

On the Weakly Nonlinear Ekman Layer: Thickness and Flux

JOSEPH PEDLOSKY

Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

(Manuscript received 24 April 2007, in final form 30 October 2007)

ABSTRACT

The first-order effects of nonlinearity on the thickness and frictionally driven flux in the Ekman layer are described for the case of an Ekman layer on a solid, flat plate driven by an overlying geostrophic flow as well as the Ekman layer on a free surface driven by a wind stress in the presence of a deep geostrophic current. In both examples, the fluid is homogeneous. Particular attention is paid to the effect of nonlinearity in determining the thickness of the Ekman layer in both cases. An analytical expression for the Ekman layer thickness as a function of Rossby number is given when the Rossby number is small. The result is obtained by insisting that the perturbation expansion of the Ekman problem in powers of the Rossby number remains uniformly valid. There are two competing physical effects. The relative vorticity of the geostrophic currents tends to reduce the width of the layer, but the vertical velocity induced in the layer can fatten or thin the layer depending on the sign of the vertical velocity. The regularized expansion is shown to give, to lowest order, expressions for the flux in agreement with earlier calculations.

1. Introduction

The theory of the Ekman layer is central to geophysical fluid dynamics and its applications to both oceanic and atmospheric phenomena (Pedlosky 1987). The fundamental theory is linear and so it is not surprising that much effort has gone into extending the theory into the nonlinear domain. Some of the early work is reviewed by Greenspan (1968). See also Benton et al. (1964). More recent analysis can be found in the work of Niiler (1974), Brink (1997), Hart (2000), and Thomas and Rhines (2002). Illuminating as these studies are, what is lacking is a clear and simple analytical formulation of the alteration in the thickness of the Ekman layer as a consequence of nonzero Rossby numbers, that is, of nonnegligible nonlinearity. In this note such a prediction is given for the two classical Ekman layer problems: the frictional layer satisfying the no-slip condition beneath a geostrophic inviscid interior and the stress-driven boundary layer on a free surface.

Since in linear theory the boundary layer thickness is $\delta = (2\nu/f)^{1/2}$, where ν is the kinematic viscosity (or its

turbulent equivalent) and f is the planetary vorticity, one might expect that the first effect of nonlinearity would be to replace f with the total vorticity, $f + \zeta$, where ζ is the relative vorticity. A positive relative vorticity would tend to make the boundary layer thinner. However, in the presence of a convergent frictional flux in the boundary layer due to that relative vorticity and the consequent vertical velocity, the boundary layer would be stretched and fattened by the vertical advection. So, the two effects of the relative vorticity are competing. What is the result? That is the object of the present study.

In section 2, the problem for the Ekman layer that satisfies the no-slip condition on a flat plane beneath a geostrophic current is formulated. Section 3 discusses the perturbation expansion and the condition that it remain uniformly valid in space and derives the correction to the layer thickness by removing secular terms in the perturbation expansion. After their removal the next-order corrections to the Ekman velocities are calculated and the order-Rossby number correction to the Ekman layer flux and vertical pumping velocity is calculated. Earlier studies, such as Hart (1995, 1996), in addition to the studies already mentioned earlier, have generally employed regular asymptotic expansions without concern about the uniformity of convergence.

Corresponding author address: Joseph Pedlosky, Woods Hole Oceanographic Institution, Woods Hole, MA 02543.
E-mail: jpedlosky@whoi.edu

Although this allows an accurate calculation of the *integrated* effects of the nonlinearity on, for example, the pumping, they cannot provide a clear picture of the role of the nonlinearity on the structure of the layer, in particular its thickness.

Section 4 describes a similar procedure for the stress-driven problem. In the first problem it is the vertical velocity in the Ekman layer that is the dominant factor in altering the linear Ekman layer thickness. In the stress-driven problem there is a competition between the vertical velocity produced by the wind stress curl and the relative vorticity of the underlying geostrophic current.

