. \tag{2.2}$$

The form (2.1) has the advantage of expressing the fact that, for small n^2-1, the effective field gradient can be identified with the gradient of the Maxwell field, since any deviation must be due to the presence of polarization, and its effect on the force must therefore be of second order in n^2-1.

If we leave the coefficients σ and τ, or alternatively a_1 and a_2, arbitrary, the relations are applicable to any linear non-magnetic medium in which dispersion is negligible.

Now consider a light pulse of any shape, of dimensions large compared to the wavelength, so that the spreading and change of shape may be neglected, as well as the curvature of the surfaces of constant phase. We may then assume the pulse to travel in the z direction with its electric vector approximately in the x direction. (An example of such a field is written out in §3.) If such a pulse advances into an undisturbed medium at rest, the mechanical momentum density at a point $\boldsymbol{r}$ and at time t is given by

$$g^{\text{mech}} = \int_{-\infty}^{t} \mathrm{d}t' \boldsymbol{f}(\boldsymbol{r},t'). \tag{2.3}$$

Consider first the longitudinal momentum, g_z. The first term in (2.1) has no z component. In the second term we may, on the stated assumptions, replace $\partial/\partial z$ by $(n/c)\partial/\partial t$. Carrying out the time integration in (2.2) and using the relation $E_x = (c/n)\,B_y = (1/nc\epsilon_0)\,H_y$, we find

$$g_x^{\text{mech}} = \frac{1}{2c}2[(n^2-1)-\sigma(n^2-1)^2]\,(\boldsymbol{E}\times\boldsymbol{H})_z \tag{2.4}$$

as in (I, 2.11). This expression for the longitudinal momentum, and hence the relation (2.12) of paper I, is therefore valid for a pulse of any shape, including a long narrow one. It is clear that the longitudinal mechanical momentum density is again reduced to zero when the pulse has passed.

Next consider a transverse component, g_x^{mech} or g_y^{mech}. Now the third term in (2.1) makes no contribution, and we find

$$\left.\begin{aligned} g_x^{\text{mech}} &= \int_{-\infty}^{t} \mathrm{d}t' \tfrac{1}{2}\epsilon_0[n^2-1+(\sigma+\tau)(n^2-1)^2]\frac{\partial}{\partial x}E_x^2(r,t'), \\ g_y^{\text{mech}} &= \int_{-\infty}^{t} \mathrm{d}t' \tfrac{1}{2}\epsilon_0[(n^2-1)+\sigma(n^2-1)^2]\frac{\partial}{\partial y}E_x^2(r,t'). \end{aligned}\right\} \tag{2.5}$$

Both these quantities are space derivatives, so that their integral over space vanishes. They do not therefore contribute to the total momentum, which is

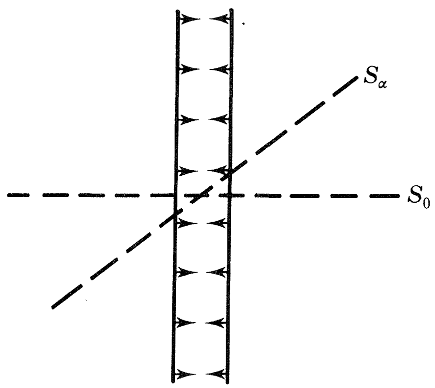

FIGURE 1. Momentum distribution after the passage of a narrow light beam. The surfaces S_0 and S_α relate to the discussion in § 5.

entirely longitudinal. However, at any given point in space, the integral does not tend to zero after a long time, so that a local momentum density remains after the passage of the pulse. The direction of this momentum is inwards from the edge of the 'trail' of the light pulse, though not normal to the edge, because of the different coefficients in the x and y components of (2.4). The situation after the passage of the pulse is shown schematically in figure 1. The subsequent behaviour depends, of course, on the mechanical properties of the medium. In general, these local momenta will be dissipated by acoustic signals. In principle this phenomenon is common to liquids and solids. In the latter there is additionally the transverse sound signal mentioned in § 7 of paper I.

