NATIONAL TECHNICAL UNIVERSITY OF ATHENS
SCHOOL OF NAVAL ARCHITECTURE AND MARINE ENGINEERING
SECTION OF SHIP & MARINE HYDRODYNAMICS

STEADY FREE SURFACE FLOWS: THE TWO-DIMENSIONAL PROBLEM OVER AN ARBITRARY TOPOGRAPHY, IN THE PRESENCE OF SUBMERGED BODIES.

DIPLOMA THESIS
EYANGELOS G. PAPAGEORGIOU

SUPERVISOR: GERASSIMOS A. ATHANASSOULIS
PROFESSOR, N.T.U.A.

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Chapter One
Introduction

1. Introduction.

1.1 General.

This work deals with the steady flow, of an inviscid, incompressible and irrotational fluid with free surface, interacting with localized disturbances (bottom topography, submerged bodies). The solutions to such problems are expected to provide at least qualitative insight into the mechanism of wave generation by the interaction of free surface flows with localized disturbances. This problem finds application to hydraulic and coastal engineering, as well as to the study of hydrodynamic characteristics (i.e. wave resistance) of floating and immersed bodies.

Free surface flows over various obstacles are subject of studied from 19th century. Kelvin (1886) considered the stationary wave pattern caused by finite elevations or depressions in the bed of a stream and also developed expressions for the hydrodynamic forces acting on these obstacles. Several authors have used concentrated singularities to model uniform flow over finite and infinite bodies in finite or infinite depth i.e. Havelock (1927), Tuck (1965). Havelock (1927) calculated a linearized solution to the problem of a dipole moving with constant velocity beneath the surface of infinitely deep fluid at rest. He then assumed that, at some first order of approximation, his solution would also describe the flow about a circular cylinder beneath the surface of an infinitely deep fluid, as well as the flow about a semicircular obstruction on the bottom of horizontal canal. The book by Kochin, Kibel' & Roze (1964) contains detailed and elegant solutions for the cases of point vortex, a point source and a dipole moving beneath the surface of infinitely deep fluid. The corresponding solutions for a fluid of fixed depth are given in Wehausen & Laitone (1960).

The difficulty which arises in all free surface flows is that the free-surface boundary of the fluid field is not known *a priori*. In mathematical terms, the calculation of the steady-state, two-dimensional, irrotational, incompressible free surface flows involves not only the determination of the potential function, satisfying the Laplace equation, but also the free-surface boundary of the solution domain. In a very limited number of situations, where both the fluid flow and the boundary conditions are simple, mathematical solutions can be found directly (by finding the potential function), using analytical methods. In order to solve much more complex free surface flow problems, numerous numerical techniques have been developed in recent years and the most common and powerful methods are the finite-difference, the boundary integral equations and the finite elements methods.

In the boundary integral method, the governing partial differential equations are integrated analytically, and then for a certain class of equations, which includes the Laplace equation, only the unknowns on the boundary of the required solution domain are involved. Then the problem is reformulated into a set of integral equations on the boundary of the surface in the region of interest.

For incompressible, inviscid and irrotational fluid flows a study of the boundary integral method, which based on Hamiltonian principle and Green's functions, is given by Athanassoulis, Voutsinas & Theodoulidis (1991), Theodoulidis (1995), Bai & Yeung (1974). Moreover, a study of the boundary integral method, which based on complex variable theory (i.e. conformal mapping), is given by Forbes & Schwartz (1982), King & Bloor (1987), (1990), Wen, Ingham & Widodo (1997).

The finite element method (F.E.M.) has been very successfully used to solve a wide range of free surface problems. In the diffraction theory of water waves by bodies or arbitrary topography, the method of finite elements, which is especially versatile in dealing with complicated geometry, has been introduced by Bai & Yeung (1974), Bai (1977), (1978), Chen & Mei (1976) -for steady flows- who replaced the boundaries at infinity by boundaries at large but finite distances. They have applied a hybrid approach with conventional finite elements near the localized variable topography, and an analytical representation for the infinite remaining region (the super-elements). This method (Hybrid F.E.M.), can be applied when the disturbances are localized, in conjunction with the appropriate representations of the wave potential in the infinite half-strips.

The present work is structured as follows: In Chapter 2 the complete, linear Neumann-Kelvin problem in water of constant depth, and the general representations of the wave potentials in the two semi-infinite strips are introduced. In Chapter 3 the steady free-surface flows obstructed by underwater steps and trenches are studied. Apart from their simple geometry these problems can be formulated as matching boundary value problems, with the aid of appropriate matching conditions at the vertical artificial interfaces. Using the conversation of mass in integral form the appropriate conditions for the disturbance current are developed. The problems are solved for various representative cases correspond to both subcritical and supercritical flows. In Chapter 4 steady free-surface flows over an arbitrary topography in the presence of submerged bodies is studied. Since the water layer extends to infinity in the horizontal directions, the assumption is made that, in the far field, the depth is eventually constant (although may be different in different directions). The problem is reformulated as a matching boundary value problem in the finite subdomain, enclosing the varying bathymetry and the fixed body, using the appropriate matching conditions at the vertical interfaces. In Section 4.2 a variational formulation of the hydrodynamic problem is presented. In Section 4.3 a complete representation of the velocity potential in intermediate subdomain is presented based on Finite Element Method, to be used in conjunction with the variational principle, leading to a Galerkin method. The approximate solution of the problem is obtained by truncating the general representations into a finite number of terms, retaining the sufficient number of evanescent modes and keeping the appropriate number of basis functions of finite elements that are required to achieve numerical convergence. Finally, numerical results are presented for various cases of bottom and submerged body geometries for both subcritical and supercritical flows.

1.2 Physical description of the problem.

Let us consider a liquid strip extending at infinity in both horizontal directions, whereas in vertical direction is bounded above by the free surface and below by a rigid bottom; see figure 1.1. Moreover, it is assumed that the fixed submerged obstacles and bottom unevenness appear in a local region.

We consider a free-surface flow of a stream -which was originally at rest everywhere- of an inviscid, incompressible and irrotational fluid, obstructed by the localized disturbances. After a sufficient time, the studied problem is considered as a steady flow, where the term *'steady'* is used to describe that the flow remains *time-independent* for all points in the liquid domain.

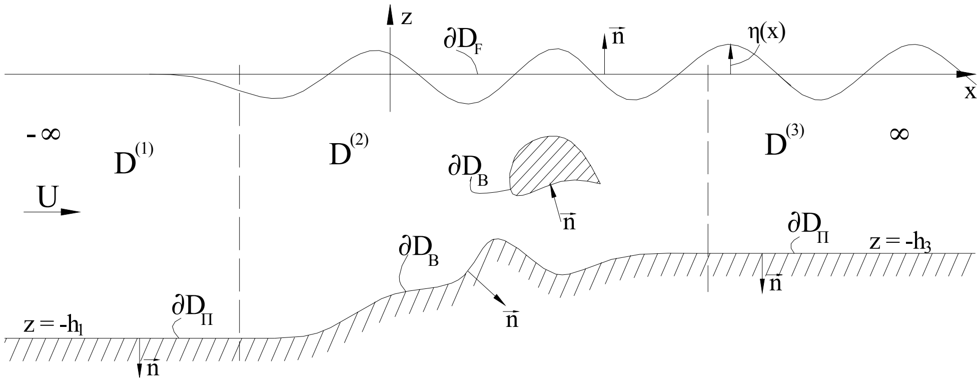

FIGURE 1.1. Geometric configuration of the general problem

We assume (without loss of generality) that the motion is along the *x*-axis of a fixed coordinate system, with its origin on the mean water level (in the variable bathymetry region), the *z*-axis pointing upwards and the *y*-axis being parallel to the bottom contours.

The physical parameters which describe the studied problem are:

- The bathymetric Froude number:

$$Fr = U / \sqrt{g \cdot h} \tag{1}$$

- The geometry of the localized disturbances.

Generally, the pattern of the flow downstream is not known a priori, but depends upon the physical parameters of the problem. Under the linear theory the flow downstream is uniquely determined by the Froude number. In case of subcritical flow ($Fr < 1$), the downstream flow is a wave – current system. In contrary, for supercritical flow$\left(Fr > 1\right)$, the disturbance decays exponentially as $x \to \infty$, thus asymptotically a uniform flow is observed. In case of critical flow$\left(Fr = 1\right)$, it seems that there is no steady flow. Stoker (1957) has shown this by studying the unsteady problem and allowing the time to tend to ∞.

1.3 General formulation of the problem.

With the assumption of irrotational motion and an incompressible fluid, a velocity potential $\Phi(x,z)$ exists which would satisfy the continuity equation i.e. Laplace equation in $D^{(\infty)} = \{(x,z) \in R^2 : -\infty < x < \infty, -h(x) < z < \eta(x)\}$:

$$\Delta\Phi(x,z) = 0, \quad (x,z) \in D^{(\infty)}. \tag{2}$$

Laplace equation accounts for the incompressibility and the irrotationality of the flow and thus it is of purely kinematic character. The Laplace equation is of elliptic type partial differential equation thus in general it is not admits solutions of wave character. In the case of free surface flows the wave character of solution arises from the dynamical and kinematical free surface boundary conditions.

At any boundary, whether it is fixed, such as the bottom, or free, such as the water surface, which is free to deform under the influence of forces, certain physical conditions must be satisfied by the fluid velocities. It is clear that there must be no flow across the surface. Under these considerations (see e.g. Stoker (1957), Wehausen & Laitone (1960)) the kinematical free surface b.c. is

$$\Phi,_x \eta,_x - \Phi,_z = 0, \quad z = \eta(x), \tag{3}$$

and the dynamical free surface b.c. (i.e. Bernoulli equation on free surface) is

$$\frac{1}{2} \cdot |\nabla\Phi|^2 + g \cdot \eta = 0, \quad z = \eta(x). \tag{4}$$

In fixed boundaries such as the bottom and the submerged bodies we have the following condition:

$$\Phi,_n = 0, \quad z = -h(x). \tag{5}$$

In addition, proper conditions at infinity must be imposed.

Considering that the motion arising from disturbances created in the uniform stream, is described by velocity potential see i.e. Stoker (1957)

$$\Phi(x,z) = U \cdot x + \varphi(x,z), \quad -\infty < x < \infty, -h < z < \eta(x), \tag{6}$$

the problem is formulated as follows:

PROBLEM $\mathscr{P}_\varphi(D^{(\infty)};\varphi,\eta)$. *Find the function* $\varphi(x,z)$ *defined in* $D^{(\infty)}=\{(x,z)\in R^2 : -\infty<x<\infty, -h(x)<z<\eta(x)\}$, *and the function* $\eta(x),\ x\in R$, *satisfying the following boundary value problem*

$$\nabla^2\varphi(x,z)=0, \qquad -\infty<x<\infty, -h<z<\eta(x), \tag{7}$$

$$\frac{1}{2}|\nabla\varphi|^2+U\varphi,_x+g\eta=0, \qquad z=\eta(x), \tag{8}$$

$$\varphi,_x\eta,_x+U\eta,_x-\varphi,_z=0 \qquad z=\eta(x), \tag{9}$$

$$\varphi,_n=-Un_x, \qquad (x,z)\in\partial D_B\cup\partial D_\Pi \tag{10}$$

and, in addition, $|\nabla\varphi|$ *is bounded at* ∞ (11)

Chapter Two
The Linear Neumann-Kelvin problem in water of Constant finite depth

2. The linear Neumann-Kelvin problem in water of constant finite depth.

2.1 Differential formulation of the problem.

In this section we consider steady water waves in an infinite strip of constant depth h, (Fig.2.1) when the stream of an inviscid incompressible and irrotational fluid has uniform velocity U in the undisturbed state. The term 'steady' is used to describe a flow, which remains time-independent for all points in the liquid domain. We assume (without loss of generality) that the motion is along the x-axis of a fixed coordinate system, see Fig.2.1.

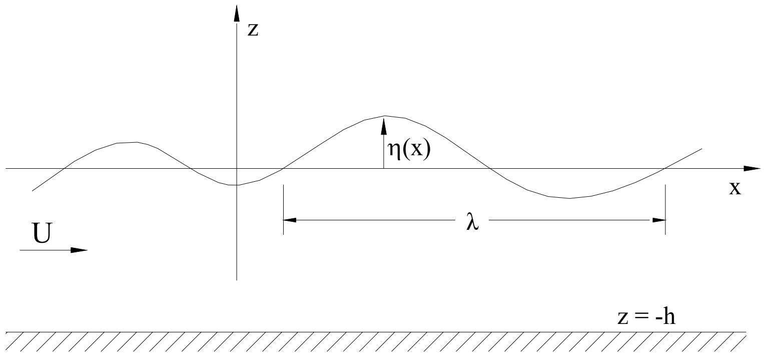

FIGURE 2.1. Waves on a running stream

We consider that the motion arising from disturbances created in the uniform stream, is described by velocity potential see i.e. Stoker (1957)

$$\Phi(x,z) = U \cdot x + \varphi(x,z), \quad -\infty < x < \infty, -h < z < \eta(x). \tag{1}$$

The function $\varphi(x,z)$ is assumed to yield a small disturbance on the uniform flow, which means that φ and its derivatives are small quantities with respect to undisturbed velocity potential, therefore the quadratic and higher-order terms can be neglected in comparison with linear (first-order) terms. We also assume that the vertical displacement of the free surface $\eta(x)$, as measured from the undisturbed level $z = 0$, is also a small quantity - with respect to the characteristic wavelength λ - of the same order as $\varphi(x,z)$. Under these assumptions the dynamic free surface boundary condition (cf. (1.3.8)) and the kinematic free surface condition (cf. (1.3.9)) can be linearized (cf. Appendix 2.A), and the problem can be formulated as follows:

PROBLEM $P_{NK}(D^{(\infty)};\varphi,\eta)$. *Find the function* $\varphi(x,z)$ *defined in* $D^{(\infty)} = \{(x,z) \in R^2 : -\infty < x < \infty, -h < z < 0\}$, *and the function* $\eta(x)$, $x \in R$, *satisfying the following boundary value problem*

$$\nabla^2 \varphi(x,z) = 0, \qquad -\infty < x < \infty, -h < z < 0, \tag{2}$$

$$U \cdot \varphi,_x + g \cdot \eta = 0, \qquad z = 0, \tag{3}$$

$$\varphi,_z - U \cdot \eta,_x = 0, \qquad z = 0, \tag{4}$$

$$\varphi,_z = 0, \qquad z = -h, \tag{5}$$

and, in addition, $|\nabla \varphi|$ *is bounded at* ∞ (6)

Eliminating $\eta(x)$ between the dynamic (3) and the kinematic (4) free surface boundary conditions, the following 'mixed' free surface boundary condition is obtained, see i.e. Stoker (1957, p.200), Wehausen & Laitone (1960, §20):

$$\varphi,_{xx} + \frac{g}{U^2} \cdot \varphi,_z = 0, \qquad z = 0, \tag{7}$$

usually referred to as the Kelvin condition.

Defining the bathymetric Froude number

$$Fr = U / \sqrt{g \cdot h}\ , \tag{8}$$

where g is the acceleration due to gravity, the above equation (7) takes the following alternative form

$$\varphi,_{xx} + \frac{1}{hFr^2} \cdot \varphi,_z = 0, \qquad z = 0. \tag{9}$$

The Froude number's physical significance is the ratio of inertial forces to gravity forces squared. It can also be interpreted as the ratio of the velocity U of uniform flow to propagation speed $\sqrt{g \cdot h}$ of infinitesimally waves propagating in shallow water. Its value determines the regime of flow -sub, super or critical flow.

Once the velocity potential $\varphi(x,z)$ has been determined the elevation of the free surface is given by

$$\eta(x) = -\frac{U}{g} \cdot \varphi,_x(x,0)\ . \tag{10}$$

We can see according to the above analysis (also cf. Appendix 2.A) the great simplifications which result through the linearization of the free surface conditions: not only does the problem become linear, but also the domain in which its solution is to be determined becomes fixed and known a priori.

2.2. General representation of the steady wave potential in semi-infinite strip.

In this paragraph we refer to the linear Neumann-Kelvin problem as in the two semi-infinite strips $D^{(U)}$ and $D^{(D)}$, where $D^{(U)} = \{(x,z):\ -\infty < x \le a,\ -h_U < z < 0\}$ is the upstream liquid domain and $D^{(D)} = \{(x,z):\ \beta \le x < \infty,\ -h_D < z < 0\}$ is the downstream liquid domain. Due to the simple geometry of the problem, the general solution of the wave potential can be represented analytically in the form of an eigenfunction expansion, as shown below. In favor of simplicity we study the problem in the downstream strip, while as for the upstream strip the basic results are introduced. The problem in downstream strip (Fig.2.2) can be formulated as follows:

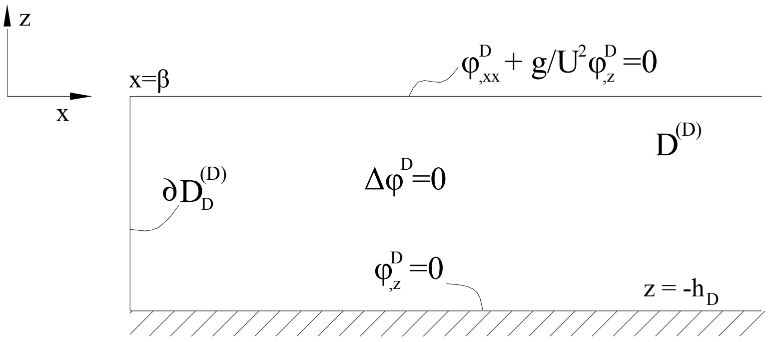

FIGURE 2.2: The problem $P_{NK}(D^{(D)};\varphi^D)$

<u>PROBLEM</u> $P_{NK}(D^{(D)};\varphi^D)$. *Find the function* $\varphi^D(x,z)$ *defined in* $D^{(D)}$, *satisfying the following boundary value problem*

$$\nabla^2\varphi^D(x,z) = 0, \quad \beta \le x < \infty, \ -h_D < z < 0, \tag{11}$$

$$\varphi,^D_{xx} + \frac{g}{U^2}\cdot\varphi,^D_z = 0, \quad z = 0, \tag{12}$$

$$\varphi,^D_z = 0, \qquad z = -h_D, \tag{13}$$

$$\left|\nabla\varphi^D\right| < \infty, \qquad x \to \infty. \tag{14}$$

As indicated from the formulation of the above problem $P_{NK}(D^{(D)};\varphi^D)$, on the vertical boundary $\partial D_D^{(D)} = \{(x,z): x = b, \ -h_D < z < 0\}$, no condition is imposed for the wave potential $\varphi^D(x,z)$. This means that we are interested in the general representation of all possible solutions of the problem.

In view of the above definitions we are proceeding to the general solution of the problem $P_{NK}(D^{(D)};\varphi^D)$.

- ***Solution of the problem*** $P_{NK}(D^{(D)};\varphi^D)$:

A convenient method for solving the above problem is applying separation of variables. The assumption behind its use is that the solution can be expressed as a product of terms, each of which is a function of only one of the independent physical variables. In our case,

$$\varphi(x,z) = X(x)\cdot Z(z) \tag{15}$$

where $X(x)$ is a function depending only on *x,* the horizontal co-ordinate, and $Z(z)$ depends only on *z*, the vertical co-ordinate. Substituting Eq.(15) into the Laplace equation (11) and dividing through by $\varphi(x,z) = X(x)\cdot Z(z)$ we have

$$\frac{X''(x)}{X(x)} + \frac{Z''(z)}{Z(z)} = 0 \tag{16}$$

Clearly, the first term of this equation depends on x alone, while the second term depends only on z. If we consider a variation in z in Eq. (16) holding x constant, the second term could conceivably vary, whereas the first term could not. This would give a nonzero sum in Eq. (16) and thus the equation would not be satisfied. The only way that the equation would hold is if each term is equal to the same constant, except for a sign difference, that is,

$$\frac{X''(x)}{X(x)} = k_n^{\ 2} \quad \Rightarrow \quad X''(x) - k_n^{\ 2} \cdot X(x) = 0 \tag{17a}$$

$$\frac{Z''(z)}{Z(z)} = -k_n^{\ 2} \quad \Rightarrow \quad Z''(z) + k_n^{\ 2} \cdot Z(z) = 0 \tag{17b}$$

In the above equations (17) k_n^2 denotes the real separation constants ($\{\, k_n^2 \,\} \in R$), although, in general, the roots of $k_n = \pm\sqrt{k_n^2}$ could be a complex numbers ($\{\, k_n \} \in \mathbb{C}$).

Substituting the relation (15) for the wave potential, into the free surface b.c. (12) and the bottom b.c. (13) give us

$$X''(x) \cdot Z(z) + \frac{g}{U^2} \cdot X(x) \cdot Z'(z) = 0, \qquad z = 0, \tag{18}$$

$$X(x) \cdot Z'(z) = 0, \qquad z = -h, \tag{19}$$

respectively.

Using the equation (17a), the free surface b.c. (18) yields

$$X(x) \cdot \left(k_n^{\ 2} \cdot Z(z) + \frac{g}{U^2} \cdot Z'(z) \right) = 0, \qquad z = 0,$$

For this equation to be true for any x, the terms within the parentheses must be identically zero, which leads to

$$Z'(z) + k_n^{\ 2} \cdot \frac{U^2}{g} \cdot Z(z) = 0, \qquad z = 0, \tag{20}$$

The vertical equation (17b), and the boundary conditions (19) and (20) constitute the following *Vertical Eigenvalue Problem*, with spectral parameter k_n^2:

PROBLEM $VE(-h,0)$. *Find the pairs eigenvalues-eigenfunctions* $(k_n, Z_n(z))$, *satisfying the following vertical eigenvalue problem:*

$$Z''(z) + k_n^{\ 2} \cdot Z(z) = 0, \qquad -h < z < 0, \tag{21a}$$

$$Z' + k_n^{\ 2} \cdot \frac{U^2}{g} \cdot Z = 0, \qquad z = 0, \tag{21b}$$

$$Z' = 0 \qquad z = -h, \tag{21c}$$

The above problem $VE(-h,0)$ is a non-self-adjoint boundary value problem – because the spectral parameter k_n^2 is contained not only in the differential equation (21a) but in the boundary condition (21b) as well. Therefore the problem $VE(-h,0)$ is not a classical *Sturm-Liouville problem*. It is a *Steklov eigenvalue problem* (after the name of the great Russian mathematician *Steklov* who introduced and studied that problem in 1902). We denote here that the eigenfunctions of the problem $VE(-h,0)$ form a *Riesz basis* for $L^2(-h,0)$, as treatment in the following paragraphs.

The general solution of the second order, ordinary differential equation (21a), is of the form

$$Z_n(z) = A_n \cdot \cos(k_n \cdot z) + B_n \cdot \sin(k_n \cdot z) \tag{22a}$$

where A_n, B_n are constant coefficients.
From the free surface boundary condition (21b), we have

$$Z'(0) + k_n{}^2 \cdot \frac{U^2}{g} \cdot Z(0) = 0 \;\Rightarrow\; B_n \cdot k_n + k_n{}^2 \cdot \frac{U^2}{g} \cdot A_n = 0 \;\Rightarrow$$

$$B_n = -k_n \cdot \frac{U^2}{g} \cdot A_n \tag{22b}$$

and from the bottom boundary condition (21c)

$$Z'(-h) = 0 \;\Rightarrow -k_n \cdot A_n \cdot \sin(-k_n \cdot h) + k_n \cdot B_n \cdot \cos(-k_n \cdot h) = 0 \;\Rightarrow$$

$$B_n = -A_n \cdot \tan(k_n \cdot h) \tag{22c}$$

By elimination of the coefficient B_n from the equations (22b) and (22c), we have the following ”dispersion relation”, see i.e. Lamb (1932, p.407),

$$k_n \cdot \frac{U^2}{g} = \tan(k_n \cdot h), \tag{23a}$$

or in dimensionless form

$$k_n \cdot h = \frac{1}{Fr^2} \tan(k_n \cdot h), \tag{23b}$$

where $Fr = U / \sqrt{g \cdot h}$ is the bathymetric Froude number.
The roots $\{k_n\}$, n = 0,1,2,… of the equation (23b), which as it will be shown in the sequel are countably infinite, are depend on the Froude number Fr and on the depth h of the semi-infinite strip.

The corresponding eigenfunctions are obtained from the equation (22a), using eq.(22b) as follows

$$Z_n(z) = A_n \frac{\cos\big(k_n(z+h)\big)}{\cos(k_n h)}, \quad n = 0,1,2,\ldots \tag{24}$$

From equation (23b) and relation (24), an obvious solution of the above problem is the pair $\left(k, Z(z)\right) = \left(0,1\right)$.

LEMMA *For each value of Froude number* $Fr \in (0,\infty)$, $Fr \neq 1$ *and for finite depth h, the relation (23b) has:*

(i) *one pure imaginary root* $k_o = i\left|k_o\right| \in I^+$ *and infinite number of discrete real roots* $\left\{k_n\right\} \in R^+$, *n = 1,2,... if* $Fr < 1$,

(ii) *infinite number of discrete real roots* $\left\{k_n\right\} \in R^+$, *n = 0,1,2,... if* $Fr > 1$,

and their conjugates.