2. The Ekman layer beneath a geostrophic flow

The boundary layer on a flat plane with a geostrophic flow in the x direction far from the plate is considered. The plate is at $z = 0$. The geostrophic flow is taken to be a function of y the coordinate across the current but, for simplicity, all variables are assumed to be independent of the x direction. The horizontal scales are non-dimensionalized with L , the characteristic scale of variation of the current, and the linear Ekman layer thickness $(2\nu/f)^{1/2}$ is chosen as the vertical scale. The velocities are scaled with the characteristic value U_0 of the geostrophic current. The vertical coordinate is ζ —that is, the vertical coordinate scaled by the Ekman layer thickness. It is useful to also consider a “slow” spatial variable related to ζ to account for the effects of nonlinearity. That coordinate is defined as $Z = \varepsilon\zeta$. All of the variables in the boundary layer are then considered to be functions of both ζ and Z so that the vertical derivatives transform as

$$\frac{\partial}{\partial \zeta} \rightarrow \frac{\partial}{\partial \zeta} + \varepsilon \frac{\partial}{\partial Z}. \tag{2.1}$$

The steady equations of motion then become, to first order in Rossby number,

$$\varepsilon(vu_y + Wu_\zeta) - v = \frac{1}{2}(u_{\zeta\zeta} + 2\varepsilon u_{\zeta Z} + Eu_{yy}), \tag{2.2a}$$

$$\varepsilon(vv_y + Wv_\zeta) + u = -\partial p/\partial y + \frac{1}{2}(v_{\zeta\zeta} + 2\varepsilon v_{\zeta Z} + Ev_{yy}), \tag{2.2b}$$

$$\varepsilon E(vW_y + WW_\zeta) = -\partial p/\partial \zeta + \frac{1}{2}E(W_{\zeta\zeta} + 2\varepsilon W_{\zeta Z} + EW_{yy}), \text{ and} \tag{2.2c}$$

$$v_y + W_\zeta + \varepsilon W_Z = 0, \tag{2.2d}$$

where

$$\varepsilon = U_0/fL \quad \text{and} \quad E = (\delta/L)^2.$$

The vertical velocity has been scaled with $(\delta/L)U$; that is, the relation between dimensional and nondimensional velocities is

$$(u^*, v^*, w^*) = U_0[u, v, (\delta/L)W]. \tag{2.3}$$

At $\zeta = 0$, all velocities vanish and for large vertical coordinate the horizontal velocity smoothly merges with $U(y)$. Far from the boundary layer, in the interior, v is $O(E)$, and so far from the boundary the vertical velocity must be independent of vertical coordinate.

The Rossby number ε is assumed to be small, and a perturbation expansion is sought in the form, for all variables,

$$u = u_0 + \varepsilon u_1 + \dots \tag{2.4}$$

The lowest-order problem is the linear problem whose solution is

$$u_0 = U - A(y, Z)e^{-\zeta} \cos \zeta + B(y, Z)e^{-\zeta} \sin \zeta, \\ v_0 = A(y, Z)e^{-\zeta} \sin \zeta + B(y, Z)e^{-\zeta} \cos \zeta. \tag{2.5}$$

Note that the “constants” A and B in the solution are functions of both y and Z . On the lower boundary one has the conditions (suppressing the dependence on y)

$$A(0) = U \quad \text{and} \quad B(0) = 0. \tag{2.6}$$

The vertical velocity is obtained from (2.2d) at lowest order in Rossby number, yielding

$$W_0 = C(Z) + \frac{1}{2} \frac{\partial A}{\partial y} e^{-\zeta} (\sin \zeta + \cos \zeta) \\ + \frac{1}{2} \frac{\partial B}{\partial y} e^{-\zeta} (\cos \zeta - \sin \zeta). \tag{2.7}$$

To satisfy the lower boundary condition on W one must have

$$C(0) = -\frac{1}{2} \frac{\partial U}{\partial y}. \tag{2.7a}$$

However, the condition that the interior vertical velocity must be independent of vertical coordinate implies that (2.7) holds for all Z . Thus,

$$C = -\frac{1}{2} U_y \tag{2.7b}$$

for all Z and

$$W_0 = -\frac{1}{2} \frac{\partial U}{\partial y} + \frac{1}{2} \frac{\partial A}{\partial y} e^{-\zeta} (\sin \zeta + \cos \zeta) \\ + \frac{\partial B}{\partial y} e^{-\zeta} (\cos \zeta - \sin \zeta). \tag{2.8}$$