3. Oblique reflexion: the electromagnetic field

We consider a light pulse of finite width and length, incident at angle α on an infinite plane mirror at $z = 0$. We assume the electric vector to be in the plane of incidence, since this is the situation to which Gordon's theorem does not apply. This suggests the following form for the field quantities:

$$\left.\begin{aligned} E_x &= I\left\{\left[-\cos\alpha+\frac{\mathrm{i}\gamma}{k}(x\cos\alpha+z\sin\alpha)\sin\alpha\right]\mathrm{e}^{\xi_+}\right.\\ &\qquad\left.+\left[\cos\alpha-\frac{\mathrm{i}\gamma}{k}(x\cos\alpha-z\sin\alpha)\sin\alpha\,\mathrm{e}^{\xi_-}\right]\right\},\\ E_y &= 0,\\ E_z &= I\left\{\left[-\sin\alpha-\frac{\mathrm{i}\gamma}{k}(x\cos\alpha+z\sin\alpha)\cos\alpha\right]\mathrm{e}^{\xi_+}\right.\\ &\qquad\left.+\left[-\sin\alpha-\frac{\mathrm{i}\gamma}{k}(x\cos\alpha-\mathrm{i}z\sin\alpha)\cos\alpha\right]\mathrm{e}^{\xi_-}\right\}, \end{aligned}\right\} \tag{3.1}$$

$$\left.\begin{aligned} B_x &= -I\frac{\mathrm{i}n}{c}\frac{\gamma}{k}y\sin\alpha\,(\mathrm{e}^{\xi_+}\,\mathrm{e}^{\xi_-}),\\ B_y &= I\frac{n}{c}(\mathrm{e}^{\xi_+}+\mathrm{e}^{\xi_-}),\\ B_z &= I\frac{\mathrm{i}n}{c}\frac{\gamma}{k}y\cos\alpha(\mathrm{e}^{\xi_+}-\mathrm{e}^{\xi_-}), \end{aligned}\right\} \tag{3.2}$$

where the exponents are defined by

$$\begin{aligned} \xi_+ &= \mathrm{i}k\left(\mp z\cos\alpha+x\sin\alpha-\frac{c}{n}t\right)\\ &\quad-\frac{\gamma}{2}(x\cos\alpha\pm z\sin\alpha)^2-\tfrac{1}{2}\gamma y^2-\tfrac{1}{2}\nu\left(\mp z\cos\alpha+x\sin\alpha-\frac{c}{n}t\right)^2. \end{aligned} \tag{3.3}$$

γ and ν are parameters which determine the dimensions of the pulse; apart from numerical factors $\gamma^{-\frac{1}{2}}$ is the width of the pulse (assumed the same in the plane of incidence as perpendicular to it, for simplicity) and $\nu^{-\frac{1}{2}}$ is the length.

We assume the width and length of the pulse very large compared to the wavelength, so that the ratios γ/k^2 and ν/k^2 are small and we shall retain in each result only the leading term in these small ratios.

It is easy to verify that the fields (3.1) and (3.2) satisfy Maxwell's equations and the boundary condition of the conducting surface, except for terms containing the factor k^{-2}. Note that the field on the mirror is appreciable over a horizontal distance of the order

$$x_{\max}\sim\frac{1}{\cos\alpha\sqrt{\gamma}}.$$

If we wish to consider our calculation as an approximate treatment for a finite mirror, the width b of the mirror should be greater than $x_{\max}$, so that

$$\gamma b^2 \cos^2\alpha > 1. \tag{3.4}$$

The total energy carried by the incident pulse, which consists of the first term on the right hand side in each of the equations (3.1) and (3.2), is

$$W_{in} = \epsilon_0 I^2 \frac{\pi n^2}{2\gamma} \sqrt{\left(\frac{\pi}{\nu}\right)}. \tag{3.5}$$