Proof. Two individual cases are introduced:

(a) In case where one of the roots of equation (23b) is a pure imaginary number of the form $i|k_n|$, the relation (23b) take the following form

$$\left|k_n\right| \cdot h = \frac{1}{Fr^2}\tanh(\left|k_n\right| \cdot h) \tag{25a}$$

We define $x = \left|k_n\right| \cdot h$, $x \in R^+$ hence the above equation take the form

$$x = \frac{1}{Fr^2}\tanh(x). \tag{25b}$$

The curves $\zeta_1 = \tanh(x)$ and $\zeta_2 = Fr^2 \cdot x$ are plotted for $Fr < 1$ and $Fr > 1$ in Fig.2.3a and Fig.2.4a respectively. The roots of (25a) are of course furnished by the intersections $x = \left|k_n\right| \cdot h$ of these curves. One can observe the following: *i)* *k*=0 is always a root, *ii)* there is one real positive root different from zero if $Fr < 1$, *iii)* there is no real root other than zero if $Fr > 1$.

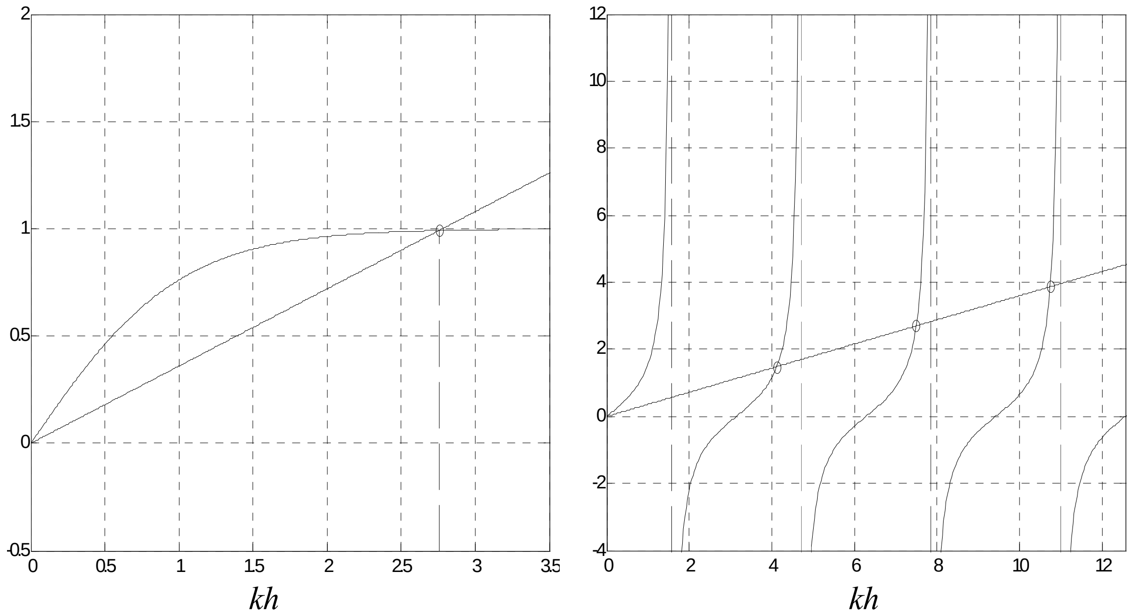

FIGURE 2.3a,b: Graphic solution of eigenvalues ($Fr < 1$)

(b) Moreover, equation (25b) has infinite number of discrete real roots $\{k_n\} \in R^+$; n = 1,2,… if $Fr < 1$, and n = 0,1,2,… if $Fr > 1$, since $\tan(k_n \cdot h)$ is periodic function in the subsets $I_n = \left[n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2} \right] \subset R^+$. Graphically, these roots are the intersections of $\gamma_1 = \tan(k_n h)$ and $\gamma_2 = Fr^2 \cdot (k_n h)$, as shown in Fig.2.3b and Fig.2.4b, for $Fr < 1$ and $Fr > 1$ respectively.

From these figures it is evident that every root of the equation (25b) belongs to the interval I_n; specifically we have that

$$k_n \in \left(\left(n\pi, \left(n + \frac{1}{2} \right) \cdot \pi \right) \subset I_n \right), \qquad \begin{cases} n = 1,2\ldots \; if \quad Fr < 1, \\ n = 0,1,2\ldots \; if \quad Fr > 1. \end{cases}$$

Asymptotically, as n becomes large, $k_n h$ approaches $\left(n + \frac{1}{2} \right) \cdot \pi$,

$$\text{or} \quad k_n \xrightarrow[n \to \infty]{} \frac{1}{h} \cdot \left(n\pi + \frac{\pi}{2} \right).$$

Consequently, the only point of accumulation of the sequence $\{k_n\}$ is $+\infty$. ■

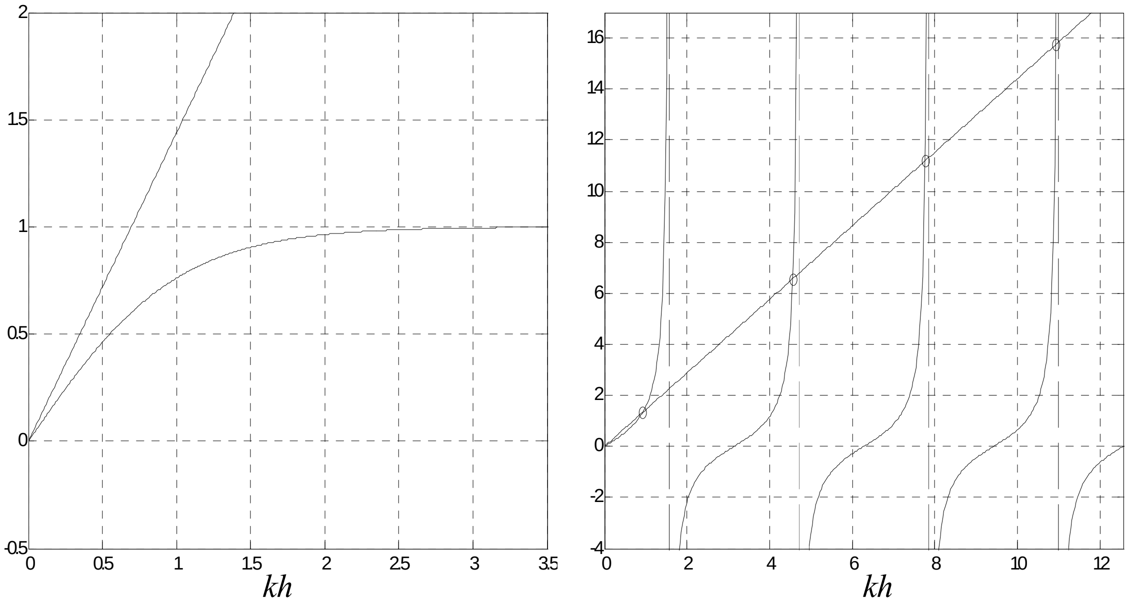

FIGURE 2.4a,b: Graphic solution of eigenvalues ($Fr > 1$)

According to the above analysis, the set S_1 of eigenfunctions of the problem $VE(-h,0)$ is: $S_1 = S \cap \{1\}$,

where $S = \{Z_n(z) : n = 0,1,2,...\}$ is the set of eigenfunctions (cf. (24)), corresponding to the roots of equation (23b).

The set S_1 of eigenfunctions of the problem $VE(-h,0)$ does not satisfy the property of orthogonality on $L^2(-h,0)$ (cf. Appendix 2.B), so there is no indication that the set S_1 is complete and has the property of basis on $L^2(-h,0)$. This arises from the fact that the spectral parameter k_n^2 is contained in the boundary condition (21b) (such problems are called *Steklov eigenvalue problems*). Physically, the appearance of the spectral parameter in the boundary condition reflects the fact that the wave character of the problem is introduced through the free-surface boundary condition.

The following theorem, which its proof (Athanassoulis (1991)) is associated with the theory of non-harmonic Fourier functions (see i.e. Young (1980) and Higgins (1976)), provides the completeness of sequences of eigenfunctions of the problem $VE(-h,0)$.

THEOREM. *The set* S *of eigenfunctions of the problem* $VE(-h,0)$ *constitutes a Riesz basis on* $L^2(-h,0)$. *In particular, the series*

$$\sum_{n=0}^{\infty} C_n Z_n(z)$$

converges on $L^2(-h,0)$ *if and only if, the series* $\sum_{n=0}^{\infty} |C_n|^2$ *converges, and for every function* $g(z) \in L^2(-h,0)$ *there exists a unique sequence of real constants* $\{C_n : n = 0,1,2,...\}$ *such that*

$$g(z) = \sum_{n=0}^{\infty} C_n Z_n(z),$$

if there exists real constants A *and* B, $0 < A \le B < \infty$ *so that* $A \cdot \sum_{n=0}^{\infty} |C_n|^2 \le \|g\|_{L^2} \le B \cdot \sum_{n=0}^{\infty} |C_n|^2$. ■

We proceed now, by the above definitions about the set $\{k_n\}$ of eigenvalues, to the general solution of horizontal equation (17a). The general solution of Eq.(17a) is of the form

$$X_n(x) = A_n \exp(ik_n x) + B_n \exp(-ik_n x) \qquad for\ n=0\ \ if\ \ Fr<1,$$

and

$$X_n(x) = A_n \exp(k_n x) + B_n \exp(-k_n x) \qquad for\ \begin{cases} n=1,2...\ if\ \ Fr<1, \\ n=0,1,2...\ if\ \ Fr>1. \end{cases}$$

where A_n and B_n are general complex constants.
We denote here that in the case where k=0 (in this eigenvalue the opposite index number is $n=-1$) the general solution is of the form

$$X_n(x) = A_n x + B_n \quad for\ n=-1\ \ , A_{-1}\ and\ B_{-1} \in R\,.$$

The imposed radiation conditions require that the potential $\varphi(x,z)$ and its derivatives up to first order to be bounded at infinity. Therefore, we have the following cases:

(i) In the case where φ is defined in the downstream liquid domain $D^{(D)}$, the solutions of the form $A_n \exp(k_n x)$ tend to infinity with exponential rate, as $x \to \infty$. Consequently, these solutions must be rejected (A_n=0 for $\begin{cases} n=1,2...\ if\ \ Fr<1, \\ n=0,1,2...\ if\ \ Fr>1. \end{cases}$).

(ii) Similarly, in the case where φ is defined in the upstream liquid domain $D^{(U)}$, the solutions of the form $B_n \exp(-k_n x)$ tend to infinity with exponential rate, as $x \to -\infty$. Hence, these solutions must be rejected (B_n=0 for $n=1,2...\ if\ \ Fr<1$, and for $n=0,1,2...\ if\ \ Fr>1$).

Accordingly to the above analysis, the general representation of disturbance velocity potential $\varphi(x,z)$ in the semi-infinite strips $D^{(U)}$ and $D^{(D)}$ respectively, is:

(a) <u>Upstream</u> ($\varphi \in D^{(U)}$)

a1. *Subcritical case* ($Fr<1$):

$$\varphi^{(U)}(x,z) = A_{-1}x + B_{-1} + \left(A_0 \exp\left(ik_0 x\right) + B_0 \exp\left(-ik_0 x\right)\right) Z_0(z) + \\ + \sum_{n=1}^{\infty} A_n \exp\left(k_n\left(x-\alpha\right)\right) Z_n(z) \qquad ((x,z)\in D^{(U)}) \tag{26a}$$

a2. *Supercritical case* ($Fr>1$):

$$\varphi^{(U)}(x,z) = A_{-1}x + B_{-1} + \sum_{n=0}^{\infty} A_n \exp\left(k_n\left(x-\alpha\right)\right) Z_n(z) \quad ((x,z)\in D^{(U)}) \tag{26b}$$

(b) Downstream ($\varphi \in D^{(D)}$)

b1. *Subcritical case* ($Fr < 1$):

$$\varphi^{(D)}(x,z) = A_{-1}x + B_{-1} + \left(A_0 \exp\left(ik_0 x\right) + B_0 \exp\left(-ik_0 x\right)\right) Z_0(z) + \\ + \sum_{n=1}^{\infty} A_n \exp\left(k_n \left(\beta - x\right)\right) Z_n(z) \qquad ((x,z) \in D^{(D)}) \tag{27a}$$

b2. *Supercritical case* ($Fr > 1$):

$$\varphi^{(D)}(x,z) = A_{-1}x + B_{-1} + \sum_{n=0}^{\infty} A_n \exp\left(k_n \left(\beta - x\right)\right) Z_n(z) \quad ((x,z) \in D^{(D)}) \tag{27b}$$

We denote here that we study time-independent flows so only the real part of the disturbance velocity potential (Eq.(26a), (27a)) is to be retained.

On the basis of the relations about the velocity potential, we are now to the point of making some general remarks.

- In subcritical case, the terms $\mathrm{Re}\left\{\exp(\pm i k_0 x)\right\}$ are periodic in x with fixed wavelength λ given by $\lambda = 2\pi / k_0$ as x tends to infinity. If we were to observe these steady waves from a system of coordinates moving in the x-direction with the constant velocity U, we would see a train of progressing waves, since the form of periodic terms would be $\mathrm{Re}\left\{\exp(\pm i k_0 (x + Ut))\right\}$. The phase speed of these waves would of course be the velocity U. According to the notation $(\pm)$ any of these propagating waves travel towards negative or positive x-direction respectively and are called *propagating modes.*
- In supercritical and subcritical cases, the terms $\exp\left(k_n \left(x - a\right)\right)$ for upstream strip and $\exp\left(k_n \left(\beta - x\right)\right)$ for downstream strip as x tends to infinity ($x \to -\infty$ and $x \to \infty$ respectively), die out with exponential rate. These terms are only of local importance and are called *evanescent modes.*
- From the general representations of the velocity potential, we have the following asymptotic forms, e.g. for downstream strip:
  - *Subcritical case* ($Fr < 1$):

    $$\varphi^{(D)}(x,z) \xrightarrow[n \to \infty]{} A_{-1}x + B_{-1} + \left(A_0 \exp(ik_0 x\right) + B_0 \exp(-ik_0 x)) \cdot Z_0(z)$$

  - *Supercritical case* ($Fr > 1$):

    $$\varphi^{(D)}(x,z) \xrightarrow[n \to \infty]{} A_{-1}x + B_{-1}$$

  According to the above relations, we denote that asymptotically no motions other than the steady flow with no surface disturbance will exist unless $Fr < 1$. These waves are then seen to have the wavelength appropriate for simple harmonic waves of propagation speed U in water of depth h, as mentioned above. We observe here the physical meaning of the roots of equation (23b), and it's strictly dependence upon the velocity U of the uniform flow.

- The coefficients A_{-1}, B_{-1}, A_n - with *n* of the appropriate set of index - in the above representations are defined by using the appropriate boundary conditions at $x = a$ (or $x = \beta$). These boundary conditions (matching conditions) are associated with the additional information related to the cause of the wave disturbance.—

  Concluding, we should point out that these coefficients have a special physical character denoting the infinite degrees of freedom of the fluid system. Consequently, certain radiation conditions (including the boundness) are imposed such that the problem will be uniquely determined, constraining the appropriate coefficients to the asymptotic form of the velocity potential. We denote for example, that an appropriate such condition is that the disturbance should die out upstream in subcritical flow; hence $A_0, B_0 = 0$. In the following sections we introduce this special character with application to certain problems.

APPENDIX 2.A. Linearization of the non-linear problem $\mathscr{P}_\varphi$

We recall here the non-linear problem $\mathscr{P}_\varphi$ (φ-formulation).

PROBLEM $\mathscr{P}_\varphi(D^{(\infty)};\varphi,\eta)$. Find the functions $\varphi(x,z)$ and $\eta(x)$ defined in $D^{(\infty)}$, satisfying the following non-linear mixed boundary value problem

$$\nabla^2\varphi(x,z) = 0, \qquad -\infty < x < \infty, -h < z < \eta(x), \tag{1}$$

$$\frac{1}{2}\left|\nabla\varphi\right|^2 + U\varphi,_x + g\eta = 0, \qquad z = \eta(x), \tag{2}$$

$$\varphi,_x \eta,_x + U\eta,_x - \varphi,_z = 0 \qquad z = \eta(x), \tag{3}$$

$$\varphi,_z = 0, \qquad z = -h, \tag{4}$$

and, in addition, $\left|\nabla\varphi\right|$ is bounded at ∞. (5)

The non-linear problem $\mathscr{P}_\varphi$ cannot be solved exactly, the principal difficulty being the fact that the domain in which the equations have to be solved is part of the solution, i.e. the free surface elevation itself is one of the unknowns. Linearization in fixed domains is performed by ignoring terms of quadratic and higher order. In the present problem, where the domain of definition is one of the unknowns, the perturbation procedure is used i.e. linearization of the problem about a particular exact solution see i.e. Wehausen & Laitone (1960, §10), Stoker (1957, §2.1). For the application of perturbation method, one must be able to select a dimensionless parameter (or parameters), say ε, which helps to determine the exact physical problem and is such that the known exact solution to be approached (in some sense) when $\varepsilon \to 0$. It is then assumed that the various functions entering into the problem may be expanded into power series with respect to parameter ε. Further, the series are substituted into the governing equation and boundary conditions and grouped according to powers of ε.

The coefficients of each power then yield a sequence of equations and boundary conditions. The linear solution will not depend on ε (first-order theory), while the second-order will, the third-order will depend on ε^2, and so on.

In this section the above perturbation approach of infinitesimal-wave theory is followed, where as an exact initial solution the uniform flow is taken. We remark that this follows from the consideration of the studied problem that the motion arises from disturbances created in the uniform flow (c.f. eq.2.1.1).

It is convenient at this point to put the governing equation and the related boundary conditions into dimensionless forms. We define the following dimensionless variables, developed in terms of g, a, h, U and k, which are gravity, the wave amplitude, the depth, the velocity of the uniform flow, and the wave number, respectively.

$$\bar{x} = kx$$

$$\bar{z} = kz$$

$$\bar{\eta} = \frac{\eta}{a}$$

$$\bar{\varphi} = \frac{k\varphi}{a\sqrt{gk}}$$

$$Fr = \frac{U}{\sqrt{gh}} .$$

The governing equation (Laplace equation) is thus

$$\bar{\varphi},_{\bar{x}\bar{x}} + \bar{\varphi},_{\bar{z}\bar{z}} = 0 . \tag{6}$$

The dynamic free surface boundary condition (2) is modified to be

$$\frac{1}{2}(ka)^2(\bar{\varphi},_{\bar{x}}^2 + \bar{\varphi},_{\bar{z}}^2) + (ka)Fr\sqrt{kh}\,\bar{\varphi},_{\bar{x}} + (ka)\bar{\eta} = 0 , \quad \bar{z} = (ka)\bar{\eta} . \tag{7}$$

We denote that if $ka = 0$, then $\bar{z} = 0$; there are no waves and therefore only the trivial solution exists. The kinematic free surface boundary condition (3) takes the following dimensionless form

$$(ka)^2\bar{\varphi},_{\bar{x}}\,\bar{\eta},_{\bar{x}} + (ka)Fr\sqrt{kh}\,\bar{\eta},_{\bar{x}} - (ka)\bar{\varphi},_{\bar{z}} = 0 , \quad \bar{z} = (ka)\bar{\eta} . \tag{8}$$

As mentioned above in the infinitesimal-wave theory, the non-linear boundary conditions are expanded about the mean water level ($\bar{z} = 0$), and then products of very small quantities are neglected, such as $\bar{\varphi},_{\bar{x}}^2$. It follows that the gradient of the free surface elevation ka is small quantity; hence the terms of order $(ka)^2$ are neglected when are compared to ka. In the perturbation approach, we assume that the solution depends on the small quantity ka, which is defined as ε.

Therefore, we decompose all quantities into a power series in ε, which is presumed to be less than unity.

$$\bar{\eta} = \bar{\eta}^{(1)} + \varepsilon\bar{\eta}^{(2)} + \varepsilon^2\bar{\eta}^{(3)} + \ldots$$
$$\bar{\varphi} = \bar{\varphi}^{(1)} + \varepsilon\bar{\varphi}^{(2)} + \varepsilon^2\bar{\varphi}^{(3)} + \ldots \tag{9}$$

Again, as we a priori do not know the location of the free surface $\bar{z} = (ka)\bar{\eta}$, we resort to expanding the non-linear free surface boundary conditions about $\bar{z} = 0$ in terms of $\varepsilon\bar{\eta}$, retaining the higher-order terms up to ε^2, denoted as $O(\varepsilon^2)$. Using the Taylor series, we have on $\bar{z} = \varepsilon\bar{\eta}$

$$\bar{\varphi}(\bar{x},\bar{z}) = \bar{\varphi}(\bar{x},0) + (\varepsilon\bar{\eta})\bar{\varphi},_{\bar{z}}(\bar{x},0) + \frac{(\varepsilon\bar{\eta})^2}{2}\bar{\varphi},_{\bar{z}\bar{z}}(\bar{x},0) + \ldots. \tag{10}$$

Substituting the relation (10) in the dynamic free surface b.c. (7) we obtain

$$\left(\frac{1}{2}\varepsilon^2(\bar{\varphi},_{\bar{x}}^2 + \bar{\varphi},_{\bar{z}}^2) + \varepsilon Fr\sqrt{kh}\,\bar{\varphi},_{\bar{x}} + \varepsilon\bar{\eta}\right) + \varepsilon\bar{\eta}\frac{\partial}{\partial\bar{z}}\left(\frac{1}{2}\varepsilon^2(\bar{\varphi},_{\bar{x}}^2 + \bar{\varphi},_{\bar{z}}^2) + \varepsilon Fr\sqrt{kh}\,\bar{\varphi},_{\bar{x}}\right) +$$
$$+\frac{\varepsilon^3\bar{\eta}^2}{2}Fr\sqrt{kh}\frac{\partial^2}{\partial\bar{z}^2}\left(\bar{\varphi},_{\bar{x}}\right) + O(\varepsilon^3) = 0\ , \quad \bar{z} = 0. \tag{11}$$

Similarly the kinematic free surface b.c. (8) takes the following form

$$(\varepsilon^2\bar{\varphi},_{\bar{x}}\,\bar{\eta},_{\bar{x}} + \varepsilon Fr\sqrt{kh}\,\bar{\eta},_{\bar{x}} - \varepsilon\bar{\varphi},_{\bar{z}}) + \varepsilon\bar{\eta}\frac{\partial}{\partial\bar{z}}(-\varepsilon\bar{\varphi},_{\bar{z}}) + O(\varepsilon^3) = 0, \qquad \bar{z} = 0. \tag{12}$$

Substituting the perturbations expansions, eqs. (9), into the Laplace equation (6) and the bottom boundary condition, we have, retaining only terms of first order in ε (the others being much smaller):

$$\nabla^2\bar{\varphi}^{(1)} + \varepsilon\nabla^2\bar{\varphi}^{(2)} + \ldots = 0,$$
$$\bar{\varphi},_{\bar{z}}^{(1)} + \varepsilon\bar{\varphi},_{\bar{z}}^{(2)} + \ldots = 0, \qquad \bar{z} = -kh. \tag{13}$$

At the free surface, we obtain from the dynamic (11) and kinematic (12) boundary conditions, respectively:

$$\frac{1}{2}\varepsilon\left((\bar{\varphi},_{\bar{x}}^{(1)})^2 + (\bar{\varphi},_{\bar{z}}^{(1)})^2\right) + Fr\sqrt{kh}\,\bar{\varphi},_{\bar{x}}^{(1)} + \varepsilon Fr\sqrt{kh}\,\bar{\varphi},_{\bar{x}}^{(2)} + \bar{\eta}^{(1)} + \varepsilon\bar{\eta}^{(2)} +$$
$$+\varepsilon\bar{\eta}^{(1)}Fr\sqrt{kh}\frac{\partial}{\partial\bar{z}}\left(\bar{\varphi},_{\bar{x}}^{(1)}\right) + O(\varepsilon^2) = 0, \qquad \bar{z} = 0, \tag{14}$$

$$\varepsilon\bar{\varphi},_{\bar{x}}^{(1)}\,\bar{\eta},_{\bar{x}}^{(1)} + Fr\sqrt{kh}\,\bar{\eta},_{\bar{x}}^{(1)} + \varepsilon Fr\sqrt{kh}\,\bar{\eta},_{\bar{x}}^{(2)} - \bar{\varphi},_{\bar{z}}^{(1)} - \varepsilon\bar{\varphi},_{\bar{z}}^{(2)} - \varepsilon\bar{\eta}^{(1)}\bar{\varphi},_{\bar{z}\bar{z}}^{(1)} + O(\varepsilon^2) = 0,$$
$$\bar{z} = 0. \tag{15}$$

The original non-linear boundary value problem $\mathscr{P}_{\varphi}$ has now been reformulated into an infinite set of linear equations of ascending orders. Before proceeding to the separation of the equations by order, we consider the following general form of the perturbed equations:

$$A^{(1)} + \varepsilon A^{(2)} + \varepsilon^2 A^{(3)} + \ldots = B^{(1)} + \varepsilon B^{(2)} + \varepsilon^2 B^{(3)} + \ldots$$

The required condition that the above equality holds for arbitrary ε is that the coefficients of like powers of ε must be equal. Therefore, we have that $A^{(1)} = B^{(1)}$, $A^{(2)} = B^{(2)}$, $A^{(3)} = B^{(3)}$, etc.