To determine the Z structure of the coefficients A and B , one needs to consider the $O(\varepsilon)$ problem. That can be written as

$$\Lambda_{1\zeta\zeta} - 2i\Lambda_1 = Ru + iRv, \tag{2.9a}$$

$$\Lambda_1 = u_1 + iv_1, \tag{2.9b}$$

$$Ru \equiv 2(v_0u_{0y} + W_0u_{0z}) - 2u_{0\zeta z}, \tag{2.9c}$$

$$Rv \equiv 2(v_0v_{0y} + W_0v_{0z}) - 2v_{0\zeta z}. \tag{2.9d}$$

The right-hand side of (2.9a) contains terms that are multiples of the solutions of the homogeneous operator

of the left-hand side, that is, of the form $\exp[-\zeta(1 + i)]$. Those terms must be removed or the solutions for u_1 and v_1 would have the form $\zeta \exp[-\zeta(1 + i)]$ and render the solution disordered when $\varepsilon\zeta$ is order unity, that is, when Z is order 1. To eliminate those terms, one can use the derivatives with respect to Z coming from the final terms in Ru and Rv . This leads to a differential equation in Z ,

$$\frac{\partial}{\partial Z}(A - iB) - (A - iB) \left[\overbrace{C}^{\text{vertical advection}} + \overbrace{\frac{(1+i)}{4}U_y}^{\text{relative vorticity}} \right] = 0. \tag{2.10}$$

The term C comes from the vertical advection in the boundary layer and the remaining term in the brackets comes from the relative vorticity term in the advection. By using (2.6) and (2.7b),

$$A - iB = Ue^{-U_y Z(1-i)/4} \tag{2.11}$$

so that one obtains for the lowest-order solutions

$$u_0 = U[1 - e^{-\zeta(1+\varepsilon U_y/4)} \cos\zeta(1 - \varepsilon U_y/4)], \tag{2.12a}$$

$$v_0 = Ue^{-\zeta(1+\varepsilon U_y/4)} \sin\zeta(1 - \varepsilon U_y/4), \text{ and } \tag{2.12b}$$

$$W_0 = -\frac{U_y}{2} \{1 - e^{-\zeta(1+\varepsilon U_y/4)} [\sin\zeta(1 - \varepsilon U_y/4) + \cos\zeta(1 - \varepsilon U_y/4)]\}. \tag{2.12c}$$

Thus, a positive value of the vorticity (i.e., $U_y < 0$) leads to a slower decay of the boundary layer and thus a thicker boundary layer. The effect of the vertical advection dominates the effect of the relative vorticity. In our nondimensional units, the boundary layer thickness is

$$\delta_\varepsilon = \frac{\delta}{(1 + \varepsilon U_y/2)^{1/2}}.$$

In dimensional units, this is equivalent to a decay scale for the Ekman layer¹

$$\delta^* = \left(\frac{2\nu}{f + U_y/2} \right)^{1/2}. \tag{2.13}$$

The same result can be obtained heuristically by considering the asymptotic matching region where the boundary layer solution blends into the interior. If (2.2a) and (2.2b) are linearized around the interior flow, that is, $u = U(y) + u'$, $v = v'$, and $W = -\frac{1}{2}U_y + W'$, where the primed variables are considered to be small, the resulting linear problem yields the same decay scale for the boundary layer as (2.12a) and (2.12b).

To obtain the correction to the cross-isobar flow at order Rossby number it is necessary to solve (2.9) *after secular terms have been removed*. This, after some algebra and using the result (2.11), yields

$$\Lambda_{1\zeta\zeta} - 2i\Lambda_1 = (1 + i)e^{-2(\zeta+\gamma Z)}UU_y - iUU_y e^{-\zeta(1-i)}e^{-\gamma Z(1+i)} \text{ and } \tag{2.14a}$$

$$\gamma \equiv U_y/4, \tag{2.14b}$$

whose solution, satisfying the no-slip boundary conditions at the plate, is

$$\Lambda_1 = UU_y \left\{ \frac{1 + 3i}{10} [e^{-2(\zeta+\gamma Z)} - e^{-\zeta(1+i)}] + \frac{1}{4} [e^{-\zeta(1-i)-\gamma Z(1+i)} - e^{-\zeta(1+i)}] \right\}. \tag{2.15}$$

The total correction to the cross-isobar flow comes from the imaginary part of Λ_1 , and its integral in ζ is, to lowest order in ε ,

$$\varepsilon \int_0^\infty v_1 d\zeta = \varepsilon \frac{12}{40} UU_y. \tag{2.16}$$

For (2.12b), the total cross-isobar flow, *including the Rossby number dependence of v_0* , is

$$\int_0^\infty (v_0 + \varepsilon v_1) d\zeta = \frac{U}{2} \left(1 + \varepsilon \frac{7}{20} U_y \right), \tag{2.17}$$

which agrees with the result of Benton et al. (1964) and Hart (2000). The vertical velocity pumped into the interior from the Ekman layer follows immediately, and, in agreement with the references cited above,

$$W(\infty) = -\frac{1}{2}U_y - \varepsilon \frac{7}{40}(U_y^2 + UU_{yy}) \tag{2.18}$$

so that in the case of a constant shear, the Rossby number correction to the Ekman pumping velocity is *independent of the sign of the vorticity*.