4. The 'prompt' impulse on the mirror

The normal stress on the mirror is given by

$$-\sigma_{zz} + p, \tag{4.1}$$

where σ_{zz} is a component of the electromagnetic stress tensor, and p the pressure in the liquid medium. The first depends on the field quantities, and therefore is of very short duration. It lasts a time of the order $n/c\sqrt{\nu}$. The pressure, on the other hand, is propagated with the velocity of sound, and since the force which gives rise to the pressure pulse is spread over a region extending a distance of the order of $1/\gamma^{\frac{1}{2}} \sin\alpha$ or $1/\nu^{\frac{1}{2}} \cos\alpha$, whichever is less, the bulk of it will not reach the mirror until a slightly later time. In the case of a small mirror, the fraction (if any) of this pressure pulse which reaches the mirror is not easy to estimate, and depends on the geometry of the apparatus.

The electromagnetic stress consists of an electric and a magnetic part given by Landau & Lifshitz (1960, eq. (16.4) and (34.2)) respectively. These equations include the mechanical stress, which should be omitted. In the electric part we use the expression for an isotropic solid rather than that for a liquid, for reasons given in the previous paper (Peierls 1976, §3). The coefficients a_1 and a_2 are given by (2.2). We again assume a non-magnetic medium, and therefore put $\mu = \mu_0$. We then find, in SI units:

$$\sigma_{ik}^{\text{e.m.}} = \epsilon_0 \{[n^2 + \tau(n^2-1)^2] E_i E_k - [1 - \sigma(n^2-1)^2] \tfrac{1}{2} E^2 \delta_{ik} + c^2 B_i B_k - \tfrac{1}{2} c^2 B^2 \delta_{ik}\}. \tag{4.2}$$

We require this stress for $z = 0$. There E_x and E_y vanish because of the boundary condition, and B_x and B_z are of order $1/k$, and therefore negligible. This leaves

$$(\sigma_{zz}^{\text{e.m.}})_{z=0} = \epsilon_0 \{[n^2 - \tfrac{1}{2} + (\tau + \tfrac{1}{2}\sigma)(n^2-1)^2] E_z^2 - \tfrac{1}{2} c^2 B_y^2\}. \tag{4.3}$$

The field quantities are given by the real parts of (3.1) and (3.2). We use the usual rule by which, on the time average,

$$\langle fg \rangle = \tfrac{1}{2} \operatorname{Re} \langle f^* g \rangle. \tag{4.4}$$

This rule is strictly correct only for sinusoidal fields, but is a sufficient approximation in our case, since the oscillatory factor is assumed to vary much faster than the secular ones. Noting also that, for $z = 0$, $\xi_+ = \xi_-$, we find, for the time average of (4.3)

$$\epsilon_0 I^2\{[(2n^2-1)+(2\tau+\sigma)(n^2-1)^2]\sin^2\alpha - n^2\} \times \exp\left\{-\gamma\left(x^2\cos^2\alpha + y^2 - \nu\left(x\sin\alpha - \frac{c}{n}t\right)^2\right\}. \quad (4.5)$$

The 'prompt' momentum imparted to the mirror is

$$p_p = \int (\sigma_{zz}^{\text{e.m.}})_{z=0}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t. \quad (4.6)$$

Integrating (4.5) and using (3.5), we find

$$p_p = \left\{\frac{2n}{c}\cos\alpha - \frac{2(n^2-1)}{nc}[1+(\sigma+2\tau)(n^2-1)]\frac{\sin^2\alpha}{\cos\alpha}\right\} W_{in}. \quad (4.7)$$

For normal incidence, $\alpha = 0$, this agrees with the 'naive' Minkowski result, and therefore with the experiment of Jones and Richards, as expected. For vacuum, $n = 1$, it gives the familar result $2W_{in}\cos\alpha/c$, for any α. For $n \neq 1$, $\alpha \neq 0$, however, the result is very different from the Minkowski formula. Observations at oblique incidence are therefore capable of verifying the arguments of the present papers. It would also be interesting to see whether one could in this way distinguish media for which the Clausius–Mosotti model is applicable, i.e. for which $\sigma = \tau = \frac{1}{5}$, from those where it is not.