Using this procedure we obtain:

- *First-Order perturbation equations*

If we gather together all the terms that do not depend on ε, the linear equations result:

$$\nabla^2 \overline{\varphi}^{(1)} = 0, \tag{16a}$$

$$Fr\sqrt{kh}\,\overline{\varphi},_{\overline{x}}^{(1)} + \overline{\eta}^{(1)} = 0, \quad \overline{z} = 0 \tag{16b}$$

$$Fr\sqrt{kh}\,\overline{\eta},_{\overline{x}}^{(1)} - \overline{\varphi},_{\overline{z}}^{(1)} = 0, \quad \overline{z} = 0 \tag{16c}$$

$$\overline{\varphi},_{\overline{z}}^{(1)} = 0, \qquad \overline{z} = -kh \tag{16d}$$

The above equations in dimensional form are the linear Neumann-Kelvin problem referred to §2.1. Moreover the general solution in dimensional form is (see i.e. Stoker 1957, §7.3)

$$\varphi(x,z) = A\cos(kx + a)\cosh(k(z+h)),$$

with A and α arbitrary constants , and k a root of the dispersion equation (cf. eq.23 §2.2).

- *Second-Order perturbation equations*

If we gather together all the terms that depend on ε, we have:

$$\nabla^2 \overline{\varphi}^{(2)} = 0, \tag{17a}$$

$$\overline{\eta}^{(2)} + Fr\sqrt{kh}\,\overline{\varphi},_{\overline{x}}^{(2)} = -\frac{1}{2}\left((\overline{\varphi},_{\overline{x}}^{(1)})^2 + (\overline{\varphi},_{\overline{z}}^{(1)})^2\right) - Fr\sqrt{kh}\,\overline{\varphi},_{\overline{x}}^{(1)} - \overline{\eta}^{(1)} Fr\sqrt{kh}\frac{\partial}{\partial \overline{z}}\left(\overline{\varphi},_{\overline{x}}^{(1)}\right), \quad \overline{z} = 0$$

(17b)

$$Fr\sqrt{kh}\,\overline{\eta},_{\overline{x}}^{(2)} - \overline{\varphi},_{\overline{z}}^{(2)} = \overline{\eta}^{(1)}\overline{\varphi},_{\overline{z}\overline{z}}^{(1)} - \overline{\varphi},_{\overline{x}}^{(1)}\overline{\eta},_{\overline{x}}^{(1)}, \quad \overline{z} = 0 \tag{17c}$$

$$\overline{\varphi},_{\overline{z}}^{(2)} = 0, \qquad \overline{z} = -kh. \tag{16d}$$

We denote that all equations and conditions are linear in the variables of interest, $\overline{\varphi}^{(2)}(\overline{x},\overline{z})$ and $\overline{\eta}^{(2)}(\overline{x})$, but the free surface boundary conditions have inhomogeneous terms that depend on the first solution. Since the first-order solution is known, the terms on the right-hand side are known also.

APPENDIX 2.B. Non-orthogonality relation of the eigenfunctions

With application of equation (21a) §2.2, for two different eigenfunctions $Z_n(z)$ and $Z_m(z)$, we obtain

$$Z_n''(z) + k_n^2 Z_n(z) = 0, \tag{1a}$$

$$Z_m''(z) + k_m^2 Z_m(z) = 0. \tag{1b}$$

We denote that the eigenfunctions $Z_n(z)$ and $Z_m(z)$ are corresponded to the eigenvalues k_n^2 and k_m^2, where $k_n^2 \neq k_m^2$. By multiplication of eq. (1a) and eq. (1b) with the eigenfunctions $Z_m(z)$ and $Z_n(z)$ respectively, and subtract by parts, we have

$$Z_m Z_n'' - Z_n Z_m'' = (k_m^2 - k_n^2) Z_n Z_m. \tag{2}$$

The above relation is defined for $z \in [-h, 0]$; hence by integration on this interval and applying integration by parts we obtain

$$\begin{aligned}< Z_n, Z_m >&= \int_{-h}^{0} Z_n(z) Z_m(z) dz = \frac{1}{(k_m^2 - k_n^2)} \int_{-h}^{0} (Z_m Z_n'' - Z_n Z_m'') dz = \\ &= \frac{1}{(k_m^2 - k_n^2)} \left\{ Z_m Z_n' \big|_{-h}^{0} - Z_n Z_m' \big|_{-h}^{0} - \int_{-h}^{0} Z_m' Z_n' dz + \int_{-h}^{0} Z_n' Z_m' dz \right\},\end{aligned} \tag{3}$$

where $< \cdot, \cdot >$ is the L^2 – inner product.

Using the bottom boundary condition, $z = -h$ eq.(21c) §2.2 of the vertical eigenvalue problem $VE(-h,0)$ we obtain from (3)

$$< Z_n, Z_m >= \frac{1}{(k_m^2 - k_n^2)} \{ Z_m(0) Z_n'(0) - Z_n(0) Z_m'(0) \}, \quad m \neq n. \tag{4}$$

With application of free surface boundary condition, $z = 0$ eq.(21b) §2.2 the above relation takes the following form

$$< Z_n, Z_m >= hFr^2, m \neq n. \tag{5}$$

According to the above, the following non-orthogonal relation of the eigenfunction of the vertical eigenvalue problem $VE(-h,0)$, is defined

$$< Z_n, Z_m >= \begin{cases} hFr^2 & , \ m \neq n \\ \|Z_n\|^2 & , \ m = n \end{cases} \tag{6}$$

where, $\|Z_n\| = \left(\int_{-h}^{0} Z_n^2(z) dz \right)^{1/2}$ is the L^2 – norm of the eigenfunction $Z_n(z)$.

Chapter Three
Steady Free-Surface Flows obstructed by underwater Steps and Trenches.

3. Steady Free-Surface Flows obstructed by underwater Steps and Trenches.

3.1 Free-Surface flow over an infinite step.

3.1.1 Differential formulation of the problem.

The steady two-dimensional free-surface flow of a stream, of an inviscid incompressible and irrotational fluid, which is obstructed by a semi-infinite step on the bottom, is considered. The wave field is excited by an incident uniform flow with direction normal to the bottom contours, and velocity *U*. The studied free-surface flow consists of a water layer D_{3D} bounded above by the free surface $\partial D_{F,3D}$ and below by a rigid bottom $\partial D_{\Pi,3D}$.

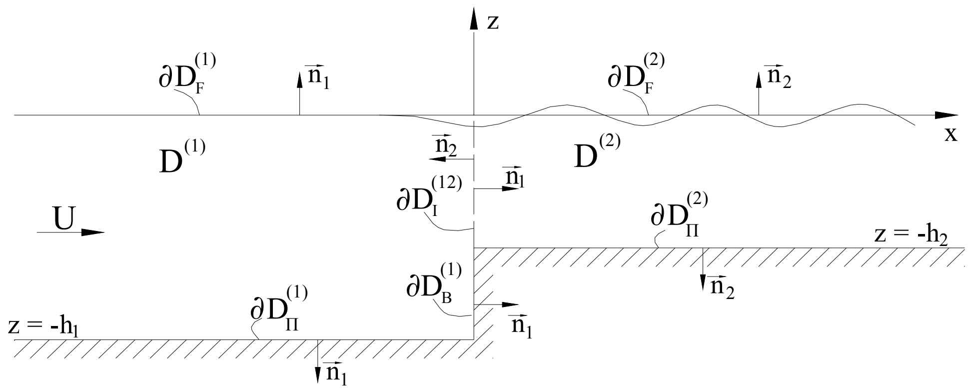

FIGURE 3.1. Domain decomposition and basic notation

Before proceeding to the formulation of the problem, we shall introduce some geometrical notation. A Cartesian coordinate system is introduced, with its origin on the mean water level (at the cut of the step), the *z*-axis pointing upwards and the *y*-axis being parallel to the bottom contours. See figure 3.1.

The liquid domain D_{3D} will be represented by $D_{3D} = D \times R$, where D is the (two-dimensional) intersection of D_{3D} by a vertical plane perpendicular to the bottom contours, and $R = (-\infty, +\infty)$, is a copy of the real line:

$D_{3D} = \{(x, y, z): \ (x, y) \in R^2, -h(x) < z < 0\}$, $D = \{(x, z): \ x \in R \ , -h(x) < z < 0\}$,

where $h(x) = \begin{cases} -h_1, & x < 0 \\ -h_2, & x \geq 0 \end{cases}$.

The liquid domain D_{3D} is decomposed in two subdomains $D_{3D}^{(i)} = D^{(i)} \times R, \ i = 1, 2,$ defined us follows: $D_{3D}^{(1)}$ is the constant-depth upstream subdomain characterized by $x < 0$, and $D_{3D}^{(2)}$ is the constant-depth downstream subdomain characterized by $x > 0$. Without loss of generality, we assume that $h_1 > h_2$.

The decomposition is also applied to the boundaries $\partial D_{F,3D} = \partial D_F \times R$ and $\partial D_{\Pi,3D} = \partial D_\Pi \times R$. The lines ∂D_F and ∂D_Π are decomposed in two pieces each, for example, $\partial D_F = \partial D_F^{(1)} \cup \partial D_F^{(2)}$, where $\partial D_F^{(1)}$ belongs to the boundary of $D^{(1)}$, and similarly for ∂D_Π. Finally, we define the artificial vertical interface $\partial D_{I,3D}^{(12)} = \partial D_I^{(12)} \times R$, which is the common vertical boundary of subdomains $D_{3D}^{(1)}$ and $D_{3D}^{(2)}$, and the vertical wall $\partial D_{B,3D}^{(1)} = \partial D_B^{(1)} \times R$ which belongs to the boundary of $D_{3D}^{(1)}$. Clearly, $\partial D_I^{(12)}$ is vertical segment (between the bottom and the mean water level) at $x = 0$. See figure 3.1.

We consider that the motion arising from disturbances created by the obstruction of the uniform flow by the step, have the velocity potential

$$\Phi(x,z) = Ux + \varphi(x,z), \qquad (x,z) \in D. \tag{1}$$

Assuming that the disturbance velocity potential $\varphi(x,z)$ and the velocity of the stream are small enough, the linearized equations of the *Neumann-Kelvin* problem can be used (cf. § 2.1). By the decomposition of the liquid domain $D = D^{(1)} \cup D^{(2)}$, the studied problem should be formulated with the aid of the general representation of the disturbance velocity potential $\varphi(x,z)$ in the semi-infinite strips $D^{(1)}$ and $D^{(2)}$ as obtained in § 2.2

Taking under consideration the decomposition of the total field in two regions $D^{(1)}, D^{(2)}$ the problem can be formulated as follows:

<u>*PROBLEM*</u> $P_M(D, \varphi^{(1)}, \varphi^{(2)})$. *Given the upstream velocity U, find the disturbance velocity potentials* $\varphi^{(1)} \in D^{(1)}$, $\varphi^{(2)} \in D^{(2)}$, *satisfying the following system of equations, boundary and matching conditions:*

$$\nabla^2 \varphi^{(i)} = 0 \qquad (x,z) \in D^{(i)}, \tag{2.1}$$

$$\varphi,_{xx}^{(i)} + \frac{g}{U^2} \varphi,_z^{(i)} = 0 \qquad (x,z) \in \partial D_F^{(i)}, \tag{2.2}$$

$$\varphi,_z^{(i)} = 0 \qquad (x,z) \in \partial D_\Pi^{(i)}, \tag{2.3}$$

$$\varphi^{(1)} = \varphi^{(2)} \qquad (x,z) \in \partial D_I^{(12)}, \tag{2.4}$$

$$\frac{\partial \varphi^{(1)}}{\partial \vec{n}^{(1)}} = -\frac{\partial \varphi^{(2)}}{\partial \vec{n}^{(2)}} \quad (x,z) \in \partial D_I^{(12)}, \quad \frac{\partial \varphi^{(1)}}{\partial \vec{n}^{(1)}} = -U \cdot n_x^{(1)} \quad (x,z) \in \partial D_B^{(1)} \tag{2.5a,b}$$

where $\vec{n}^{(i)} = (n_x^{(i)}, n_z^{(i)})$ *is the unit normal vector to the boundary* $\partial D^{(i)}$ *directed to the exterior of* $D^{(i)}$ $i = 1,2$.

However, appropriate conditions at infinity $|x| \to \infty$ must imposed in order the problem to be well-defined. These conditions are usually called 'radiations conditions', although in some cases there may be no real 'radiation at infinity'. However, in the subcritical case ($Fr < 1$) a radiation condition – such as the requirement of the disturbance to die out upstream – is imposed. The formulation of the above radiation condition upstream has been discussed by Lamb (1932, p.406) and Stoker (1957, p.209); see also Wehausen & Laitone (1960, p. 569). It appears from the definition about the velocity potential $\Phi \in D$, that the conservation of mass is not valid, since the velocity of the uniform flow remains constant; hence it is independent from the variation of the depth. Consequently, proper condition at infinity downstream must be imposed for the conservation of mass. The formulation of the radiation boundary condition downstream using the continuity equation in integral form is introduced in § 3.1.2.

Before proceeding to the reformulation of general representation of the disturbance velocity potential $\varphi(x,z)$ in $D^{(1)}$ and $D^{(2)}$ according to the above definitions, let us define the following sets of index with respect to the studied cases:

– *Subrcritical case* ($Fr < 1$): $N_f = \{n: \ n = 1, 2, ...\}$

– *Supercritical case* ($Fr > 1$): $N_t = \{n: \ n = 0, 1, 2, ...\}$

Hence, taking the real part of equations (2.2.26) and (2.2.27), (cf. § 2.2), and imposing the appropriate radiation conditions, as indicated above, we obtain:

(a) <u>Upstream</u> ($\varphi \in D^{(1)}$)

a1. *Subcritical case* ($Fr_1 < 1$):

$$\varphi^{(1)}(x,z) = \sum_{n=1}^{\infty} C_n^{(1)} \exp(k_n^{(1)} x) Z_n^{(1)}(z), \quad ((x,z) \in D^{(1)}), \tag{3a}$$

a2. *Supercritical case* ($Fr_1 > 1$):

$$\varphi^{(1)}(x,z) = \sum_{n=0}^{\infty} C_n^{(1)} \exp(k_n^{(1)} x) Z_n^{(1)}(z), \quad ((x,z) \in D^{(1)}), \tag{3b}$$

(b) <u>Downstream</u> ($\varphi \in D^{(2)}$)

b1. *Subcritical case* ($Fr_2 < 1$):

$$\varphi^{(2)}(x,z) = A_{-1}^{(2)} x + B_{-1}^{(2)} + (A_0^{(2)} \cos(k_0^{(2)} x) + B_0^{(2)} \sin(k_0^{(2)} x)) Z_0^{(2)}(z) + \\ + \sum_{n=1}^{\infty} C_n^{(2)} \exp(-k_n^{(2)} x) Z_n^{(2)}(z), \quad ((x,z) \in D^{(2)}), \tag{4a}$$

b2. *Supercritical case* ($Fr_2 > 1$):

$$\varphi^{(2)}(x,z) = A_{-1}^{(2)}x + B_{-1}^{(2)} + \sum_{n=0}^{\infty} C_n^{(2)} \exp(-k_n^{(2)}x) Z_n^{(2)}(z)\,,\ ((x,z)\in D^{(2)})\,, \tag{4b}$$

where $A_{-1}^{(2)}, B_{-1}^{(2)}$, $A_0^{(2)}, B_0^{(2)}$ and $C_n^{(i)}$, $n \in N_f$ or $n \in N_t$, $i = 1,2$, are real constants.

In the expansions (3) and (4) the sets of numbers
$\left\{k_n^{(1)}, n \in N_f\right\} \cap \{0\}$, $\left\{k_0^{(2)}, k_n^{(2)}, n \in N_f\right\} \cap \{0\}$, $\left\{k_n^{(i)}, n \in N_t\right\} \cap \{0\}$ $i = 1,2$,
and the sets of vertical functions
$\left\{Z_n^{(1)}(z), n \in N_f\right\} \cap \{1\}$, $\left\{Z_0^{(2)}(z), Z_n^{(2)}(z), n \in N_f\right\} \cap \{1\}$, $\left\{Z_n^{(i)}(z), n \in N_t\right\} \cap \{1\}$ $i = 1,2$,
are the eigenvalues and the corresponding eigenfunctions of the vertical eigenvalue problems $VE(-h_i,0)$, $i = 1,2$ obtained by separation of variables in the half strips $D^{(1)}$ and $D^{(2)}$ (cf. § 2.2). The eigenvalues are given as the roots of the relations

$$k_n^{(i)}h = \frac{1}{Fr_i^2}\tan(k_n^{(i)}h_i) \quad (n \in N_t \text{ or } n \in N_f\,, i = 1,2), \tag{5a}$$

$$k_0^{(2)}h = \frac{1}{Fr_2^2}\tanh(k_0^{(2)}h_2)\,, \quad \text{where} \quad Fr_i = \frac{U}{\sqrt{gh_i}} \quad i = 1,2\,,$$

and the eigenfunctions are given by

$$Z_0^{(2)}(z) = \frac{\cosh\left(k_0^{(2)}(z+h_2)\right)}{\cosh(k_0^{(2)}h_2)}\,,\ Z_n^{(i)}(z) = \frac{\cos\left(k_n^{(i)}(z+h_i)\right)}{\cos(k_n^{(i)}h_i)}\,,\ (n \in N_t \text{ or } n \in N_f\,, i = 1,2). \tag{5b}$$

The correctness (completeness) of the expansions (3) and (4) follows by the theorem, which is introduced in § 2.2.

We remark that in the general representations of disturbance potential (eq. (3)), the coefficient B_{-1} represents a part of the potential difference between far upstream and downstream, due to the fluid acceleration and deceleration by the obstruction, and is essentially the so-called *blockage parameter* which is discussed by Newman (1969). Without loss of generality we define as essential condition upstream that $B_{-1}^{(1)} = 0$, i.e. the radiation condition upstream is

$$\varphi(x,z) \to 0 \ \text{ as } x \to -\infty\,. \tag{6}$$

Given the upstream velocity *U*, and imposing the downstream radiation condition for conservation of mass, the half strip potentials $\varphi^{(1)}$ and $\varphi^{(2)}$ are uniquely determined by the means of the real coefficients $\left\{C_n^{(1)}\right\}$, and $B_{-1}^{(2)}$ $A_0^{(2)}, B_0^{(2)}$, $\left\{C_n^{(2)}\right\}$ $n \in N_f$ or $n \in N_t$, respectively.

3.1.2 Far field downstream condition using the conservation of mass in integral form.

As mentioned above a radiation condition must be imposed for the conservation of mass. Let us consider the problem illustrated in Fig. 3.2 of a fluid occupies the region D with boundary ∂D where $\partial D = \partial D_F \cup \partial D_{-\infty} \cup \partial D_\Pi \cup \partial D_B \cup \partial D_\infty$. We denote that the boundary segment ∂D_∞ ($\partial D_{-\infty}$) is an artificial boundary, chosen to be far downstream (upstream). Also, we assume that the disturbance velocity potential $\varphi(x,z)$ on this boundary is maximum or minimum. As we will show in § 3.2.2, the validity of the following result is independent from this consideration.

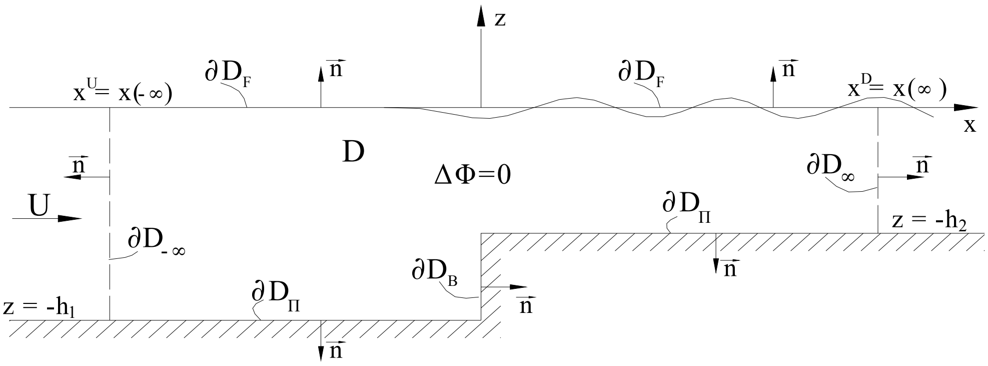

FIGURE 3.2. The domain of definition D of the velocity potential Φ

Without loss of generality we assume that the flow is subcritical in the region D. We recall that the velocity potential in domain D is

$$\Phi\left(x,z\right) = Ux + \varphi(x,z) \qquad \left(x,z\right) \in D, \tag{7}$$

where, $\varphi(x,z)$ is the disturbance velocity potential.

Let us consider the velocity potential Φ, which is finite, singled valued and differentiable at all points of the connected region D bounded by the closed surface ∂D. Hence, the following form of Green's theorem may be obtained for the above region D, i.e. Lamb (1932, p.43):

$$\int_{\partial D} \frac{\partial \Phi}{\partial \vec{n}} dS = \int_{D} \nabla^2 \Phi dV, \tag{8}$$

where $\vec{n} = (n_x, n_z)$ is the unit normal vector to the boundary ∂D directed to the exterior of D.

Based on the fact that the function Φ satisfies the Laplace equation in region $D^{(i)}$, the Green's theorem takes the following form:

$$\int_{\partial D} \frac{\partial \Phi}{\partial \vec{n}} dS = 0, \tag{9}$$

which simply express the conservation of mass in integral form for steady irrotational motion of an ideal fluid. Its physical significance is in any region occupied wholly by liquid, the total flux across the boundary is zero.

The function φ cannot be a maximum or a minimum at a point in the interior of the fluid; for, if it were, we should have $\partial\varphi/\partial n$ everywhere positive, or everywhere negative a fact that is inconsistent with (9), therefore the function φ can be maximum or minimum at the boundary, see i.e. Lamb (1932, p.38).

Applying the equation (9) in region D, with the aid of Eq.(7) we obtain

$$\int_{\partial D} \frac{\partial\Phi}{\partial\vec{n}} dS = \int_{\partial D} (U \cdot n_x + \frac{\partial\varphi}{\partial\vec{n}}) dS = 0 . \tag{10}$$

The disturbance velocity potential φ satisfies the boundary conditions of the problem P_M ; therefore, the above integral gives

$$-\int_{\partial D_F} \frac{U^2}{g} \varphi,_{xx} dS + \underbrace{\int_{\partial D_B \cup \partial D_\Pi} (Un_x + \varphi,_n) dS}_{0} - \int_{\partial D_{-\infty}} (U + \varphi,_x) dS + \int_{\partial D_\infty} (U + \varphi,_x) dS = 0 . \tag{11}$$

As $x \to -\infty$, one can easily verify from Eq.(3a) that $\varphi,_x \to 0$, then Eq.(11) obtains $-\frac{U^2}{g} \int_{x=x^U}^{x=x^D} \varphi,_{xx} dx - U \int_{-h_1}^{0} dz + U \int_{-h_2}^{0} dz + \int_{\partial D_\infty} \varphi,_x dz = 0$. (12)

From Eq.(4a), assuming that φ is maximum in ∂D_∞, one can verify that as $x \to \infty$ $\varphi,_x \to A_{-1}^{(2)}$. Hence, Eq.(12) yields (defining $u \equiv A_{-1}^{(2)}$)

$$-\frac{U^2}{g} \int_{x=x^U}^{x=x^D} \varphi,_{xx} dx + uh_2 + U(h_2 - h_1) = 0 \quad \Rightarrow \tag{13}$$

$$-\frac{U^2}{g} (\varphi,_x \Big|_{x=x^D} - \varphi,_x \Big|_{x=x^U}) + uh_2 + U(h_2 - h_1) = 0 \quad \Rightarrow \text{(using the asymptotic forms),}$$

$$u(h_2 - \frac{U^2}{g}) + U(h_2 - h_1) = 0 \quad \Rightarrow \quad u = U \frac{h_1 - h_2}{h_2 - \frac{U^2}{g}} \tag{14}$$

Defining the shoaling ratio $s = h_2 / h_1$, which is taken the values $s < 1$ if the flow is from deep to shallow water and $s > 1$ if the flow is from shallow to deep water, and denoting that $Fr_1 = U / \sqrt{gh_1}$, the above relation takes the form

$$u(s;U) = \frac{U(1-s)}{s - Fr_1^2} , \tag{15a}$$

or in non-dimensional form

$$\frac{u+U}{U}=\frac{1-Fr_1^2}{s-Fr_1^2}\,. \tag{15b}$$

With reference to the asymptotic form of the potential field downstream

$$\Phi \xrightarrow{x\to\infty} (U+u)x+B_{-1}^{(2)}+(A_0^{(2)}\cos(k_0^{(2)}x)+B_0^{(2)}\sin(k_0^{(2)}x))Z_0^{(2)}(z)\,,$$

we observe that the velocity u implies a difference to the uniform velocity of a stream, such that the conversation of mass to be satisfied. As a consequence of the linearity of the problem the downstream velocity must be of the same order as upstream velocity i.e. the quantity $\frac{u+U}{U}$ must be of order $O(1)$. Hence, the disturbance velocity u must be a small quantity; a fact, which is consistent with the linearization of the problem about the uniform flow as well as that the disturbance velocity field $\varphi,_x$ must be a small quantity (cf. Appendix 2.A).

In the case of subcritical flow (downstream, $s > Fr_1^2 \Leftrightarrow Fr_2 < 1$) and for $s<1$, equation (15a) indicates that $u>0$, so the velocity of the uniform flow downstream is greater than upstream. Similarly, as for supercritical flow ($Fr_2 > 1 \Leftrightarrow s < Fr_1^2$) we have that $u<0$, so the velocity of the uniform flow upstream is greater than downstream. If the shoaling ratio is greater than unity ($s>1$), in case of subcritical flow equation (15a) indicates that $u<0$, so the velocity of the uniform flow upstream is greater than downstream. Further, for supercritical flow $u>0$, so the velocity of the uniform flow downstream is greater than upstream.