¹ The approximation $(1+\varepsilon)^{1/2} \approx 1 + \varepsilon/2$ has been used.

3. The stress-driven Ekman layer

The classical problem of the Ekman layer on a free surface, driven by an applied stress, has often been studied [see, e.g., Stern (1965) and more recently Thomas and Rhines (2002)]. As in the example discussed in section 2, of particular interest is obtaining a simple analytic prediction for the nonlinear correction to the Ekman layer thickness. For simplicity, it is again assumed that the flow is independent of the x direction and is homogeneous, that is, that the Burger number based on the horizontal scale of the stress and the depth of the fluid is negligibly small.

The governing equations again are (2.2a)–(2.2d). One can imagine a wind stress in the x direction with a characteristic magnitude τ_0 . The upper surface is at $z = 0$ and, scaling the depth with the linear Ekman layer thickness, the dimensional depth variable lies in the range $-\infty \leq \zeta \leq 0$. One can choose to scale the horizontal velocities with

$$U_{\text{scale}} = \frac{2\tau_0}{\rho f_0 \delta}, \quad \text{where} \quad \delta = \left(\frac{2\nu}{f}\right)^{1/2}. \quad (3.1)$$

It is also assumed that beneath the Ekman layer, at large negative ζ , there is a geostrophic current in the direction of the stress. It is in cases in which the geostrophic flow is strong enough to affect the Ekman layer structure directly that alignment between the stress and the geostrophic velocity is most common. See, for example, the aforementioned example of Thomas and Rhines (2002). The boundary conditions are now

$$u_\zeta = \tau(y), v_\zeta = 0, \quad \text{and} \quad W = 0 \quad \text{for} \quad \zeta = 0 \quad (3.2a)$$

and

$$u \rightarrow u_g(y), v \rightarrow 0, \quad \text{and} \quad W \rightarrow W_\infty(y) \quad \text{for} \quad \zeta \rightarrow -\infty, \quad (3.2b)$$

where W_∞ must be determined.

A similar expansion in powers of ε yields, for the lowest-order problem,

$$\Lambda_{0\zeta\zeta} - 2i\Lambda_0 = 0, \quad (3.3a)$$

$$\Lambda_0 = u_0 + iv_0, \quad (3.3b)$$

$$\Lambda_0 = A(Z)e^{\zeta(1+i)}, \quad (3.3c)$$

leading to

$$u_0 = u_g(y) + e^\zeta(A_r \cos \zeta - A_i \sin \zeta), \quad (3.3d)$$

$$v_0 = e^\zeta(A_i \sin \zeta + A_r \cos \zeta), \quad (3.3e)$$

where A_r and A_i are the real and imaginary parts of $A(Z)$. The boundary conditions on $\zeta = Z = 0$ imply that

$$A_r(0) = \tau/2, \quad (3.4a)$$

$$A_i(0) = -\tau/2. \quad (3.4b)$$

The vertical velocity can be found from (2.2d), and, with the condition that the interior vertical velocity is independent of Z , one obtains

$$W_0 = -\frac{\tau_y}{2} - \frac{e^\zeta}{2} \frac{\partial}{\partial y} [(A_i - A_r) \cos \zeta + (A_i + A_r) \sin \zeta]. \quad (3.5)$$

The order- ε problem again has the structure of (2.9). Carrying out the indicated calculations yields

$$\begin{aligned} \frac{1}{2} \Lambda_{1\zeta\zeta} - i\Lambda_1 &= -e^{\zeta(1+i)} \frac{\partial}{\partial Z} [(A_r - A_i) + i(A_r + A_i)] + e^{\zeta(1+i)} \left\{ -\frac{\tau_y}{2} [(A_r - A_i) + i(A_r + A_i)] + \frac{u_{gy}}{2i} (A_r + iA_i) \right\} \\ &+ e^{\zeta(1-i)} \frac{u_{gy}}{2i} (-A_r + iA_i) + e^{2\zeta} \frac{(1+i)}{2} [A_i(A_{ry} + A_{iy}) + A_r(A_{ry} - A_{iy})]. \end{aligned} \quad (3.6)$$