For $n \neq 1$, our result (4.7) becomes singular for glancing incidence, $\alpha = \frac{1}{2}\pi$. In this limit, however, our calculation is valid only for an *infinite* mirror, because of the requirement (3.4). There is nothing unreasonable in the fact that a light pulse travelling along an infinite mirror at a glancing angle, (so that it remains in contact with the mirror for an unlimited time) should transfer to it an unlimited amount of momentum. In that limit the momentum must be balanced by an equally infinite and opposite amount transferred to the liquid, and we shall verify in §5 that this is the case.

5. The momentum deposited in the liquid and the delayed force on the mirror

We next determine the momentum deposited in the liquid near the mirror. This is done most easily by using the equation of momentum conservation (Landau & Lifshitz 1960, eq. (56.17)) in the form

$$f_k = \sum_i \frac{\partial}{\partial x_i}\sigma_{ik}^{\text{e.m.}} - \frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B})_k, \quad (5.1)$$

where f_k is the kth component of the force density, and $\sigma_{ik}^{\text{e.m.}}$ is again the electromagnetic stress (4.2). The hydrostatic pressure has again been omitted; this is justified for short times, of the order of the duration of the light pulse, since the motion, and hence the deformation, of the fluid during such short times is negligible.

The momentum density is the time integral of (5.1), and if this is taken over a period including the passage of the light pulse, the last term makes no contribution. The space integral of the first term can be expressed as a surface integral over the boundary of the volume in question so that we have

$$P_z^{\text{dep}} = \int_{-\infty}^{\infty} \mathrm{d}t \int_V \mathrm{d}^3 \boldsymbol{r} f_z = \int_{-\infty}^{\infty} \mathrm{d}t \int_S \mathrm{d}^2 \boldsymbol{r} \sigma_{nz}^{\text{e.m.}}, \tag{5.2}$$

where S is the surface bounding the volume V, and n indicates the component in the direction of the normal to S.

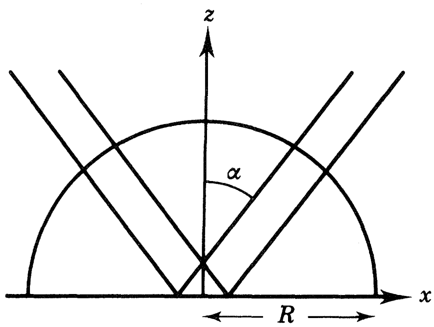

FIGURE 2. The integration region (5.3).

It would seem natural to choose for the volume V a slab parallel to the reflector, $0 < z < h$, with some suitably large h. However, this would give a misleading answer, as can be seen by reference to figure 1. As was shown in § 2, the passage of the light pulse leaves in the medium a momentum distribution, along the surface of the 'trail', with the momentum density equal and opposite on opposite sides of the trail. This momentum distribution therefore makes no contribution to the total momentum. If we consider only the momentum in a half-space bounded by a plane at right angles to the direction of propagation, as in S_0 in figure 1, the transverse momenta also cancel separately in each half-space.

However, if the dividing surface is inclined to the direction of propagation, as is S_α in figure 1, there are parts of the transverse momentum on one side of the dividing surface, whose counterparts lie on the other side. If in our reflexion problem we choose an integration volume bounded by a plane parallel to the reflector, this intersects the incoming and the reflected light signals obliquely, and therefore contains a spurious contribution to the momentum in V, which remains finite for infinite h. This difficulty can be avoided if we choose for the volume V a half-cylinder of large radius R:

$$z > 0, \quad x^2 + z^2 < R^2 \tag{5.3}$$

which intersects the incoming and outgoing pulses normally (figure 2).