Some peculiarities occur when the flow downstream is critical ($Fr_2 = 1$). In this case, the critical shoaling ratio is $s_c = Fr_1^2$, and the denominator of equation (15a) is vanished, so the velocity u becomes infinite. This, from the physical point of view and under the considerations of the problem, denotes that if the flow downstream is critical the disturbance potential and the velocity become infinite. However, Stoker (1957) has shown that in this case there is no steady state motion, by studying the unsteady problem and allowing the time to tend to ∞.

In addition, some remarks may be obtained for the free surface elevation $\eta(x)$. Recalling that

$$\eta(x)=-\frac{U}{g}\varphi,_x(x,0) \tag{16}$$

we have that as x tends to infinite far upstream $\eta \xrightarrow{x\to -\infty} 0$, and far downstream

$$\eta \xrightarrow{x\to\infty} -\left(\frac{U\cdot u}{g}+\frac{U\cdot k_0^{(2)}}{g}\left(B_0^{(2)}\cos(k_0^{(2)}x)-A_0^{(2)}\sin(k_0^{(2)}x)\right)\right) \text{ if } Fr_2<1, \tag{17a}$$

$$\eta \xrightarrow{x\to\infty} -\frac{U\cdot u}{g} \text{ if } Fr_2>1\,. \tag{17b}$$

If the flow is subcritical downstream and $u > 0$ (i.e. $s < 1$), as per asymptotic form of η the first term of (17a) is negative, hence an oscillating steady motion about the level $z = -U \cdot u / g$, with amplitude

$$a = \frac{U \cdot k_0^{(2)}}{g} \sqrt{\left(A_0^{(2)}\right)^2 + \left(B_0^{(2)}\right)^2} \tag{18}$$

and wavelength $\lambda = 2\pi / k_0^{(2)}$ is predicted.

On the other hand, if $u < 0$ (i.e. $s > 1$) the mean water level rises -according to the first term of (17a)- so an oscillating steady motion about the level $z = U \cdot u / g$ is predicted. In case of supercritical flow downstream and $u < 0$, according to (17b) we obtain that the free surface rises over the step and is asymptotically flat. Similarly, as for $u > 0$ the free surface drops and asymptotically becomes flat.

Concluding, we should point out that considering the potential field arising from the potential of uniform flow and disturbance potential (cf. Eq.(7)), the continuity of the uniform flow is provided. Moreover, the conditions expressing the continuity of the disturbance potential φ and the induced velocity field $\partial\varphi / \partial n$ (matching conditions) are satisfied, by using the proper radiation condition downstream as follows from the conservation of mass.

3.1.3 Weak formulation of matching -B.V.P. – Composition of the solution matrix.

In this subsection we proceed to derivation of the linear system, which arises from the matching boundary value problem P_M. We recall that the matching conditions at the cut of the step are:

$$\varphi^{(1)} = \varphi^{(2)} \qquad -h_2 < z < 0, \quad x = 0, \tag{19a}$$

$$\frac{\partial \varphi^{(1)}}{\partial x^{(1)}} = \begin{cases} \dfrac{\partial \varphi^{(2)}}{\partial x^{(2)}}, & -h_2 < z < 0, \quad x = 0 \\ -U \cdot n_x, & -h_1 < z < -h_2, \quad x = 0 \end{cases}, \tag{19b}$$

where, $\varphi^{(i)}(x,z)$ $i = 1,2$ is the disturbance velocity potential, given by the expansions (3) and (4). Let us define a function $f(z)$ to be equal with the right side of equation (19b), so the above condition may be rewrite to the following form

$$\varphi^{(1)} - \varphi^{(2)} = 0 \qquad -h_2 < z < 0, \quad x = 0, \tag{19΄a}$$

$$\frac{\partial \varphi^{(1)}}{\partial x^{(1)}} - f(z) = 0 \qquad -h_1 < z < 0, \quad x = 0. \tag{19΄b}$$

According to the theorem of § 2.2 the set of eigenfunctions $Z_n^{(1)}(z)$ and $Z_n^{(2)}(z)$ constitute a *Riesz basis* on $L^2(-h_1,0)$ and $L^2(-h_2,0)$, respectively. Using this definition, the above system of equations is equivalent on *weak sense* to the following:

$$\left\langle \varphi^{(1)} - \varphi^{(2)}, Z_m^{(2)}(z) \right\rangle = 0\,, \qquad x = 0 \qquad \forall m \tag{20a}$$

$$\left\langle \frac{\partial \varphi^{(1)}}{\partial x^{(1)}} - f(z), Z_m^{(1)}(z) \right\rangle = 0\,, \quad x = 0 \qquad \forall m \tag{20b}$$

where $\langle \cdot,\cdot \rangle$ is the L^2- inner product .

Without loss of generality, we introduce the derivation of linear system which constitutes the above equations, in the case where the flow is subcritical in the region D. Similar procedure is following in case of supercritical flow. We denote here, that in the expansion of $\varphi^{(1)}$ (Eq. (3a)), the terms that imply a disturbance on upstream region may be obtained, such that the number of equations to be equal with the number of unknowns. After the composition of the system's matrix, the radiation conditions shall be imposed. Truncating the series of expansions (3) and (4) to a finite number of terms (modes), and denoting by N the number of evanescent modes retained, with above definitions we have

$$\varphi^{(1)}(x,z) = (A_0^{(1)} \cos(k_0^{(1)} x) + B_0^{(1)} \sin(k_0^{(1)} x)) Z_0^{(1)}(z) + \sum_{n=1}^{N} C_n^{(1)} \exp(k_n^{(1)} x) Z_n^{(1)}(z)\,,$$
$$((x,z) \in D^{(1)})\,, \tag{21a}$$

$$\varphi^{(2)}(x,z) = A_{-1}^{(2)} x + B_{-1}^{(2)} + (A_0^{(2)} \cos(k_0^{(2)} x) + B_0^{(2)} \sin(k_0^{(2)} x)) Z_0^{(2)}(z)$$
$$+ \sum_{n=1}^{N} C_n^{(2)} \exp(-k_n^{(2)} x) Z_n^{(2)}(z) \qquad ((x,z) \in D^{(2)})\,, \tag{21b}$$

Taking the derivatives of the above expansions with respect to x we have

$$\varphi,_x^{(1)}(x,z) = k_0^{(1)}(-A_0^{(1)} \sin(k_0^{(1)} x) + B_0^{(1)} \cos(k_0^{(1)} x)) Z_0^{(1)}(z) +$$
$$+ \sum_{n=1}^{N} k_n^{(1)} C_n^{(1)} \exp(k_n^{(1)} x) Z_n^{(1)}(z) \qquad ((x,z) \in D^{(1)})\,, \tag{22a}$$

$$\varphi,_x^{(2)}(x,z) = A_{-1}^{(2)} + k_0^{(2)}(-A_0^{(2)} \sin(k_0^{(2)} x) + B_0^{(2)} \cos(k_0^{(2)} x)) Z_0^{(2)}(z) -$$
$$- \sum_{n=1}^{N} k_n^{(2)} C_n^{(2)} \exp(-k_n^{(2)} x) Z_n^{(2)}(z) \qquad ((x,z) \in D^{(2)})\,. \tag{22b}$$

Substituting the expansions (21) in equation (20a), we obtain

$$\int_{-h_2}^{0} (\varphi^{(1)} - \varphi^{(2)}) \cdot Z_m^{(2)}(z)dz = 0 \Leftrightarrow$$

$$\Leftrightarrow A_0^{(1)} \int_{-h_2}^{0} Z_0^{(1)}(z) Z_m^{(2)}(z)dz + \sum_{n=1}^{N} C_n^{(1)} \int_{-h_2}^{0} Z_n^{(1)}(z) Z_m^{(2)}(z)dz - B_{-1}^{(2)} \int_{-h_2}^{0} Z_m^{(2)}(z)dz -$$

$$-A_0^{(2)} \int_{-h_2}^{0} Z_0^{(2)}(z) Z_m^{(2)}(z)dz - \sum_{n=1}^{N} C_n^{(2)} \int_{-h_2}^{0} Z_n^{(2)}(z) Z_m^{(2)}(z)dz = 0,$$

$$m = -1, 0, 1, ..., N, N+1. \qquad (20΄a)$$

Similar, from expansions (22), and equation (20b) we have

$$\int_{-h_2}^{0} (\varphi,_x^{(1)} - \varphi,_x^{(2)}) \cdot Z_m^{(1)}(z)dz + \int_{-h_1}^{-h_2} (\varphi,_x^{(1)} + U) \cdot Z_m^{(1)}(z)dz = 0 \Leftrightarrow$$

$$\Leftrightarrow \int_{-h_1}^{0} \varphi,_x^{(1)} Z_m^{(1)}(z)dz - \int_{-h_2}^{0} \varphi,_x^{(2)} Z_m^{(1)}(z)dz = -U \int_{-h_1}^{-h_2} Z_m^{(1)}(z)dz \Leftrightarrow$$

$$\Leftrightarrow k_0^{(1)} B_0^{(1)} \int_{-h_1}^{0} Z_0^{(1)}(z) Z_m^{(1)}(z)dz + \sum_{n=1}^{N} k_n^{(1)} C_n^{(1)} \int_{-h_1}^{0} Z_n^{(1)}(z) Z_m^{(1)}(z)dz -$$

$$-A_{-1}^{(2)} \int_{-h_2}^{0} Z_m^{(1)}(z)dz - k_0^{(2)} B_0^{(2)} \int_{-h_2}^{0} Z_0^{(2)}(z) Z_m^{(1)}(z)dz + \sum_{n=1}^{N} k_n^{(2)} C_n^{(2)} \int_{-h_2}^{0} Z_n^{(2)}(z) Z_m^{(1)}(z)dz =$$

$$= -U \int_{-h_1}^{-h_2} Z_m^{(1)}(z)dz \qquad m = -1, 0, 1, ..., N, N+1. \qquad (20΄b)$$

We denote that the term on the right side of (20΄b) is the forcing of the system.
The number of the unknown coefficients of expansions (21) and (22) is $\# 2(N+3)$, and the total number of equations from coupling the equations (20΄a) and (20΄b) is $\# 2(N+3)$.

3.1.4 Numerical results and discussion.

In this paragraph a detailed presentation of the numerical results obtained, according to the present method. The numerical calculations were performed for a step of variable height and for a wide range of upstream Froude numbers when the flow is subcritical or supercritical.

The present method is compared with the method developed by King & Bloor (1987), based on complex variable theory. These authors used a direct conformal transformation of the physical plane onto a half-plane. The transformation used is a generalization of the Schwartz-Cristoffel transformation.

The physical investigation of the influence of step height and of Froude number – the physical parameters of the problem – to the numerical solutions of the problem is introduced, in order to define the range of its values for which the linear solution is valid. The numerical accuracy of the problem is interpreted to satisfaction of the matching conditions at the cut of the step, i.e. the continuity of the pressure field and the continuity of the velocity field. This is concerned with the appropriate number of evanescent modes *N*, such that the week solution of the problem P_M to converge to the exact solution in the L^2-sense.

1. Subcritical flows $(Fr < 1)$

Numerical solutions have been obtained for a wide range of the physical parameters. However, in order to illustrate the results obtained we first investigate in detail the situations when the upstream depth is $1m$, and the step heights h_s are $0.01m$, $0.1\ m$, and $0.156m$, at upstream $Fr = 0.5$.

The figure 3.3 shows, as one would expect according to theoretical investigation of the present linear model (cf. § 3.2.2), that the level of the free surface falls as it approaches the step, while downstream of the step a periodic steady wave motion is predicted. The amplitude of these waves as well as the variation of mean water level depend upon both step height and Froude number whereas the wavelength depends only upon Froude number as shown in figure 3.3.

Figure 3.3 also shows a comparison of the free surface profiles with the method developed by King & Bloor (1987). In their analysis, they satisfy the exact free surface condition, which yields a nonlinear integro-differential equation for the free surface angle. They obtained approximate linear solutions depend on the order of the step height $O(h_s)$. For very small step ($h_s = 0.01m$) the results showing that the amplitude and wavelength of the present linear solution and non-linear solution obtained by King & Bloor (1987) are in good agreement as shown in figure 3.3a. In particular the amplitude of the non-linear solution is of same order with the amplitude of the present linear solution (about the same level), whereas a small phase shifting occurs between the non-linear and linear wavelengths. On the contrary, the present linear solution exhibits a deviation from the linear solution obtained by King & Bloor (1987). This is consistent with the error $O(h_s)$ between their non-linear and linear solution.

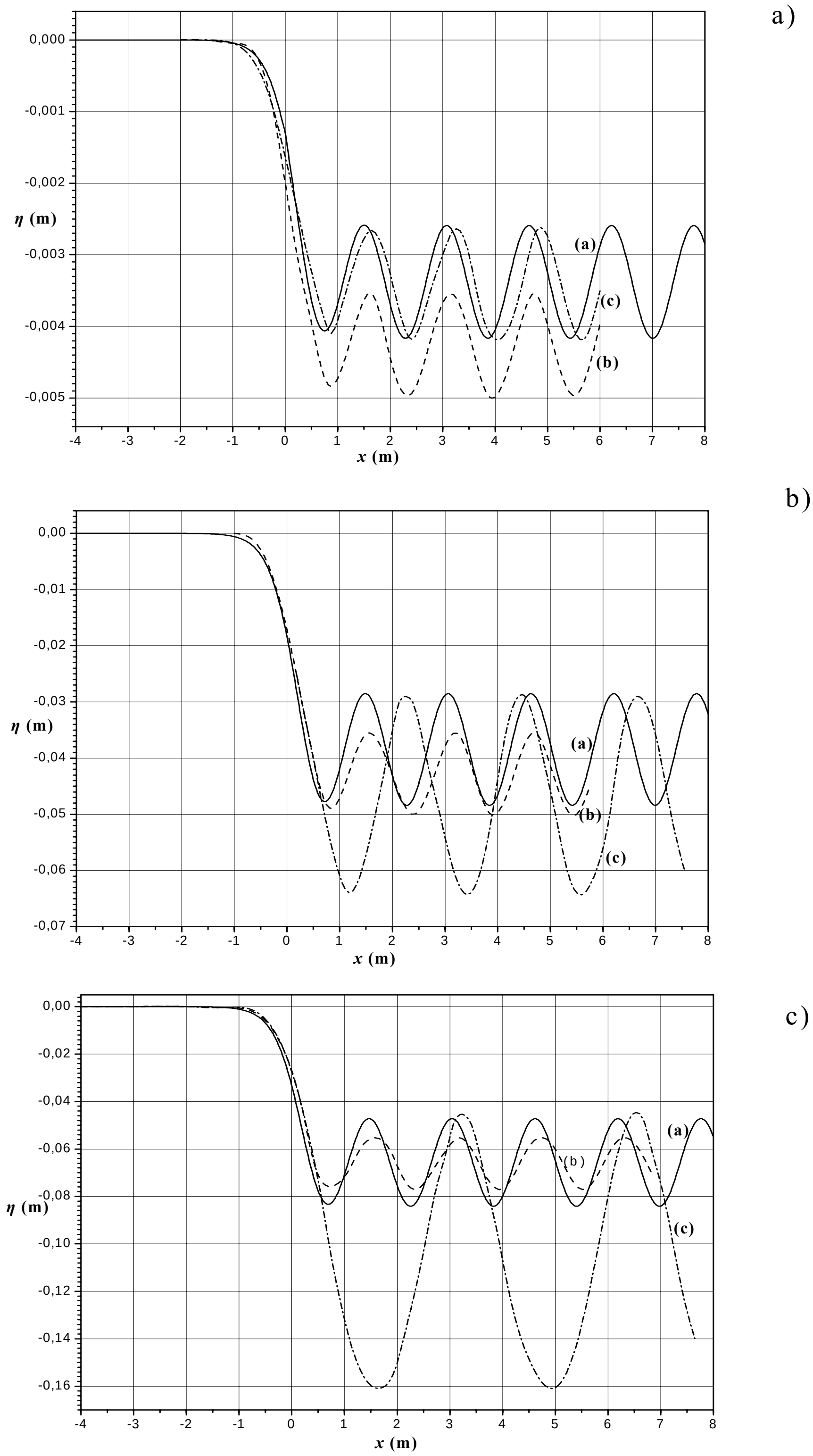

FIGURE 3.3. The free-surface elevation (a) of present model for $Fr = 0.5$ compared with linear (b) and non-linear (c) solutions obtained by King & Bloor (1987), for three values of step height: a) $h_s = 0.01$, b) $h_s = 0.1$, c) $h_s = 0.156$.

As the step height is increased a distinct difference appears between the solutions as shown in figure 3.3. At h_s = 0.1 the amplitude of the non-linear solution is $O(h_s)$ different from the linear amplitude whereas the non-linear wavelength is $O(1)$ different from the linear wavelength. For h_s = 0.156 the amplitude and wavelength of the linear solution to be grossly in error. In particular, the wavelength of the non-linear solution is two times the wavelength of the present linear solution as well as the wave amplitude and the mean water level. The non-linear solution is showing the classical narrow-peak, broad-through characteristics of the Stokes' theories and values of wave amplitude and wavelength, when adjusted to the local depth of fluid under the wave, are in good agreement with that predicted by fifth order Stokes' theory (see e.g. Wehausen & Laitone 1960, p.660). This is consistent with the considerations, which are adopted for the linearization of the non-liner problem P_{φ} (cf. for details Appendix 2.A). From these considerations such as the gradient of the free surface elevation ka and mainly the disturbance velocity field $\varphi,_x$ to be small quantities, it follows that the difference between the downstream $(U+u)$ and upstream U uniform velocity must be small (cf. the discussion in §3.1.2). We remark that this restriction arises mainly from the representation of the velocity potential i.e. $\Phi(x,z)=Ux+\varphi(x,z)$, which is taken for the formulation of the problem. Hence, suitable representation of the velocity field i.e. linearization of the problem about the proper uniform flow such us to permit large differences between the velocities upstream and downstream, can be provided a basis for better linear solutions.

In figure 3.4a the variation of the rate of downstream to upstream velocity of the flow, against the shoaling ratio $s=h_2/h_1$ ($s<1$) for various Froude numbers, is presented. We recall that u is the disturbance velocity, which arises from the conversation of mass, and U is the velocity in the undisturbed state. In reference to the discussion of §3.1.2, figure 3.4a shows that as the shoaling ratio s tends to its critical value $s_c = Fr_1^2$ the critical conditions downstream are approached. Under the considerations of the present linear model i.e. $\dfrac{U+u}{U} \approx O(1)$, figure 3.4a shows that for a small region of the parameters values the linear solution is valid.

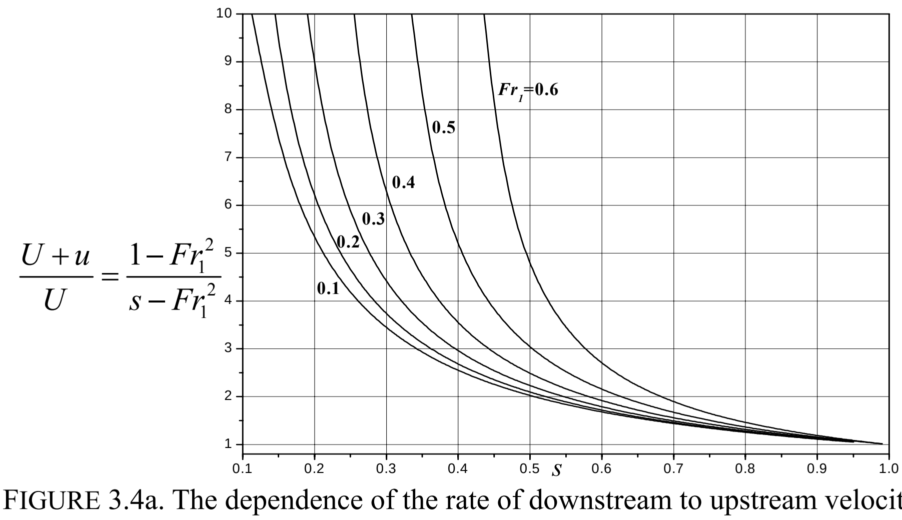

FIGURE 3.4a. The dependence of the rate of downstream to upstream velocity upon the upstream Froude number Fr_1 and upon the shoaling ratio $s=h_2/h_1$, ($h_2<h_1$).

On the contrary, if the flow is from shallow to deep water i.e. $s > 1$, figure 3.4b shows, as one would expect according to theoretical investigation (cf. § 3.1.2), that the downstream velocity tends to zero as the depth becomes large.

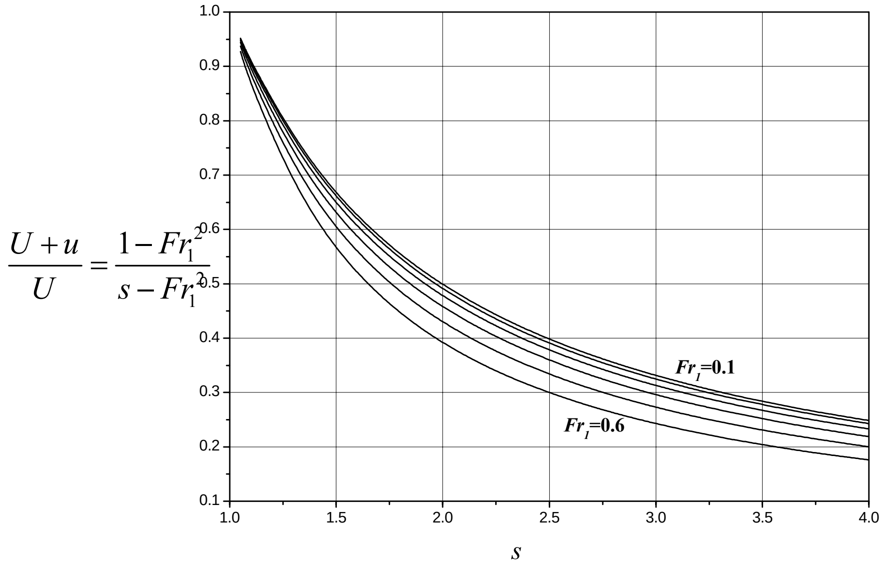

FIGURE 3.4b. The dependence of the rate of downstream to upstream velocity upon the upstream Froude number Fr_1 and upon the shoaling ratio $s = h_2 / h_1$, $(h_1 < h_2)$.

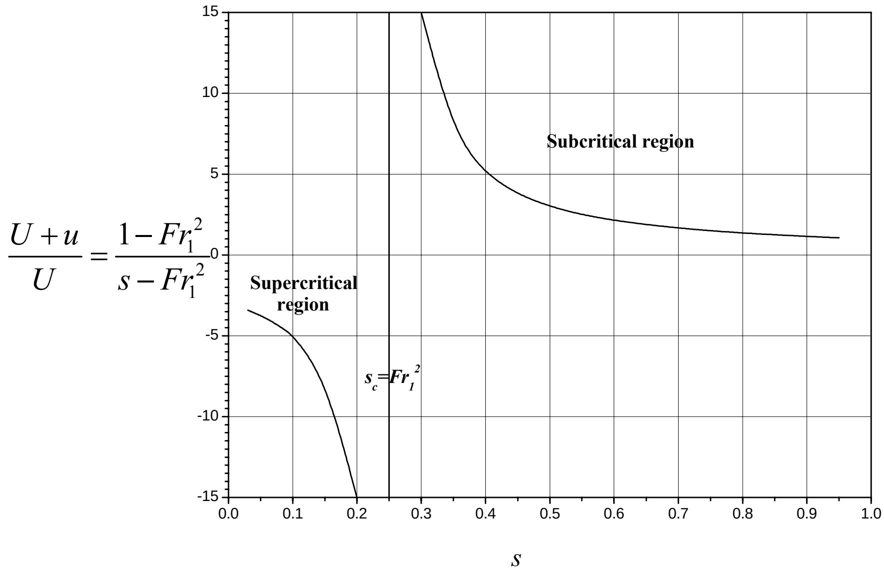

FIGURE 3.4c. The transition from subcritical flow to supercritical flow as the downstream depth reduces for Fr_1=0.5.

It is of interest to note that for finite-depth water the mean potentials far upstream and far downstream are different. This mean potential jump between the two infinities is due to the fluid acceleration and deceleration by the obstruction and defines the so-called *blockage parameter*, which is discussed by Newman (1969) in connection with channel flow. This blockage effect also denotes, if the condition of no flow normal to the cut of the step (or general of bottom obstruction) is satisfied exactly.

In figures 3.5a and 3.5b the variation of the blockage parameter C against the Froude number, for $s<1$ and $s>1$ respectively, is presented. One observes that for $s<1$ the blockage parameter takes larger values than for $s>1$.