The first and second terms on the right-hand side both have the form in ζ of the homogeneous operator on the left-hand side. To keep the expansion in ε uniformly valid those terms must vanish. This leads to a simple differential equation for the Z structure of the coefficients A_r and A_i whose solution is

$$A_r = \frac{\tau}{2^{1/2}} e^{-Z(\tau_y/2 + u_{gy}/4)} \cos(Zu_{gy}/4 + \pi/4) \quad \text{and}$$

$$A_i = -\frac{\tau}{2^{1/2}} e^{-Z(\tau_y/2 + u_{gy}/4)} \sin(Zu_{gy}/4 + \pi/4). \quad (3.7)$$

The order-1 velocity fields can then be written as

$$\begin{aligned} u_0 &= u_g + \frac{\tau}{2^{1/2}} e^{\zeta \left[1 - \varepsilon \left(\frac{\tau_y}{2} + \frac{u_{gy}}{4} \right) \right]} \cos[\zeta(1 - \varepsilon u_{gy}/4) \\ &\quad - \pi/4] \quad \text{and} \end{aligned} \quad (3.8a)$$

$$v_0 = \frac{\tau}{2^{1/2}} e^{\zeta \left[1 - \varepsilon \left(\frac{\tau_y}{2} + \frac{u_{gy}}{4} \right) \right]} \sin[\zeta(1 - \varepsilon u_{gy}/4) - \pi/4]. \quad (3.8b)$$

Note that the geostrophic vorticity affects both the decay scale and the oscillatory variation in the Ekman

layer. The wind stress curl affects *only* the thickness of the layer since it enters the dynamics solely in the vertical advection of the momentum. The vorticity of the geostrophic flow does not contribute to the vertical advection. This would require a strongly stratified flow whose geostrophic shear is large enough to produce a viscous stress of the same order as the applied stress.

The Ekman layer thickness—that is, the characteristic decay scale—is now

$$\delta_\varepsilon = \frac{\delta}{[1 - \varepsilon(\tau_y + u_{gy}/2)]^{1/2}} \quad (3.9)$$

or, in dimensional units,

$$\delta_* = \left(\frac{2\nu}{f - \frac{2\tau_{*y*}}{\rho f \delta} - \frac{u_{*gy*}}{2}} \right)^{1/2}. \quad (3.10)$$

If τ_y is negative, that is, if the wind stress curl is positive, the vertical velocity will be positive and the vertical momentum advection will upward and this will make the boundary layer thinner. If the geostrophic vorticity is positive, that is, if $u_{gy} < 0$, the boundary layer is again thinner although this time it is due to the enhanced total vorticity and not the vertical advection effect. Note that the relative size of the two correction terms in the denominator of (3.10) depends on the horizontal scales of the wind stress and the geostrophic current, and these need not be the same.

If (3.6) is integrated after potentially resonant terms have been removed and the stress condition on $\zeta = 0$ is used,

$$\Lambda_{1\zeta} + \Lambda_{0Z} = 0, \quad \zeta = Z = 0, \quad (3.11)$$

one obtains

$$\begin{aligned} -i \int_0^\infty (u_1 + iv_1) d\zeta &= (1 + i) \frac{\tau\tau_y}{8} + \frac{i u_{gy}}{2^{3/2}(1 - i)} e^{i\pi/4} \\ &+ \frac{1}{2} \Lambda_{0Z}, \end{aligned} \quad (3.12)$$

from which one obtains

$$\int_{-\infty}^0 \varepsilon v_1 d\zeta = -\frac{3}{8} \varepsilon \tau u_{gy}. \quad (3.13)$$

When this is added to the vertical integral of v_0 , using (3.8b) and keeping terms to order ε , one obtains for the total Ekman flux perpendicular to the applied stress

$$\int_{-\infty}^0 (v_0 + \varepsilon v_1) d\zeta = -\frac{\tau}{2} - \varepsilon \tau \left(\frac{\tau_y}{8} + \frac{u_{gy}}{2} \right). \quad (3.14)$$

It is noteworthy that this important result for the flux can be obtained directly by integrating (2.2a) with the nonlinear terms written in flux form, and using only the naïve linear solution (i.e., without including the variation in Z) to evaluate the nonlinear terms while using the boundary conditions to evaluate the stress term at the surface and the vertical advection at the base of the Ekman layer.