The contribution to (5.2) from the plane surface, $z = 0$, is then the 'prompt' impulse on the mirror (equation (4.7)).

The contribution from the curved surface is easily evaluated, provided R is large enough for the incident and reflected parts of the light pulse to be separated in time ($R \gg \gamma^{-\frac{1}{2}}$). In that case the cross products of the ingoing and outgoing parts of (2.1) and (2.2) are negligible. Assuming also that R is a large compared to the width of the pulse, $R \gg \gamma^{-\frac{1}{2}}$, the normal n can be taken to form the angle α with the z direction.

The z component of the momentum transport through the curved surface is now easily evaluated and comes to

$$1/c[(n^2+1)-\sigma(n^2-1)^2]\, W_{in} \cos\alpha \tag{5.4}$$

which is just the balance of momentum of the incident and reflected pulse according to equation (2.2) of paper I.

The deposited momentum is the difference between (3.8) and (4.6)

$$P_z^{\text{dep}} = -\left\{[(n-1)^2-\sigma(n^2-1)^2]\cos\alpha + \frac{2(n^2-1)}{n}[1+(\sigma+2\tau)(n^2-1)]\frac{\sin^2\alpha}{\cos\alpha}\right\}\frac{W_{in}}{c}. \tag{5.5}$$

The question of the further propagation of this momentum is now dependent on the geometry. It will propagate according to hydrodynamics, and it is plausible that half of it will travel in the positive and half in the negative z direction. This conjecture will be verified in the appendix.

It follows therefore that one-half of the momentum (5.5) reaches the mirror in the form of a sound pulse. Assuming the acoustic impedance of the mirror to be high compared to that of the liquid, the signal will be reflected, with the reflected signal carrying opposite and equal momentum. In that case the momentum transmitted to the mirror is then the whole of (5.5), in addition to the prompt impulse (4.7), so that the momentum change of the mirror equals the momentum change of the light signal.

The statements in the last paragraph are, however, strongly dependent on the assumption of a large mirror. In practice one would be likely to use a small mirror in order to facilitate detection of the impulse. If the mirror exceeds the width of the light pulse, our result for the prompt impulse is still likely to be a good approximation. In general the length of the light pulse is, however, likely to exceed the width of the mirror substantially, and in that case some of the acoustic signal will miss the mirror. The determination of the exact fraction of (5.5) received by the mirror involves a calculation of considerable complexity, which will not be undertaken here, particularly because it must be based on a full specification of the geometry.

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Appendix: propagation of the residual momentum

The hydrodynamic equations for the liquid medium are, to first order in the velocity, and neglecting viscosity:

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \nabla p = \boldsymbol{f}, \tag{A 1}$$

$$\frac{\partial p}{\partial t} + \rho \nabla . \boldsymbol{u} = 0, \tag{A 2}$$

where ρ is the liquid density, $\boldsymbol{u}$ the velocity, p the pressure, and $\boldsymbol{f}$ the force density due to the electromagnetic field (given by (5.1)). If we write

$$\rho = \rho_0(1+\phi), \tag{A 3}$$

where ρ_0 is the normal density, and if

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = s^2 \tag{A 4}$$

so that s is the sound velocity, we have

$$\left.\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + s^2 \nabla \phi &= \boldsymbol{f}, \\ \frac{\partial \phi}{\partial t} + \nabla . \boldsymbol{u} &= 0. \end{aligned}\right\} \tag{A 5}$$

These equations are to be solved with the initial conditions

$$\boldsymbol{u} = 0, \quad \phi = 0, \quad t = -\infty \tag{A 6}$$

and the boundary condition

$$u_z = 0, \quad z = 0. \tag{A 7}$$

We can replace (A 7) by extending the definition of all quantities to negative z, with the convention

$$\left.\begin{matrix} u_x, u_y, f_y, f_y, \phi & \text{are even in } z, \\ u_z, f_z & \text{are odd in } z. \end{matrix}\right\} \tag{A 8}$$