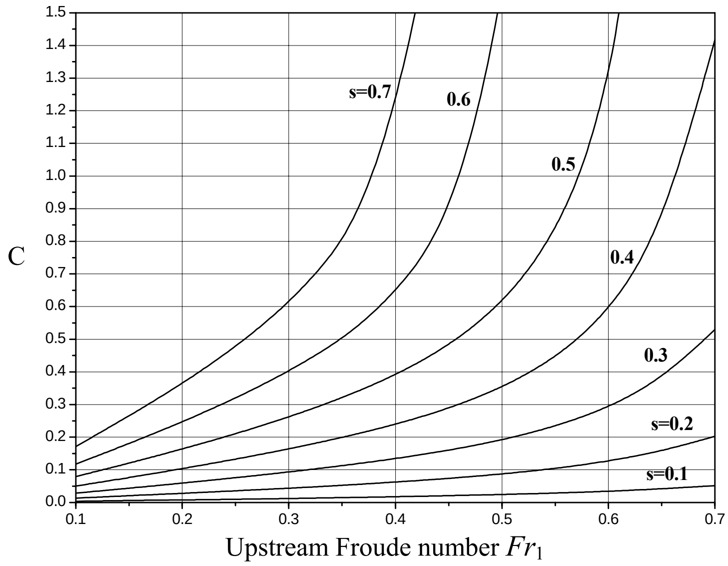

FIGURE 3.5a. The dependence of blockage parameter C upon the upstream Froude number Fr_1 and upon the shoaling ratio $s = h_2 / h_1$, ($h_2<h_1$).

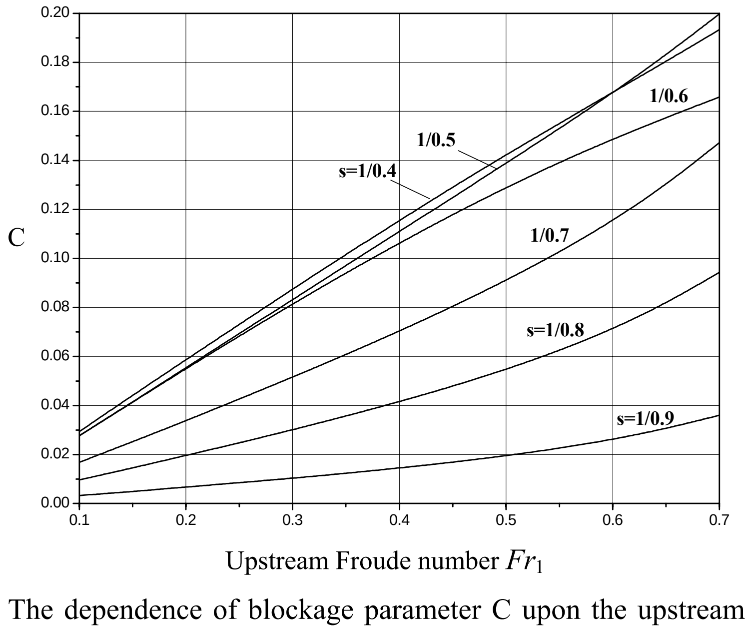

FIGURE 3.5b. The dependence of blockage parameter C upon the upstream Froude number Fr_1 and upon the shoaling ratio $s = h_2 / h_1$, ($h_1<h_2$).

The numerical solutions is obtained by retaining 51 evanescent modes ($N=51$) in the representations of disturbance wave potential, which is enough for numerical converge. In order to examine on detail the effect of the number of evanescent modes to the solution of the problem P_M, we study the continuity of the potential field Φ (C^0- continuity) and the continuity of the velocity field $\Phi,_x$ (C^1- continuity), at the junction of the step ($x=0$).

The potential field $\Phi(x,z)\in D$ is said to be continuous in *weak sense* at $x=0$, $-h_2 \le z \le 0$, if $\left\|\Phi^{(1)}-\Phi^{(2)}\right\|_{L^2} \to 0$ as $n\to\infty$,

where $\left\|\Phi^{(1)}-\Phi^{(2)}\right\|_{L^2} = \sqrt{\int_{-h_2}^{0} (\Phi^{(1)}(z)-\Phi^{(2)}(z))^2 dz}$ is the norm on $L^2(-h_2,0)$.

Similarly the velocity field $\Phi,_x(x,z)\in D$ is said to be continuous in *weak sense* at $x=0$, $-h_2 \le z \le 0$, if $\left\|\Phi,_x^{(1)}-\Phi,_x^{(2)}\right\|_{L^2} \to 0$ as $n\to\infty$.

In figure 3.6 the C^0- continuity and C^1- continuity at the junction of the step, according to above definitions, are presented.

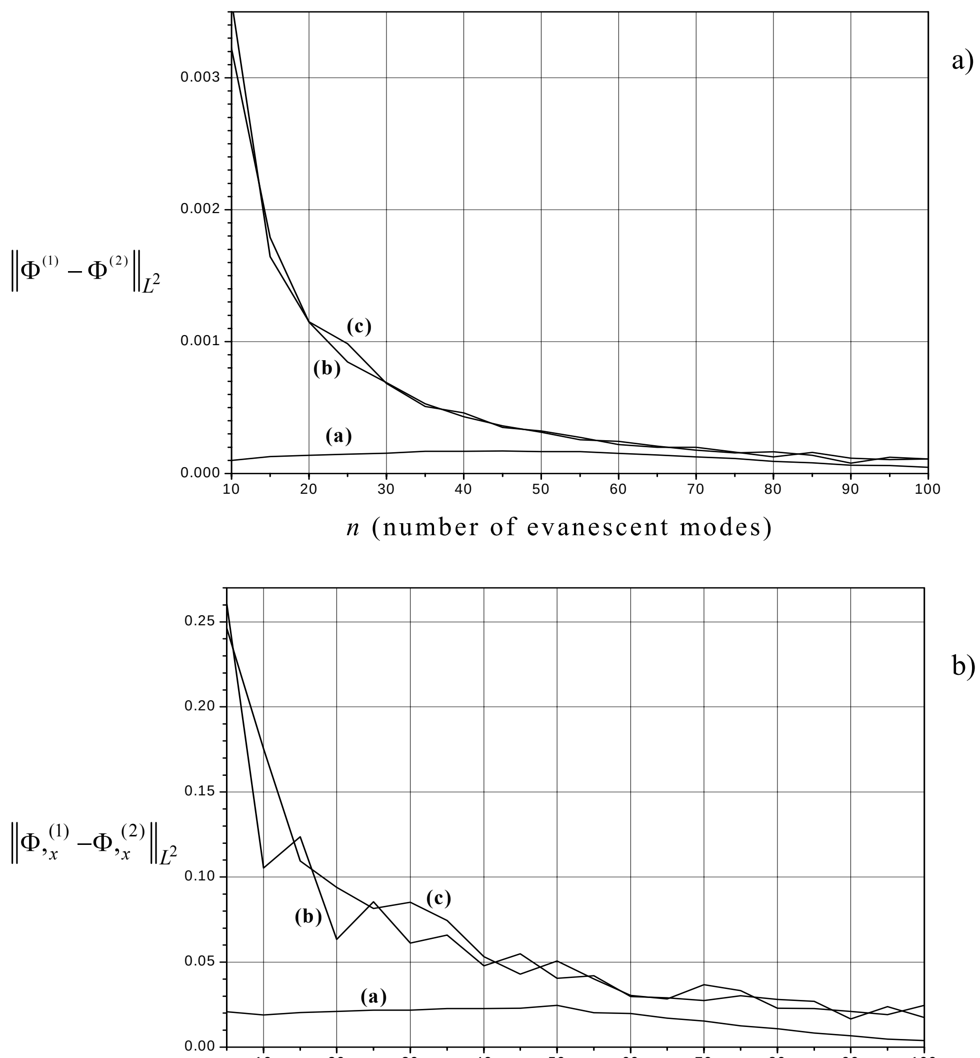

FIGURE 3.6. C^0 - continuity a) and C^1 - continuity b) *vs*. number of evanescent modes at the cut of the step ($x=0$), for three values of step height: (a) h_s = 0.01, (b) h_s = 0.1, (c) h_s = 0.156, at $Fr = 0.5$.

The figure 3.6 shows that for very small step height, the convergence rate is rapidly. On contrary as the step height increases the rate of converge is slower, and a number of 40 evanescent modes at least is needed, such that the error to be minimized. A comparison of the order of the error between C^0- continuity and C^1- continuity, denotes that there is a difference of order 10^{-2}, see figure 3.6. Indeed, i.e. for $N = 51$ and h_s = 0.1, the errors are $\left\|\Phi^{(1)} - \Phi^{(2)}\right\|_{L^2} = 3\cdot 10^{-4}$, and $\left\|\Phi,_x^{(1)} - \Phi,_x^{(2)}\right\|_{L^2} = 5\cdot 10^{-2}$. Similarly, in figure 3.7 the effectiveness of C^0- continuity and C^1- continuity, for various Froude numbers and for fixed step height, are presented.

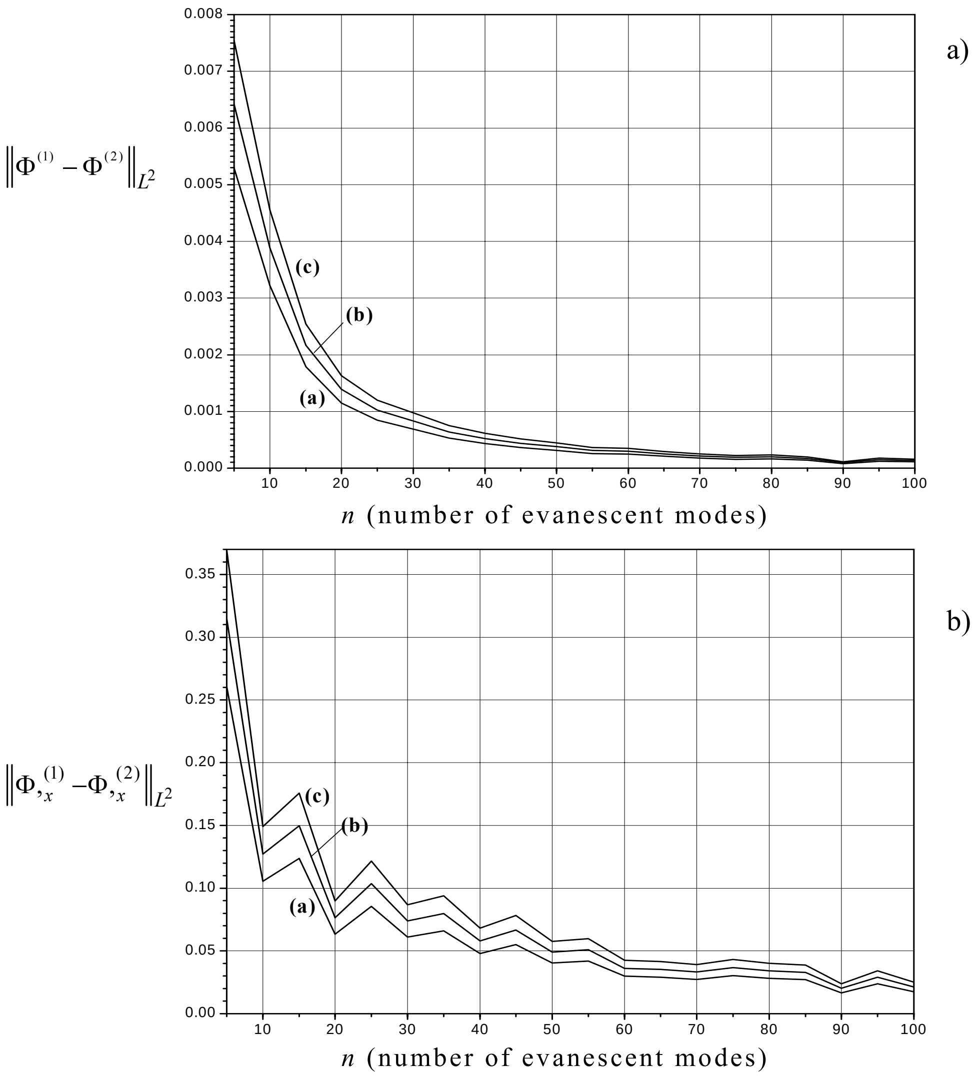

FIGURE 3.7. C^0- continuity a) and C^1- continuity b) *vs*. number of evanescent modes at the cut of the step ($x = 0$), for three values of Froude number: (a) Fr = 0.5, (b) Fr = 0.6, (c) Fr = 0.7, and for step height h_s = 0.1.

As mentioned above, the amplitude of the steady waves as well as the variation of mean water level depend upon both step heigth and Froude number. In figure 3.8 the variation of the wave amplitude (about the mean water level) and of the mean water level, against the upstream Froude number Fr_1 and shoaling ratio $s = h_2 / h_1$, are presented. This figure shows , that as the Froude number increase the wave amplitude and the mean water level become infinite. This is consistent with that as the upstream Froude number Fr_1 approaches the value $\sqrt{s_c}$ the downstream Froude number Fr_2 becomes critical (cf. the discussion in §3.1.2). As shown in figure 3.8, for small values of the shoaling ratio this critical limit is achieved for small values of Fr_1. Figure 3.8 also shows that there is a region of parameter's values (about the critical limits) which is inconsistent with the physical considerations, i.e the wave amplitude exceeds the downstream depth. This is expected since the numerical results are obtained by the linear problem P_M , as well as the critical conditions are approached the nonlinear effects dominate.

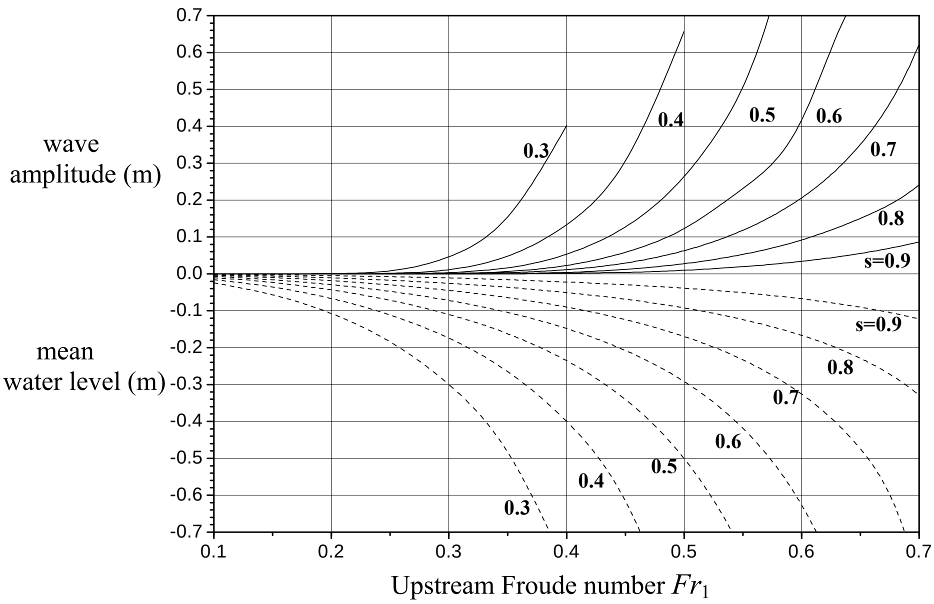

FIGURE 3.8. The dependence of wave amplitude and mean water level upon the upstream Froude number Fr_1 and upon the shoaling ratio $s = h_2 / h_1$, $(h_2 < h_1)$.

In figure 3.9 the variation of the wavelength against the upstream Froude number and the shoaling ratio, is presented. This figure shows that for low values of the Froude number, the wavelength λ is independent of the shoaling ratio. This is consistent with the figure 3.3, as one can be observes. Further, the wavelength λ denotes the same behavior with the wave amplitude α; it becomes infinite as the upstream Froude number Fr_1 approaches the critical value $\sqrt{s_c}$. Also, using this figure associated with the figure 3.8, the validity of the linear wave theory -in conjunction with figure 3.4a (cf. the discussion of fig. 3.4a)- may be predicted i.e. the gradient of the free surface elevation $k \cdot a$ to be small quantity (cf. for details Appendix 2.A).

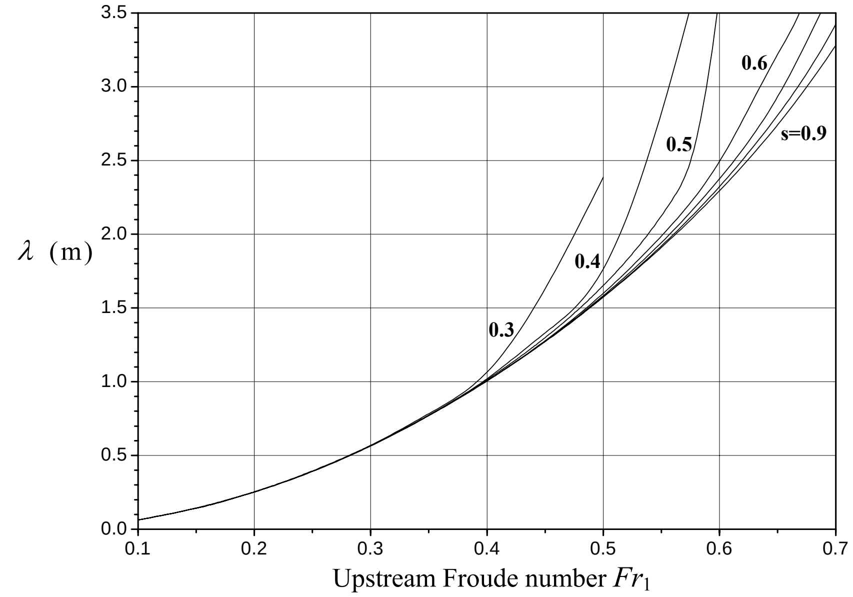

FIGURE 3.9. The dependence of wavelength upon the upstream Froude number Fr_1 and upon the shoaling ratio $s = h_2 / h_1$, $(h_2<h_1)$.

In figure 3.10a the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model. The flow is from deep to shallow water ($s<1$), the step height is h_s = 0.1 and the upstream Froude number is $Fr_1 = 0.5$. This figure shows, as one would expect, that the disturbance potential field becomes infinite at the singular point $(x,z)=(0,-h_2)$. In figures 3.10.b-c the continuity of the disturbance potential field φ (C^0 - continuity) and the continuity of the disturbance velocity field $\varphi_{,x}$ (C^1 - continuity), at the junction of the step ($x=0$) have been plotted. This figure is obtained by retaining 51 evanescent modes ($N=51$).

Similarly, in figure 3.11a the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model. The flow is from shallow to deep water ($s>1$), the upstream depth is h_1 = 0.6 and the upstream Froude number is $Fr_1 = 0.68$. Also, the corresponding figures 3.11.b-c for the continuity are presented.

In figure 3.12a the equipotential lines of the potential field $\Phi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model. The flow is from deep to shallow water ($s<1$), the step height is h_s = 0.4 and the upstream Froude number is $Fr_1 = 0.5$. This figure shows that the equipotential lines intersect the step perpendicularly, as they ought. Also, the corresponding figures 3.12.b-c for the continuity are presented.

Similarly, in figure 3.13a the equipotential lines of the potential field $\Phi(x,z)$ have been plotted for the flow from shallow to deep water ($s>1$), where the upstream depth is h_1 = 0.3 and the upstream Froude number is $Fr_1 = 0.60$.

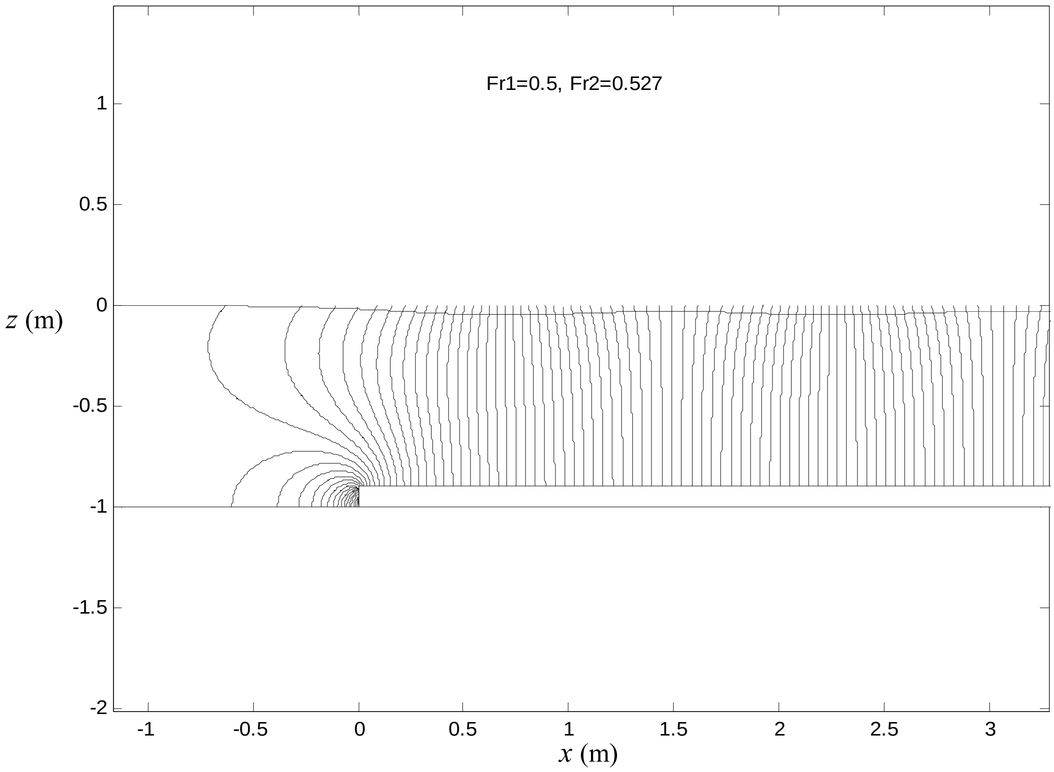

FIGURE 3.10a. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model. The flow is from deep to shallow water ($s < 1$).

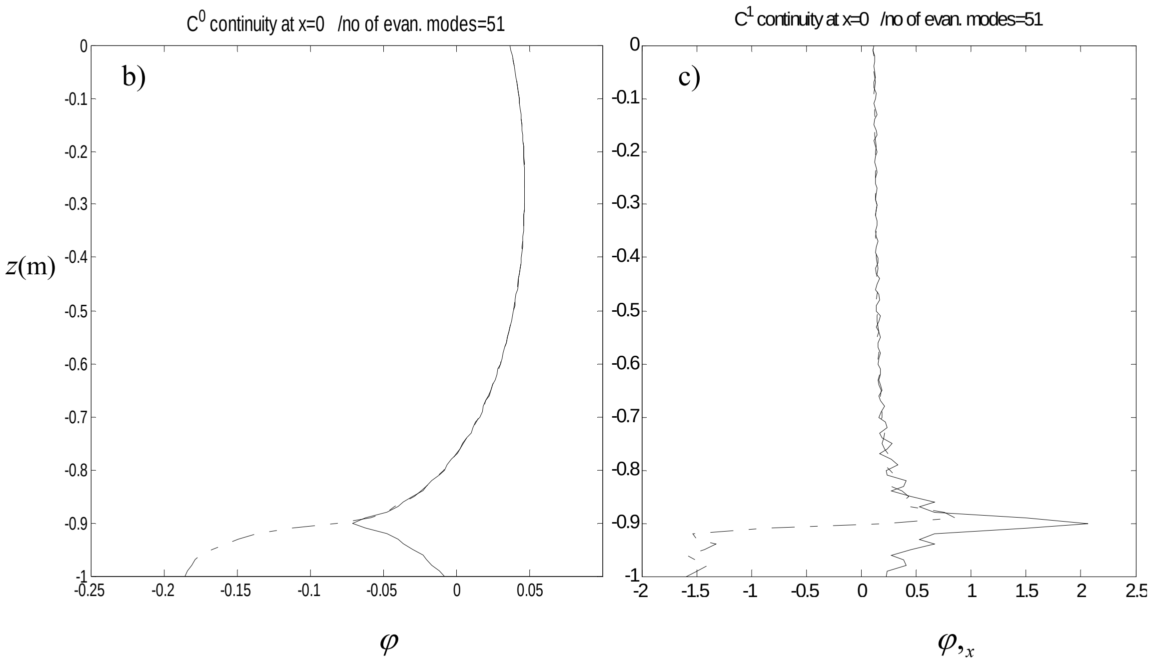

FIGURE 3.10b-c. C^0 - continuity b) and C^1 - continuity c) of the disturbance potential at the cut of the step ($x = 0$), for upstream Froude number Fr_1=0.5 and shoaling ratio $s = 0.9$. Number of evanescent modes $N = 51$.