The vertical velocity at the base of the Ekman layer, to order ε , is obtained from

$$W(-\infty) = \frac{\partial}{\partial y} \left(\int_{-\infty}^0 v_0 + \varepsilon v_1 \right) d\zeta \quad (3.15)$$

and is

$$W(-\infty) = -\frac{1}{2} \frac{\partial}{\partial y} \left[\frac{\tau}{1 - \varepsilon(u_{gy} + \tau_y/4)} \right], \quad (3.16)$$

and these results for the Ekman flux and pumping velocity agree with Thomas and Rhines (2002).

4. Discussion

The effects of nonlinearity on the Ekman layer's thickness have been studied in the weakly nonlinear limit—that is, small Rossby number. Although one might be simply tempted to replace the planetary vorticity by the total, absolute vorticity to estimate the thickness, this neglects the relatively powerful effect of the vertical advection on the boundary layer thickness. For the boundary layer that allows a geostrophic flow to satisfy the no-slip condition on a solid boundary, the vertical advection dominates the vortex force effect in determining the thickness. A positive relative vorticity, although it augments the total vorticity, leads to a thickening of the boundary layer.

For a boundary layer driven by stress on a free surface the situation is even more extreme. In the absence of a strong geostrophic current beneath the layer, the *only* effect of nonlinearity is either to thicken or to thin the layer depending on the sign of the vertical advection produced by the wind stress curl. A strong geostrophic flow, that is, with velocity as large as boundary layer flow driven directly by the stress, does not affect the vertical advection and only contributes a vortex force such that positive vorticity makes the layer thinner as would be intuited by the naïve replacement of f by the total vorticity including the geostrophic flow.

As a check to the calculation, the fluxes in the Ekman layer and the vertical velocity pumped into or out of the layer have been calculated within the formalism described above, and the results agree with the more naïve (and straightforward) calculations that ignore the role of nonlinearity on the layer thickness.

It is important to note the important simplification employed in this study. The geostrophic flow is rectilinear and is an exact solution of the inviscid quasigeostrophic equations. As such there is no advection of relative vorticity by the geostrophic flow and the effect of the geostrophic flow on the layer thickness and the pumping remains local. If the geostrophic flow is more complex, that is, two dimensional and a function of both lateral coordinates, this will no longer be the case. One expects then the effects of the flow to be nonlocal and a shift between the local vorticity and the pumping to occur. Some results that employ a regular iteration expansion are reported in Hart (1995). An approach similar to the present paper for such cases would lead to a partial differential equation for the alteration of layer thickness, and the results of that study are not yet concluded.

Acknowledgments. This research was supported in part by NSF Grant OCE-0451086. Helpful suggestions from Ken Brink and Leif Thomas are gratefully acknowledged.

REFERENCES

- Benton, G. S., F. B. Lipps, and W.-Y. Tuann, 1964: The structure of the Ekman layer for geostrophic flows with lateral shear. *Tellus*, **16**, 186–199.
- Brink, K. H., 1997: Time-dependent motions and the nonlinear bottom Ekman layer. *J. Mar. Res.*, **55**, 613–631.
- Greenspan, H. P., 1968: *The Theory of Rotating Fluids*. Cambridge University Press, 327 pp.
- Hart, J. E., 1995: Nonlinear Ekman suction and ageostrophic effects in rapidly rotating flows. *Geophys. Astrophys. Fluid Dyn.*, **79**, 201–222.
- , 1996: On nonlinear Ekman surface layer pumping. *J. Phys. Oceanogr.*, **26**, 1370–1374.
- , 2000: A note on the nonlinear corrections to the Ekman layer pumping velocity. *Phys. Fluids*, **12**, 131–135.
- Niiler, P. P., 1974: On the Ekman divergence in an oceanic jet. *J. Geophys. Res.*, **74**, 7048–7052.
- Pedlosky, J., 1987: *Geophysical Fluid Dynamics*. Springer Verlag, 710 pp.
- Stern, M. E., 1965: Interaction of a uniform wind stress with a geostrophic vortex. *Deep-Sea Res.*, **12**, 355–367.
- Thomas, L. N., and P. B. Rhines, 2002: Nonlinear stratified spinup. *J. Fluid Mech.*, **473**, 211–244.