We introduce Fourier transforms:

$$\left.\begin{aligned} \boldsymbol{u}(\boldsymbol{r},t) &= \int \mathrm{d}^3\boldsymbol{k}\, \mathrm{d}\omega \boldsymbol{v}(\boldsymbol{k},\omega)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{r}-\mathrm{i}\omega t}, \\ \phi(\boldsymbol{r},t) &= \int \mathrm{d}^3\boldsymbol{k}\, \mathrm{d}\omega \mu(\boldsymbol{k},\omega)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{r}-\mathrm{i}\omega t}, \\ \boldsymbol{f}(\boldsymbol{r},t) &= \int \mathrm{d}^3\boldsymbol{k}\, \mathrm{d}\omega \boldsymbol{g}(\boldsymbol{k},\omega)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{r}-\mathrm{i}\omega t}. \end{aligned}\right\} \tag{A 9}$$

Then (A 5) becomes

$$\left.\begin{aligned} -\mathrm{i}\boldsymbol{v} + \mathrm{i}s^2\boldsymbol{k}\mu &= \boldsymbol{g}, \\ -\mathrm{i}\mu + \mathrm{i}\boldsymbol{k}.\boldsymbol{v} &= 0 \end{aligned}\right\} \tag{A 10}$$

with the solution

$$\mu = -\frac{\mathrm{i}\boldsymbol{k}.\boldsymbol{g}}{\rho_0(a^2 - s^2k^2)}. \tag{A 11}$$

(Strictly speaking, we should add a small imaginary quantity in the denominator, to guide the integration of (A 9) at the pole of the integrand in conformity with the initial condition (A 6). However, this makes no difference to our use of (A 11).)

We require the total impulse on the mirror, which is minus the time and area integral of the pressure:

$$\begin{aligned} P &= -\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,p(x,y,0,t) \\ &= -(2\pi)^3 \rho_0 s^2 \int \mu(0,0,k_z,0)\,\mathrm{e}^{\mathrm{i}k_z z}\,\mathrm{d}k_z. \end{aligned} \tag{A 12}$$

From (A 11)

$$P = -(2\pi)^3\,\mathrm{i}\int \mathrm{d}k_z \frac{1}{k_z} g_z(0,0,k_z,0).$$

Inverting (A 9) we find

$$g_z(0,0,k_z,0) = (2\pi)^{-4}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t\,f_z(x,y,z,t)\,\mathrm{e}^{\mathrm{i}k_z z}$$

so that

$$P = -\frac{\mathrm{i}}{2\pi}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t\int \frac{\mathrm{d}k_z}{k_z}\mathrm{e}^{\mathrm{i}k_z z} f_z(x,y,z,t). \tag{A 13}$$

We know from (A 8) that f_z is odd in z, so the even part of $\mathrm{e}^{\mathrm{i}k_z z}$ gives no contribution, and the k_z integration reduces to

$$\mathrm{i}\int \frac{\sin k_z z}{k_z}\,\mathrm{d}k_z = \begin{cases} \mathrm{i}\pi, & z > 0, \\ -\mathrm{i}\pi, & z < 0. \end{cases}$$

Therefore (A 13) becomes

$$\begin{aligned} P &= \frac{1}{2}\int_{z>0} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t[f_z(x,y,z,t) - f_z(x,y,-z,t)] \\ &= \int_{z>0} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t\,f_z(x,y,z,t). \end{aligned} \tag{A 14}$$

The last equality again uses the odd character of f_z.

The total impulse on the mirror (counting it as positive in the positive z direction, away from the mirror) equals the total momentum deposited in the liquid, as stated in the text.

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