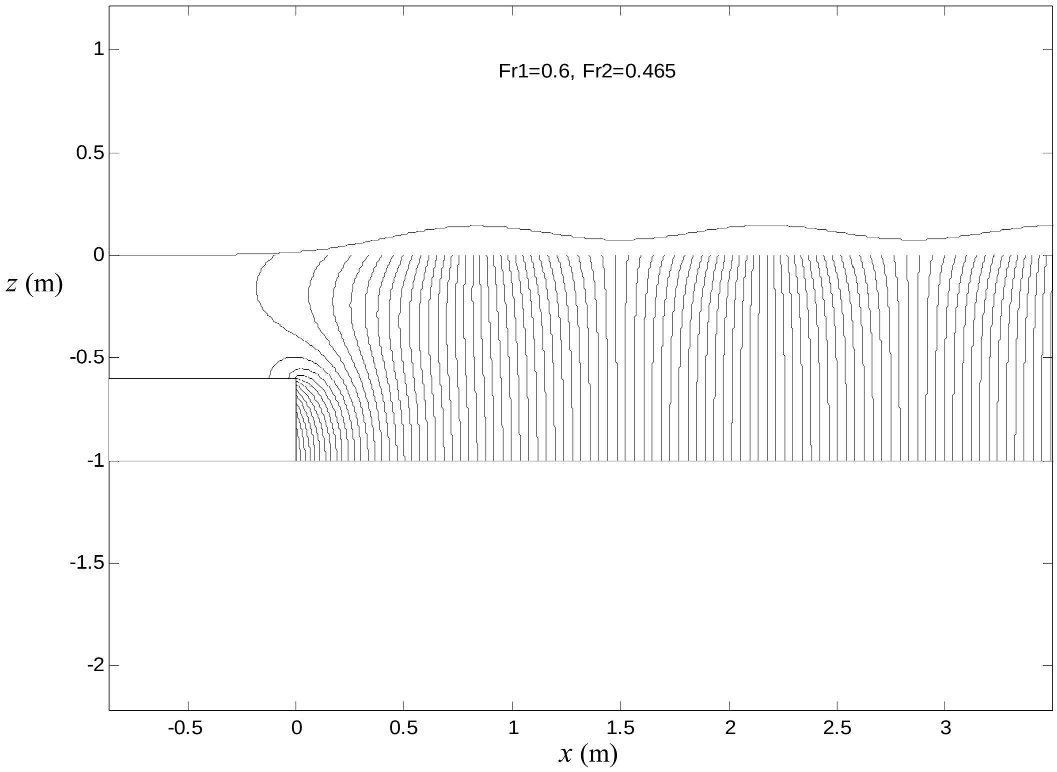

FIGURE 3.11a. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model. The flow is from shallow to deep water ($s > 1$).

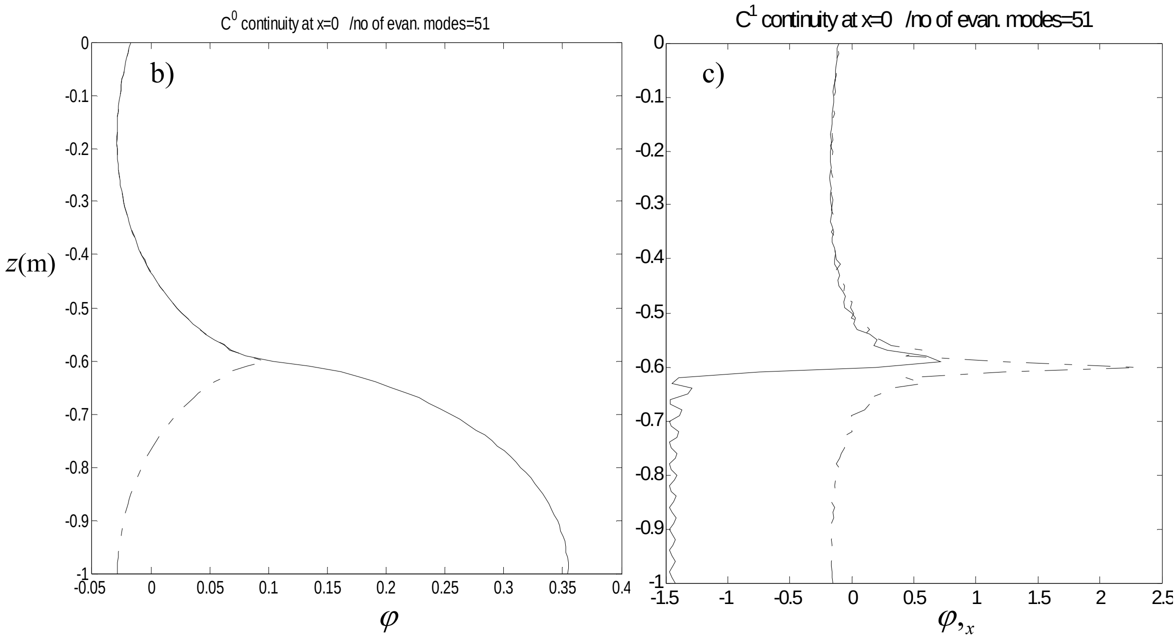

FIGURE 3.11b-c. C^0 - continuity b) and C^1 - continuity c) of the disturbance potential at the cut of the step ($x = 0$), for upstream Froude number Fr_1=0.6 and shoaling ratio $s = 1.66$. Number of evanescent modes N = 51.

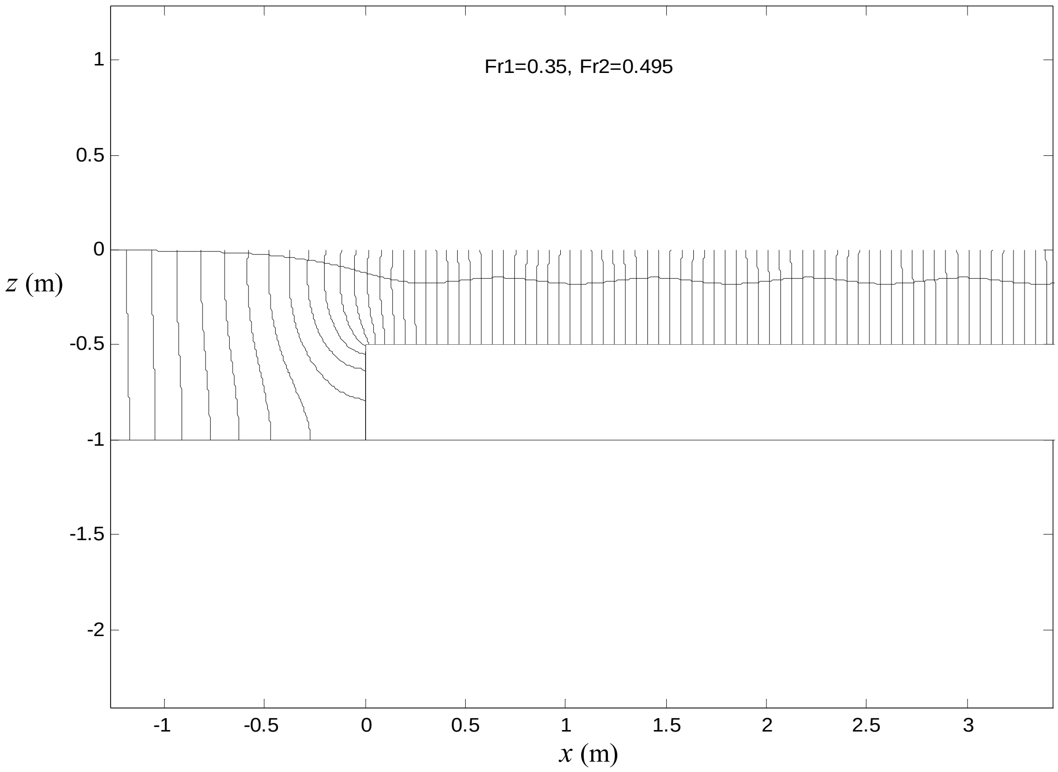

FIGURE 3.12a. Equipotential lines of the velocity potential field and free-surface elevation as obtained by present model. The flow is from deep to shallow water ($s < 1$).

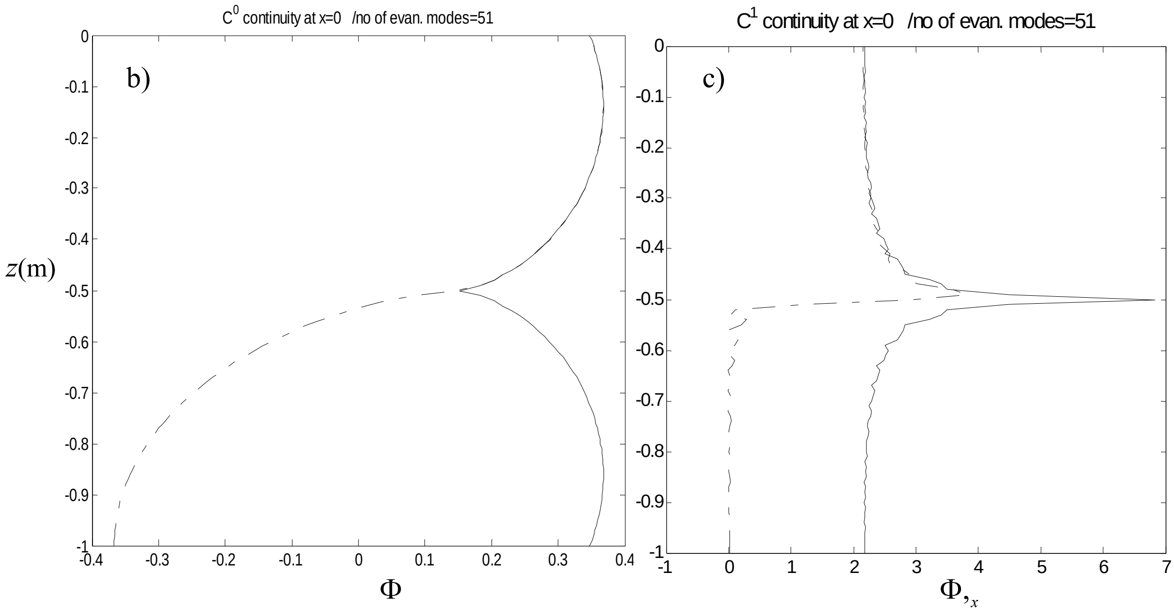

FIGURE 3.12b-c. C^0 - continuity b) and C^1 - continuity c) of the velocity potential at the cut of the step ($x = 0$), for upstream Froude number Fr_1=0.35 and shoaling ratio $s = 0.5$. Number of evanescent modes $N = 51$.

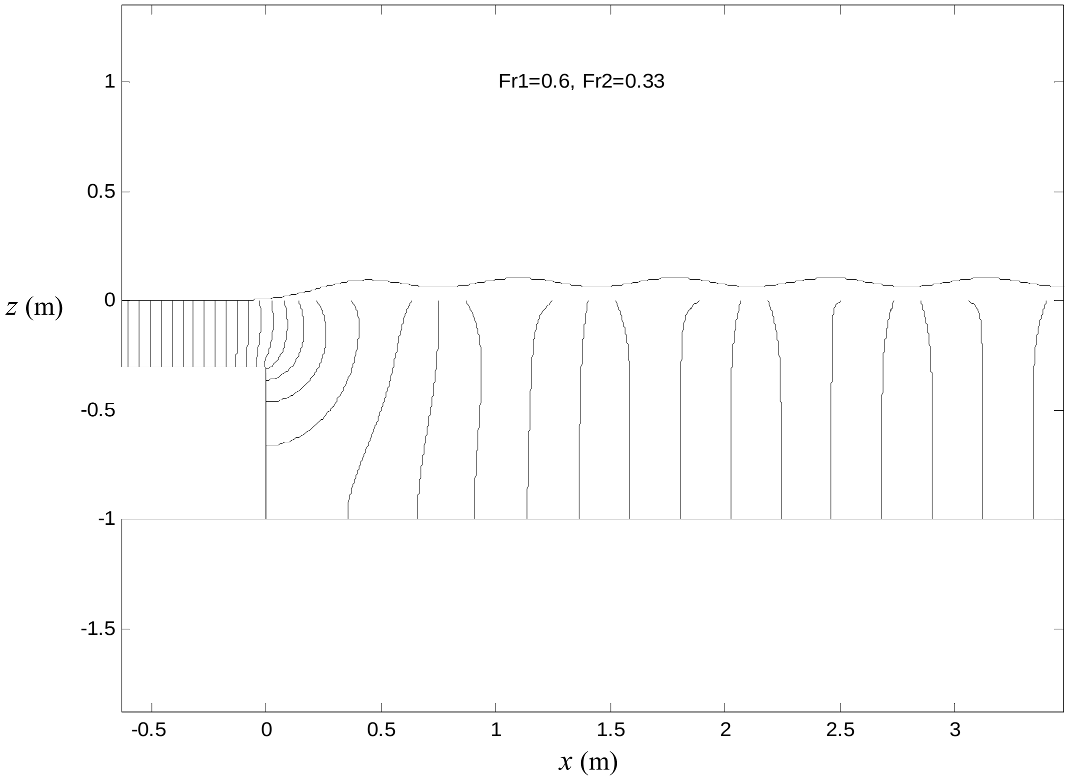

FIGURE 3.13a. Equipotential lines of the velocity potential field and free-surface elevation as obtained by present model. The flow is from shallow to deep water ($s > 1$).

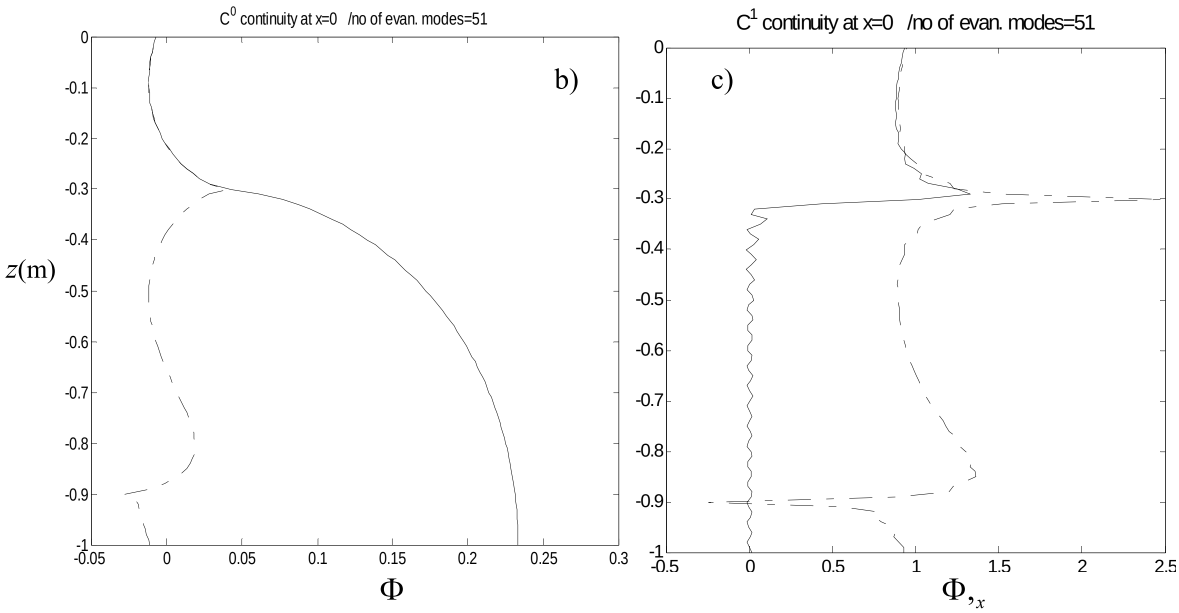

FIGURE 3.13b-c. C^0 - continuity b) and C^1 - continuity c) of the velocity potential at the cut of the step ($x = 0$), for upstream Froude number Fr_1=0.6 and shoaling ratio s = 3.33 . Number of evanescent modes N = 51.

2. *Supercritical flows* $(Fr > 1)$

The numerical calculations were performed for a step of variable height and of various Froude numbers. However, in order to illustrate the results obtained we first investigate in detail the situations when the upstream depth is $1m$, and the step heights h_s are $0.2m$, and $0.4m$, at upstream $Fr = 2$.

The figure 3.14 shows, as one would expect according to theoretical investigation of the present linear model (cf. § 3.1.2), that in supercritical flow the level of the free surface rises monotonically as it approaches the step, the slope of the surface becoming more gradual, until far downstream of the step is asymptotically flat. The slope of the free surface as well as the variation of mean water level depend both upon step height and Froude number.

Figure 3.14 also shows a comparison of the free surface profiles with the method developed by King & Bloor (1987). For small step ($h_s = 0.2m$) the results showing that the slope of the free surface of the present linear solution is more steeper than the non-linear and linear solutions obtained by King & Bloor (1987). Also a small difference occurs between the far downstream levels and this deviation becoming larger when the present model compared with non-linear solution.

As the step height is increased a distinct difference appears between the solutions as shown in figure 3.14. This is consistent with the considerations of the studied problem. As mentioned above in subcritical flow, the difference between the downstream $(U + u)$ and upstream U uniform velocity must be small (cf. the discussion in § 3.1.2).

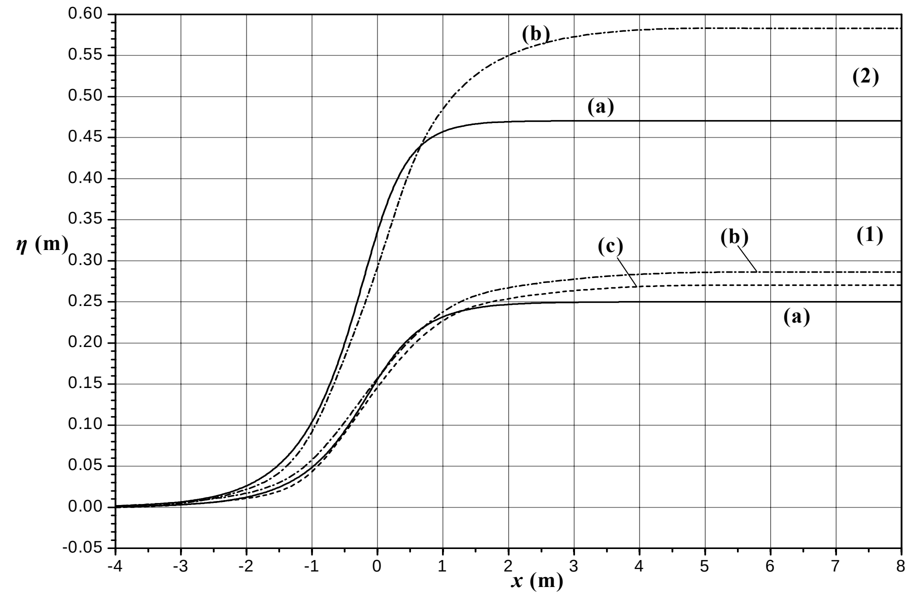

FIGURE 3.14. The free-surface elevation (a) of present model for $Fr = 2$ compared with non-linear (b) and linear (c) solutions obtained by King & Bloor (1987), for two values of step height: 1) $h_s = 0.2$, 2) $h_s = 0.4$.

In figure 3.14a the variation of the rate of downstream to upstream velocity of the flow, against the shoaling ratio $s = h_2 / h_1$ ($s < 1$) for various Froude numbers, is presented. Under the considerations of the present linear model i.e. $\frac{U+u}{U} \approx O(1)$, figure 3.14a shows that for a small region of the parameters values the linear solution is valid and this occurs for large Froude numbers.

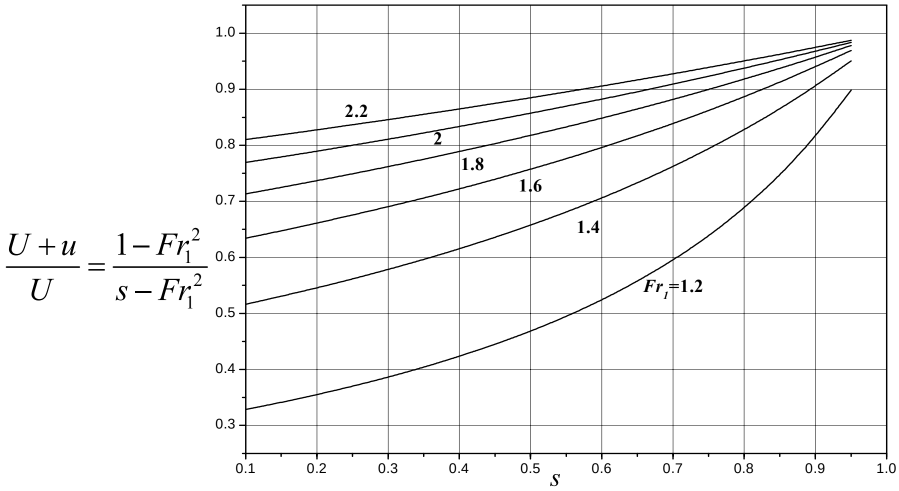

FIGURE 3.14a. The dependence of the rate of downstream to upstream velocity upon the upstream Froude number Fr_1 and upon the shoaling ratio $s = h_2 / h_1$, ($h_2 < h_1$).

On the contrary, if the flow is from shallow to deep water i.e. $s > 1$, figure 3.14b shows, as one would expect according to theoretical investigation (cf. § 3.1.2), that as the shoaling ratio s tends to its critical value $s_c = Fr_1^2$ the critical conditions downstream are approached.

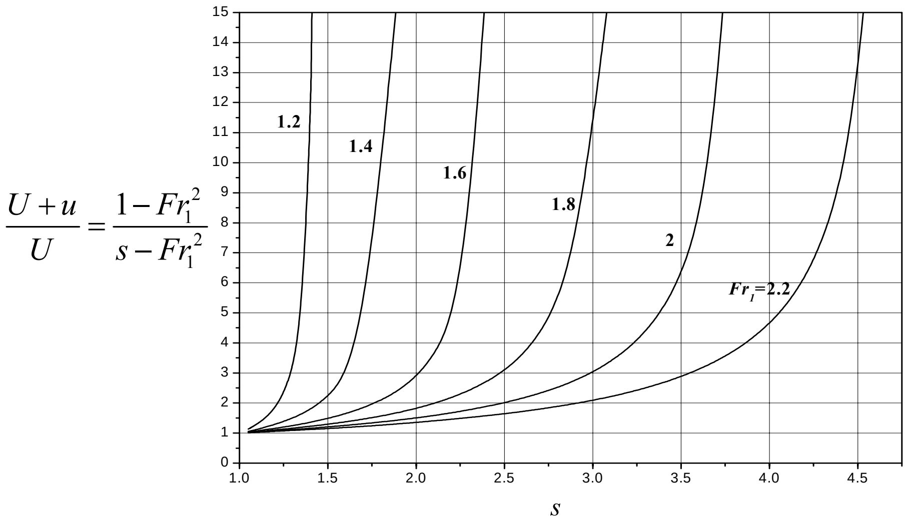

FIGURE 3.14b. The dependence of the rate of downstream to upstream velocity upon the upstream Froude number Fr_1 and upon the shoaling ratio $s = h_2 / h_1$, ($h_1 < h_2$).

$$\frac{U+u}{U}=\frac{1-Fr_1^2}{s-Fr_1^2}$$

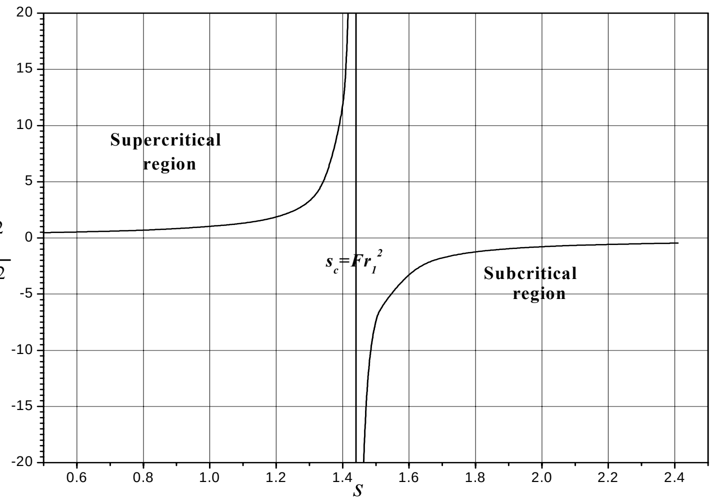

FIGURE 3.14c. The transition from supercritical flow to subcritical flow as the downstream depth increases for Fr_1=1.2.

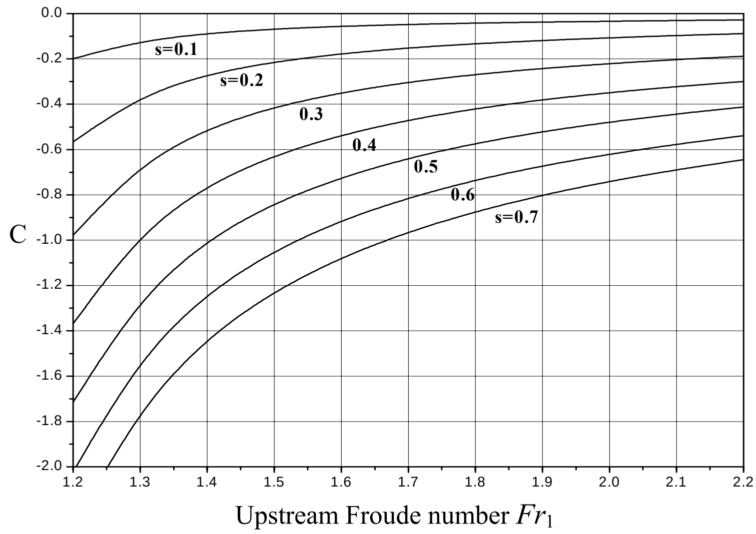

FIGURE 3.15a. The dependence of blockage parameter C upon the upstream Froude number Fr_1 and upon the shoaling ratio $s = h_2 / h_1$, ($h_2 < h_1$).

In figures 3.15a and 3.15b the variation of the blockage parameter C against the Froude number, for $s < 1$ and $s > 1$ respectively, is presented. One observes that for $s > 1$ the blockage parameter takes larger values than for $s < 1$. Comparing with subcritical flow the blockage parameter in supercritical flow takes negative values.

The numerical solutions is obtained by retaining 51 evanescent modes ($N = 51$) in the representations of disturbance wave potential, which is enough for numerical converge. In order to examine on detail the effect of the number of evanescent modes to the solution of the problem P_M, we study the continuity of the potential field Φ (C^0- continuity) and the continuity of the velocity field $\Phi,_x$ (C^1- continuity), at the junction of the step ($x = 0$).

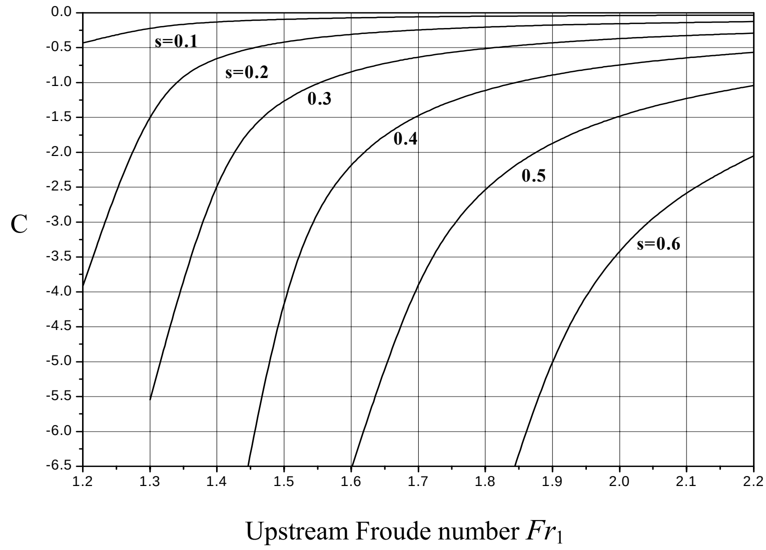

FIGURE 3.15b. The dependence of blockage parameter C upon the upstream Froude number Fr_1 and upon the shoaling ratio $s = h_2 / h_1$, $(h_1 < h_2)$.

In figure 3.16 the C^0- continuity and C^1- continuity at the junction of the step are presented. The figure 3.16 shows that for very small step height, the convergence rate is rapidly. On contrary as the step height increases the rate of converge is slower, and a number of 40 evanescent modes at least is needed, such that the error to be minimized. A comparison of the order of the error between C^0- continuity and C^1- continuity, denotes that there is a difference of order 10^{-2}, see figure 3.16. Similarly, in figure 3.17 the effectiveness of C^0- continuity and C^1- continuity, for various Froude numbers and for fixed step height, are presented.

In figure 3.18a the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model. The flow is from deep to shallow water ($s<1$), the step height is h_s = 0.2 and the upstream Froude number is $Fr_1 = 2$. This figure shows, as one would expect, that the disturbance potential field becomes infinite at the singular point $(x,z) = (0,-h_2)$. In figures 3.18.b-c the continuity of the disturbance potential field φ (C^0- continuity) and the continuity of the disturbance velocity field $\varphi,_x$ (C^1- continuity), at the junction of the step ($x=0$) have been plotted. This figure is obtained by retaining 51 evanescent modes ($N=51$).

Similarly, in figure 3.19a the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model. The flow is from shallow to deep water ($s>1$), the upstream depth is h_1 = 0.8 and the upstream Froude number is $Fr_1 = 2$. Also, the corresponding figures 3.19.b-c for the continuity are presented.

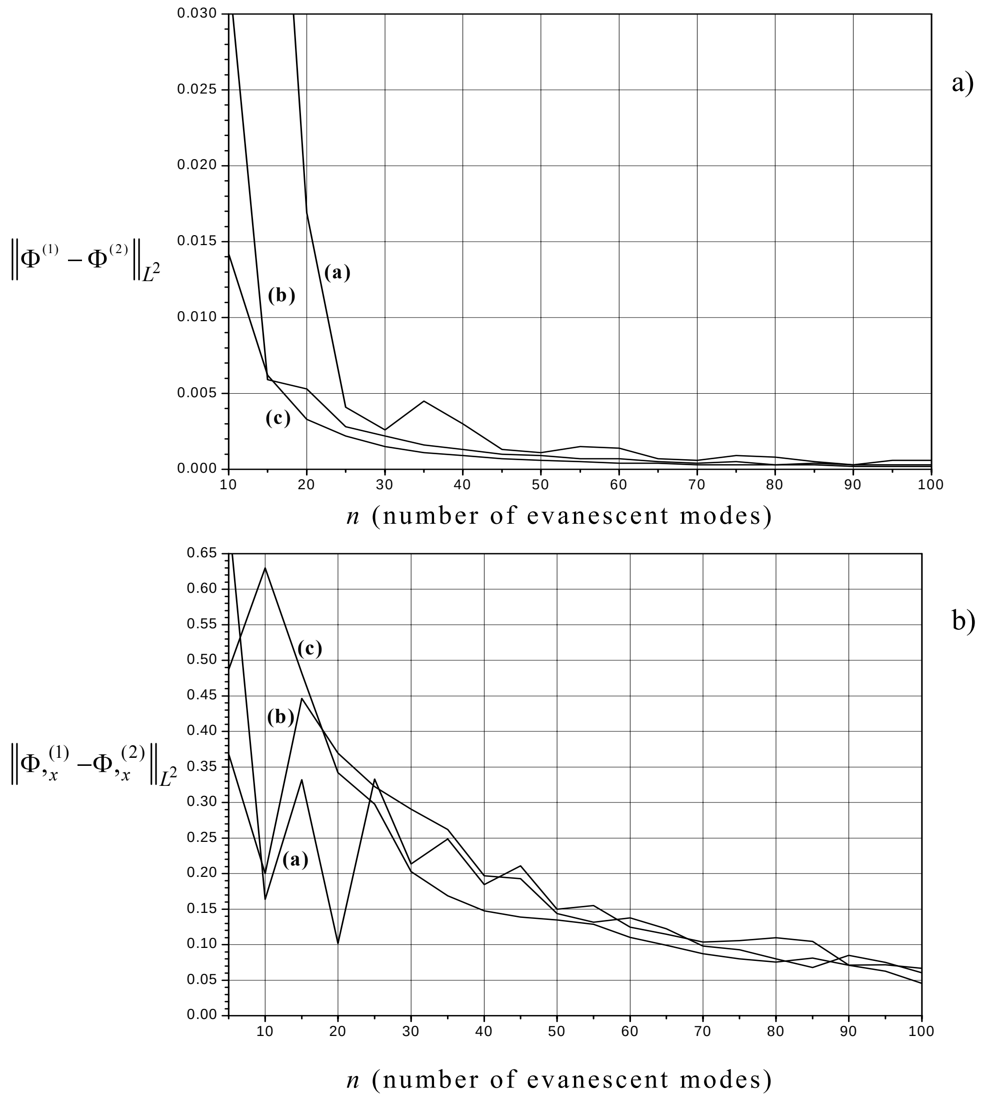

FIGURE 3.16. C^0 - continuity a) and C^1 - continuity b) *vs*. number of evanescent modes at the cut of the step ($x = 0$), for three values of step height: (a) h_s = 0.1, (b) h_s = 0.2, (c) h_s = 0.4, at $Fr = 2$.

In figure 3.20a the equipotential lines of the potential field $\Phi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model. The flow is from deep to shallow water ($s < 1$), the step height is h_s = 0.4 and the upstream Froude number is $Fr_1 = 1.5$. This figure shows that the equipotential lines intersect the step perpendicularly, as they ought. Also, the corresponding figures 3.20.b-c for the continuity are presented.

Similarly, in figure 3.21a the equipotential lines of the potential field $\Phi(x,z)$ have been plotted for the flow from shallow to deep water ($s > 1$), where the upstream depth is h_1 = 0.6 and the upstream Froude number is $Fr_1 = 2.4$.

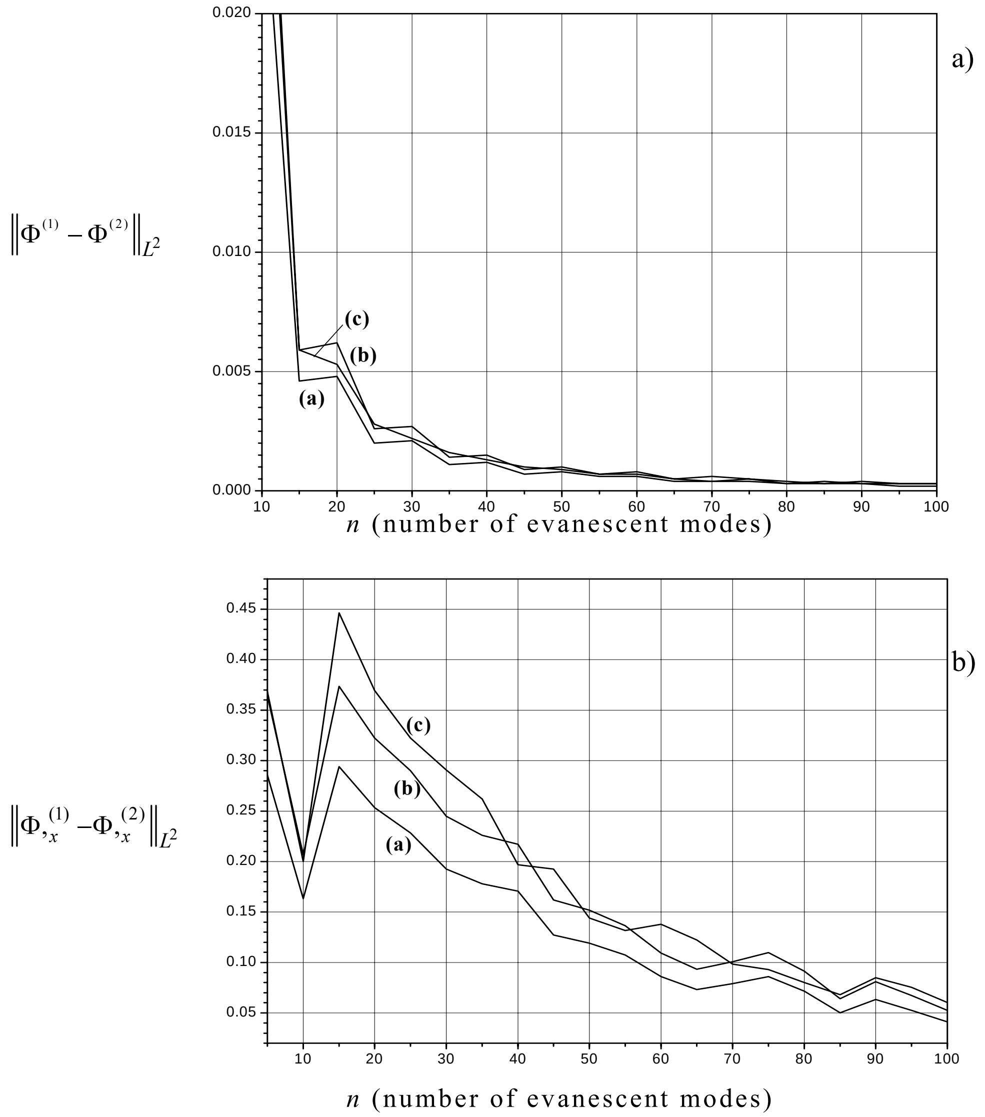

FIGURE 3.17. C^0 - continuity a) and C^1 - continuity b) *vs*. number of evanescent modes at the cut of the step ($x = 0$), for three values of Froude number: (a) Fr = 1.4, (b) Fr = 1.7, (c) Fr = 2, and for step height h_s = 0.2.

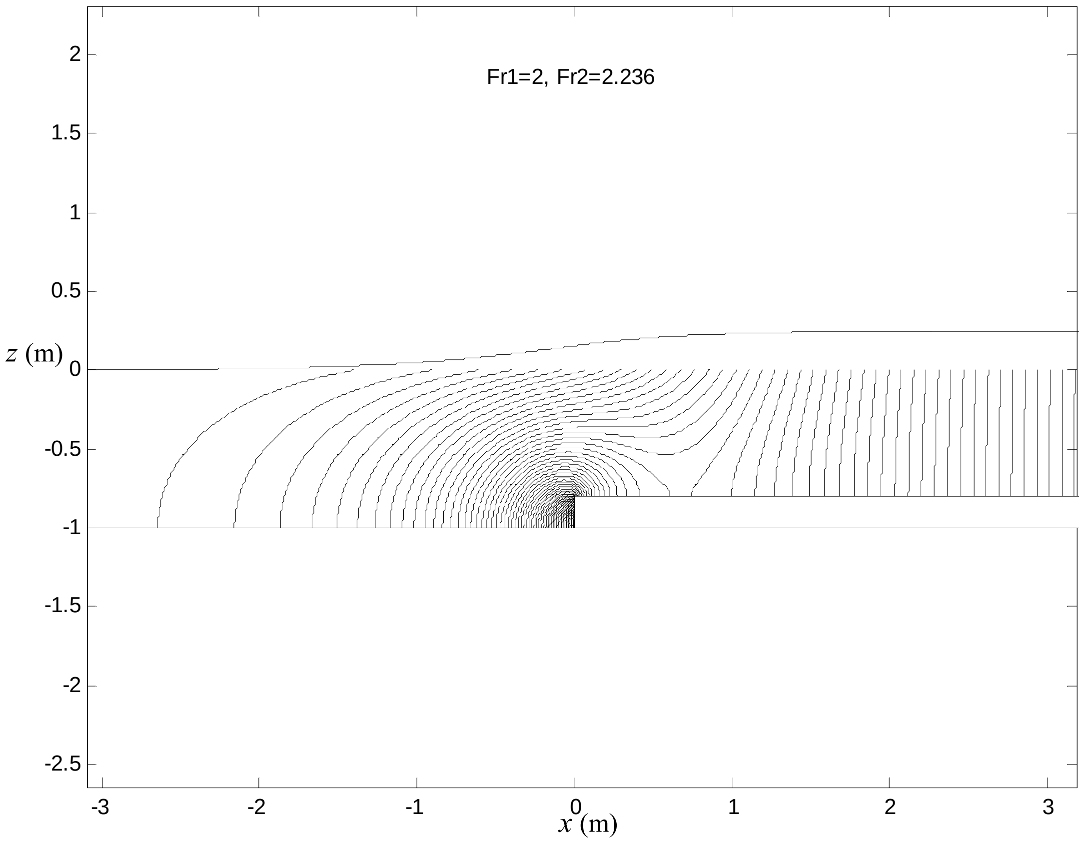

FIGURE 3.18a. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model. The flow is from deep to shallow water ($s < 1$).

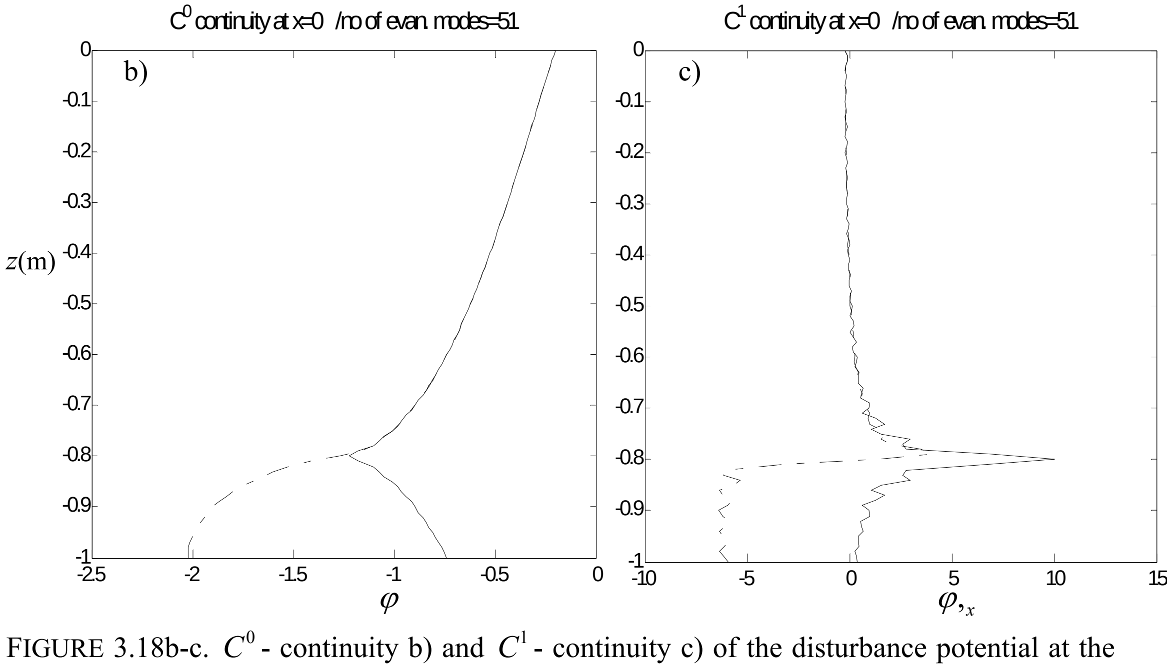

FIGURE 3.18b-c. C^0 - continuity b) and C^1 - continuity c) of the disturbance potential at the cut of the step ($x = 0$), for upstream Froude number Fr_1=2 and shoaling ratio $s = 0.8$. Number of evanescent modes $N = 51$.

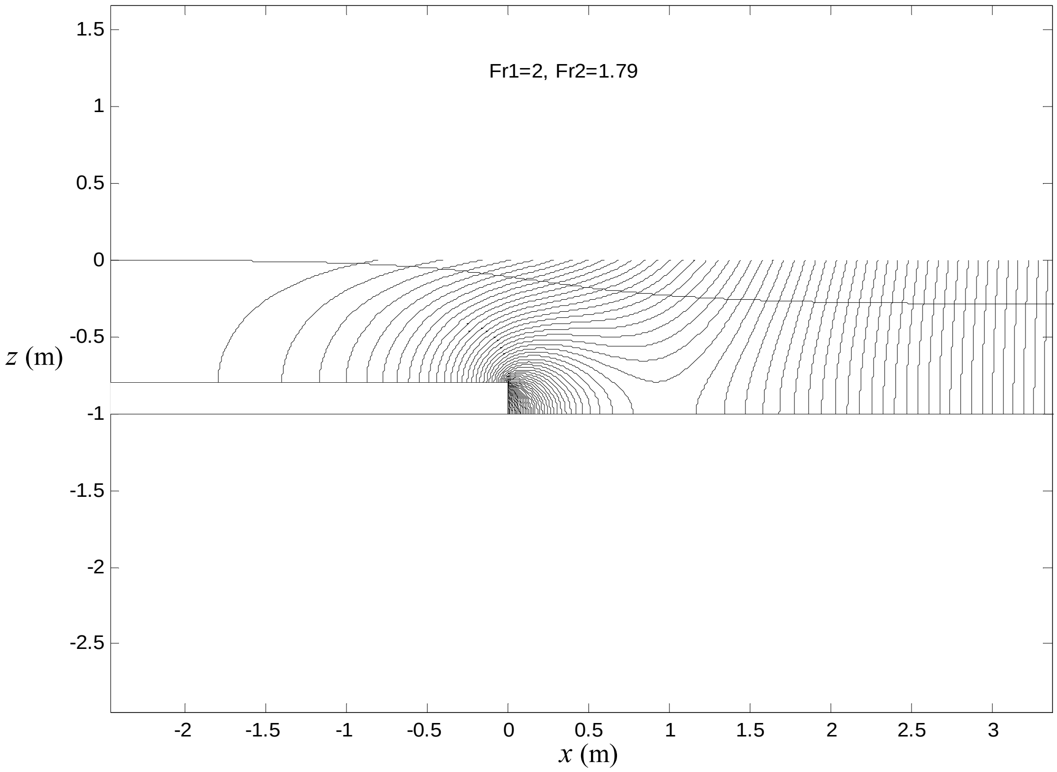

FIGURE 3.19a. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model. The flow is from shallow to deep water ($s>1$).

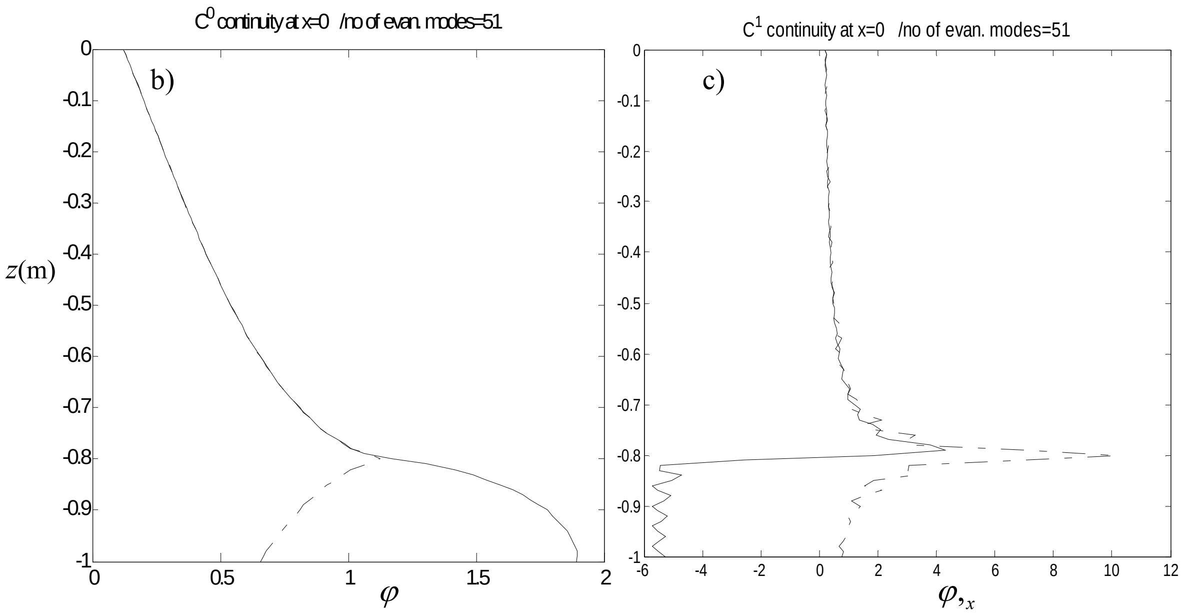

FIGURE 3.19b-c. C^0 - continuity b) and C^1 - continuity c) of the disturbance potential at the cut of the step ($x=0$), for upstream Froude number Fr_1=2 and shoaling ratio $s=1.25$. Number of evanescent modes N = 51.

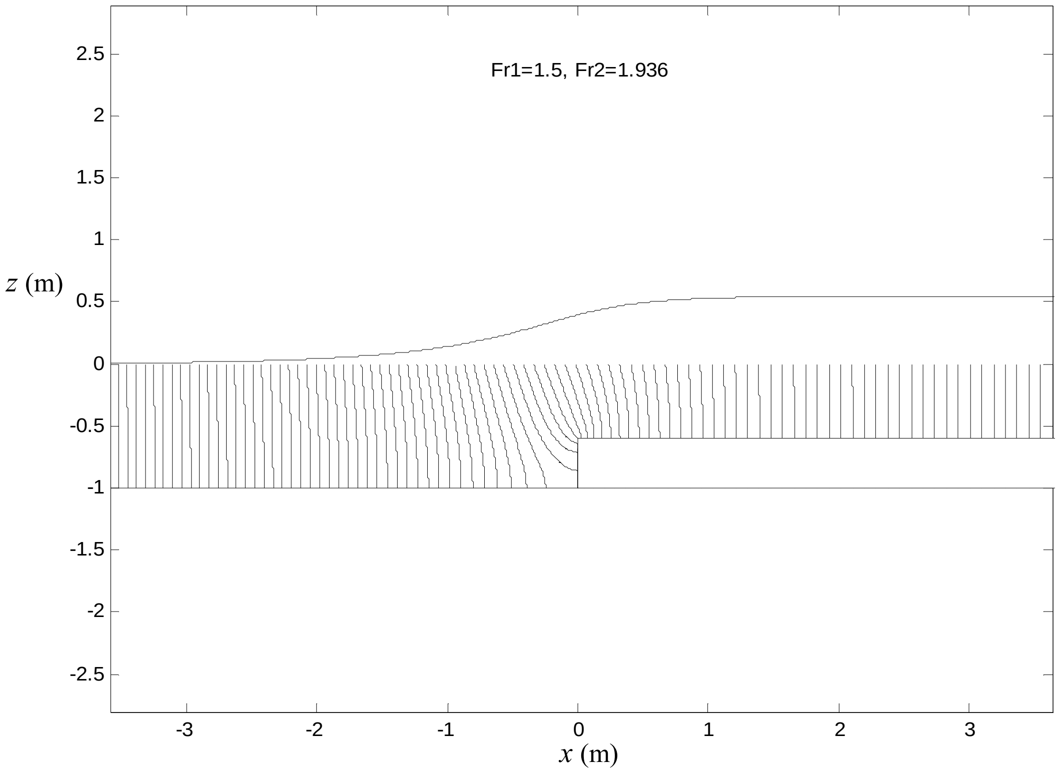

FIGURE 3.20a. Equipotential lines of the velocity potential field and free-surface elevation as obtained by present model. The flow is from deep to shallow water ($s < 1$).

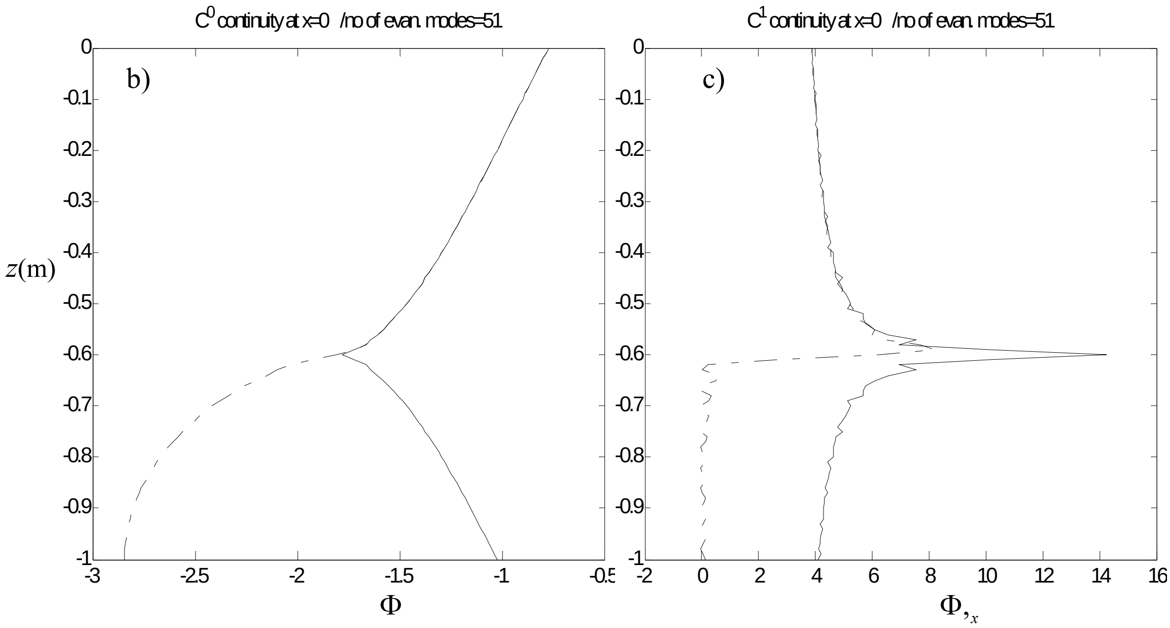

FIGURE 3.20b-c. C^0 - continuity b) and C^1 - continuity c) of the velocity potential at the cut of the step ($x = 0$), for upstream Froude number Fr_1=1.5 and shoaling ratio $s = 0.6$. Number of evanescent modes $N = 51$.

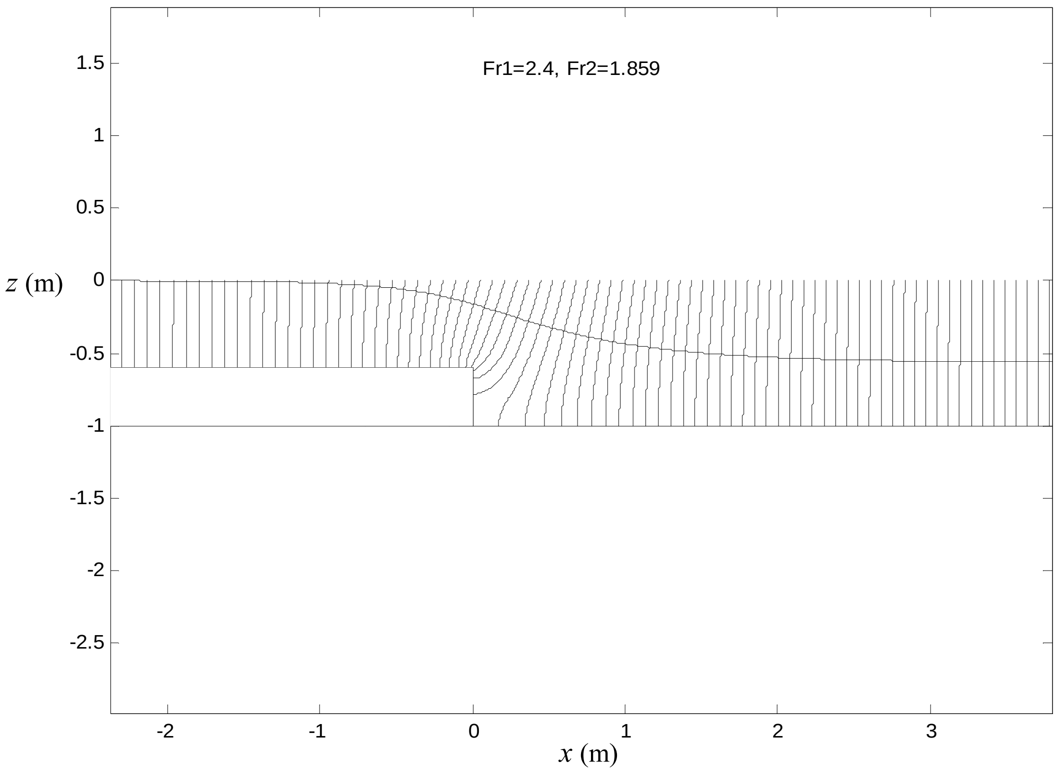

FIGURE 3.21a. Equipotential lines of the velocity potential field and free-surface elevation as obtained by present model. The flow is from shallow to deep water ($s > 1$).

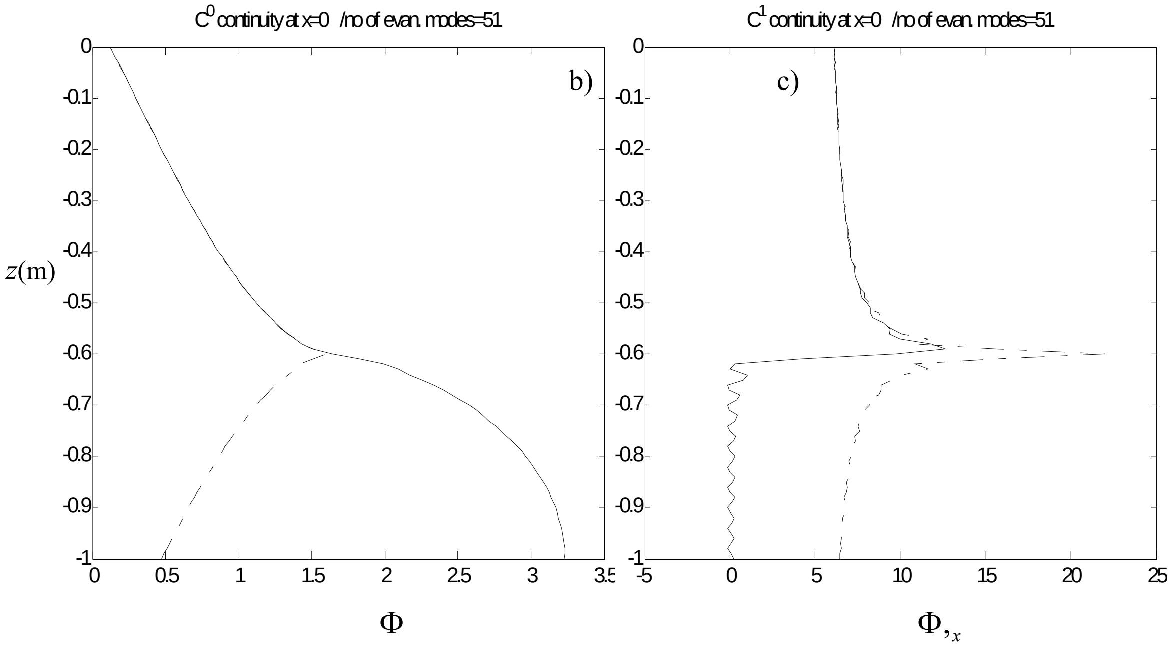

FIGURE 3.21b-c. C^0 - continuity b) and C^1 - continuity c) of the velocity potential at the cut of the step ($x = 0$), for upstream Froude number Fr_1=2.4 and shoaling ratio $s = 1.66$. Number of evanescent modes N = 51.

3.2 Free-Surface flow over a finite step or trench.

3.2.1 Differential formulation of the problem.

The steady two-dimensional free-surface flow of a stream, of an inviscid incompressible and irrotational fluid, which is obstructed by finite step (or trench) on the bottom, is considered. The wave field is excited by an incident uniform flow with direction normal to the bottom contours, and velocity U. The studied free-surface flow consists of a water layer D_{3D} bounded above by the free surface $\partial D_{F,3D}$ and below by a rigid bottom $\partial D_{\Pi,3D}$. Without loss of generality we introduce as obstruction the finite step. See figure 3.22.

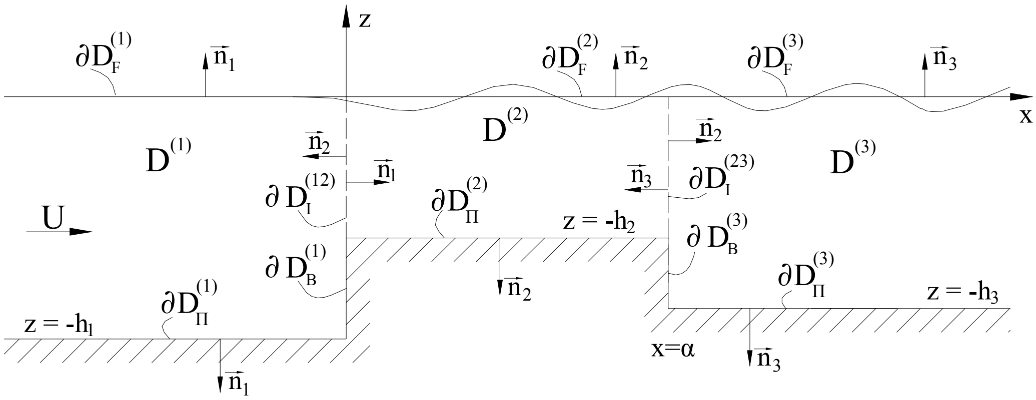

FIGURE 3.22. Domain decomposition and basic notation

Before proceeding to the formulation of the problem, we shall introduce some geometrical notation. A Cartesian coordinate system is introduced, with its origin on the mean water level (at the upstream cut of the step), the z-axis pointing upwards and the y-axis being parallel to the bottom contours. See figure 3.22.

The liquid domain D_{3D} will be represented by $D_{3D} = D \times R$, where D is the (two-dimensional) intersection of D_{3D} by a vertical plane perpendicular to the bottom contours, and $R = (-\infty, +\infty)$, is a copy of the real line: $D_{3D} = \{(x, y, z):\ (x, y) \in R^2, -h(x) < z < 0\}$, $D = \{(x, z):\ x \in R, -h(x) < z < 0\}$. The function $h(x)$, appearing in the above definitions, represents the local depth, measured from the mean water level.

$$h(x) = \begin{cases} h_1, & x \le a \\ h_2, & a < x \le b \\ h_3, & x > b \end{cases}.$$

The liquid domain D_{3D} is decomposed in three subdomains $D_{3D}^{(i)} = D^{(i)} \times R,\ i = 1,2,3,$ defined us follows: $D_{3D}^{(1)}$ is the constant-depth upstream subdomain characterized by $x < a$, $D_{3D}^{(3)}$ is the constant-depth downstream subdomain characterized by $x > b$, and $D_{3D}^{(2)}$ is the variable bathymetry subdomain lying between $D_{3D}^{(1)}$ and $D_{3D}^{(3)}$. Without loss of generality, we assume that $h_1 > h_3 > h_2$.

The decomposition is also applied to the boundaries $\partial D_{F,3D} = \partial D_F \times R$ and $\partial D_{\Pi,3D} = \partial D_\Pi \times R$. The lines ∂D_F and ∂D_Π are decomposed in three pieces each, for example, $\partial D_F = \partial D_F^{(1)} \cup \partial D_F^{(2)} \cup \partial D_F^{(3)}$, where $\partial D_F^{(1)}$ belongs to the boundary of $D^{(1)}$, and similarly for ∂D_Π. Finally, we define the artificial vertical interfaces $\partial D_{I,3D}^{(12)} = \partial D_I^{(12)} \times R$ and $\partial D_{I,3D}^{(23)} = \partial D_I^{(23)} \times R$ and the vertical walls $\partial D_{B,3D}^{(1)} = \partial D_B^{(1)} \times R$ and $\partial D_{B,3D}^{(3)} = \partial D_B^{(3)} \times R$, which is the common vertical boundaries of subdomains $D_{3D}^{(1)}$ and $D_{3D}^{(2)}$, and $D_{3D}^{(2)}$ and $D_{3D}^{(3)}$, respectively. Clearly, $\partial D_I^{(12)}$ and $\partial D_I^{(23)}$ are vertical segments (between the bottom and the mean water level) at $x = 0$ and $x = a$, respectively. See figure 3.22.

We consider that the motion arising from disturbances created by the obstruction of the uniform flow by the finite step, have the velocity potential

$$\Phi(x,z) = Ux + \varphi(x,z), \qquad (x,z) \in D. \tag{1}$$

Assuming that the disturbance velocity potential $\varphi(x,z)$ and the velocity of the stream are small enough, the linearized equations of the *Neumann-Kelvin* problem can be used (cf. §2.1). By the decomposition of the liquid domain $D = D^{(1)} \cup D^{(2)} \cup D^{(3)}$, the studied problem should be formulated with the aid of the general representation of the disturbance velocity potential $\varphi(x,z)$ in the semi-infinite strips $D^{(1)}$ and $D^{(3)}$, and in the finite subdomain $D^{(2)}$ with respect to the studied cases.

Taking under consideration the decomposition of the total field in three regions $D^{(1)}$, $D^{(2)}$ and $D^{(3)}$ the problem can be formulated us follows:

<u>PROBLEM</u> $P_{MM}(D,\varphi^{(1)},\varphi^{(2)},\varphi^{(3)})$. *Given the upstream velocity U, find the disturbance velocity potentials* $\varphi^{(1)} \in D^{(1)}$, $\varphi^{(2)} \in D^{(2)}$ *and* $\varphi^{(3)} \in D^{(3)}$, *satisfying the following system of equations, boundary and matching conditions:*

$$\nabla^2 \varphi^{(i)} = 0 \qquad (x,z) \in D^{(i)}, \tag{2.1}$$

$$\varphi,_{xx}^{(i)} + \frac{g}{U^2}\varphi,_z^{(i)} = 0 \qquad (x,z) \in \partial D_F^{(i)}, \tag{2.2}$$

$$\varphi,_z^{(i)} = 0, \qquad (x,z) \in \partial D_\Pi^{(i)}, \tag{2.3}$$

$$\frac{\partial \varphi^{(1)}}{\partial \vec{n}^{(1)}} = -Un_x, \ (x,z) \in \partial D_B^{(1)}, \ \frac{\partial \varphi^{(3)}}{\partial \vec{n}^{(3)}} = -Un_x, \ (x,z) \in \partial D_B^{(3)}, \tag{2.4a,b}$$

$$\varphi^{(2)} = \varphi^{(1)}, \ \frac{\partial \varphi^{(2)}}{\partial \vec{n}^{(2)}} = -\frac{\partial \varphi^{(1)}}{\partial \vec{n}^{(1)}} \ (x,z) \in \partial D_I^{(12)}, \tag{2.5a,b}$$

$$\varphi^{(2)} = \varphi^{(3)}, \ \frac{\partial \varphi^{(2)}}{\partial \vec{n}^{(2)}} = -\frac{\partial \varphi^{(3)}}{\partial \vec{n}^{(3)}} \quad (x,z) \in \partial D_I^{(23)}, \tag{2.6a,b}$$

where $\vec{n}^{(i)} = (n_x^{(i)}, n_z^{(i)})$ *is the unit normal vector to the boundary* $\partial D^{(i)}$ *directed to the exterior of* $D^{(i)}$ $i = 1,2,3$.

In addition we require the following radiation conditions (cf. the discussion in §3.1.1 and §3.1.2)*:*

$$\varphi \to 0 \qquad as\ x \to -\infty, \tag{3a}$$

$$|\nabla \varphi| < \infty \qquad as\ x \to \infty, \tag{3b}$$

$$A_{-1}^{(3)} = u \qquad as\ x \to \infty, \tag{3c}$$

We denote that the condition (3c) required such as the conversation of mass to be satisfied. In reference to §3.1.2

$$u = \frac{U(1-s)}{s - Fr_1^2}, \quad \text{where } s = h_3/h_1 . \tag{4}$$

However, appropriate condition (essential condition) must be imposed in the representation of disturbance potential $\varphi^{(2)}$ for the conversation of mass. In this case there is no indication that the result of Green theorem of §3.1.2 (eq.(4)) is valid. In the following section we will see that the result of §3.1.2 is generalized.

Hence, the general representations of the disturbance potentials in $D^{(1)}, D^{(2)}$ and $D^{(3)}$ according to the above definitions, with respect to the studied cases, are:

(a) ($\varphi \in D^{(1)}$)

a1. *Subcritical case* ($Fr_1 < 1$):

$$\varphi^{(1)}(x,z) = \sum_{n=1}^{\infty} C_n^{(1)} \exp(k_n^{(1)} x) Z_n^{(1)}(z), \quad ((x,z) \in D^{(1)}), \tag{5a}$$

a2. *Supercritical case* ($Fr_1 > 1$):

$$\varphi^{(1)}(x,z) = \sum_{n=0}^{\infty} C_n^{(1)} \exp(k_n^{(1)} x) Z_n^{(1)}(z), \quad ((x,z) \in D^{(1)}), \tag{5b}$$

(b) ($\varphi \in D^{(2)}$)

b1. *Subcritical case* ($Fr_2 < 1$):

$$\begin{aligned}\varphi^{(2)}(x,z) = {} & A_{-1}^{(2)} x + B_{-1}^{(2)} + (A_0^{(2)} \cos(k_0^{(2)} x) + B_0^{(2)} \sin(k_0^{(2)} x)) Z_0^{(2)}(z) + \\ & + \sum_{n=1}^{\infty} \left(A_n^{(2)} \exp(k_n^{(2)}(x-a)) + B_n^{(2)} \exp(-k_n^{(2)} x) \right) Z_n^{(2)}(z), ((x,z) \in D^{(2)})\end{aligned} \tag{6a}$$

b2. *Supercritical case* ($Fr_2 > 1$):

$$\varphi^{(2)}(x,z) = A_{-1}^{(2)} x + B_{-1}^{(2)} + \sum_{n=0}^{\infty} \left(A_n^{(2)} \exp(k_n^{(2)}(x-a)) + B_n^{(2)} \exp(-k_n^{(2)} x) \right) Z_n^{(2)}(z),$$

$$((x,z) \in D^{(2)}), \tag{6b}$$

(c) $(\varphi \in D^{(3)})$

c1. *Subcritical case* ($Fr_3 < 1$):

$$\varphi^{(3)}(x,z) = A_{-1}^{(3)}x + B_{-1}^{(3)} + (A_0^{(3)}\cos(k_0^{(3)}x) + B_0^{(3)}\sin(k_0^{(3)}x))Z_0^{(3)}(z) + \\ + \sum_{n=1}^{\infty} C_n^{(3)} \exp(-k_n^{(3)}(x-a))Z_n^{(3)}(z), \quad ((x,z) \in D^{(3)}), \tag{7a}$$

c2. *Supercritical case* ($Fr_3 > 1$):

$$\varphi^{(3)}(x,z) = A_{-1}^{(3)}x + B_{-1}^{(3)} + \sum_{n=0}^{\infty} C_n^{(3)} \exp(-k_n^{(3)}(x-b))Z_n^{(3)}(z), ((x,z) \in D^{(3)}), \tag{7b}$$

where , $A_{-1}^{(2)}, B_{-1}^{(2)}, A_{-1}^{(3)}, B_{-1}^{(3)}$ and $C_n^{(i)}$, $n \in N_f$ or $n \in N_t$, $i = 1,2,3$, are real constants. The sets of index N_f , N_t are defined as follows:

- *Subrcritical case* ($Fr < 1$): $N_f = \{n: \ n = 1,2,...\}$
- *Supercritical case* ($Fr > 1$): $N_t = \{n: \ n = 0,1,2,...\}$

Given the upstream velocity *U*, and imposing the downstream radiation condition and the essential condition in (6) for conservation of mass, the potentials $\varphi^{(1)}$, $\varphi^{(2)}$ and $\varphi^{(3)}$ are uniquely determined by the means of the real coefficients $\{C_n^{(1)}\}$, $B_{-1}^{(2)}$, $\{C_n^{(2)}\}$, $B_{-1}^{(3)}$ and $\{C_n^{(3)}\}$ $n \in N_f$ or $n \in N_t$, respectively.

3.2.2 Essential condition on the representation of disturbance potential in the finite subdomain using the conversation of mass in integral form.

As mentioned above an essential condition must be imposed in the general representation of $\varphi \in D^{(2)}$ for the conservation of mass. Let us consider the problem illustrated in Fig. 3.23 of a fluid occupies the region D with boundary ∂D where $\partial D = \partial D_F \cup \partial D_{-\infty} \cup \partial D_\Pi \cup \partial D_B \cup \partial D_I$. We denote that the boundary segment ∂D_∞ is an artificial boundary chosen to be far upstream, and the boundary segment ∂D_I is an artificial boundary chosen at $x = x^I$.

Without loss of generality we assume that the flow is subcritical in the region D. We recall that the velocity potential in domain D is

$$\Phi(x,z) = Ux + \varphi(x,z) \qquad (x,z) \in D, \tag{8}$$

where, $\varphi(x,z)$ is the disturbance velocity potential.

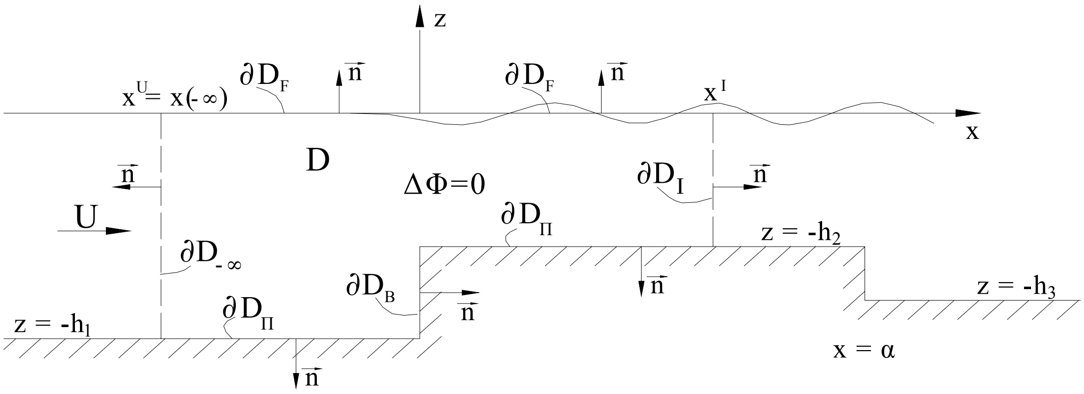

FIGURE 3.23. The domain of definition D of the velocity potential Φ

Let us consider the velocity potential Φ, which is finite, singled valued and differentiable at all points of the connected region D bounded by the closed surface ∂D. Hence, the following form of Green's theorem may be obtained for the above region D, i.e. Lamb (1932, p.43):

$$\int_{\partial D} \frac{\partial \Phi}{\partial \vec{n}} dS = \int_{D} \nabla^2 \Phi dV \,, \tag{9}$$

where $\vec{n} = (n_x, n_z)$ is the unit normal vector to the boundary ∂D directed to the exterior of D.

Based on the fact that the function Φ satisfies the Laplace equation in region $D^{(i)}$, the Green's theorem takes the following form:

$$\int_{\partial D} \frac{\partial \Phi}{\partial \vec{n}} dS = 0 \,, \tag{10}$$

which simply express the conservation of mass in integral form for steady irrotational motion of an ideal fluid.

Applying the equation (10) in region D, with the aid of Eq.(8) we obtain

$$\int_{\partial D} \frac{\partial \Phi}{\partial \vec{n}} dS = \int_{\partial D} (U \cdot n_x + \frac{\partial \varphi}{\partial \vec{n}}) dS = 0 \,. \tag{11}$$

The disturbance velocity potential φ satisfies the boundary conditions of the problem P_M; therefore, the above integral gives

$$-\int_{\partial D_F} \frac{U^2}{g} \varphi,_{xx} dS + \underbrace{\int_{\partial D_B \cup \partial D_\Pi} (U n_x + \varphi,_n) dS}_{0} - \int_{\partial D_{-\infty}} (U + \varphi,_x) dS + \int_{\partial D_I} (U + \varphi,_x) dS = 0 \,. \tag{12}$$

As $x \to -\infty$, one can easily verify from Eq.(5a) that $\varphi,_x \to 0$, then Eq.(12) obtains $-\dfrac{U^2}{g}\int\limits_{x=x^U}^{x=x^I} \varphi,_{xx} dx - U\int\limits_{-h_1}^{0} dz + U\int\limits_{-h_2}^{0} dz + \int\limits_{\partial D_I} \varphi,_x dz = 0$. (13)

By differentiation of eq.(6a), for $x = x^I$ we obtain:

$$\varphi,_x^{(2)}(x^I,z) = A_{-1}^{(2)} + k_0^{(2)}(-A_0^{(2)}\sin(k_0^{(2)}x^I) + B_0^{(2)}\cos(k_0^{(2)}x^I))Z_0^{(2)}(z) +$$

$$+\sum_{n=1}^{\infty} k_n^{(2)}\left(A_n^{(2)}\exp(k_n^{(2)}(x^I - a)) - B_n^{(2)}\exp(-k_n^{(2)}x^I)\right)Z_n^{(2)}(z). \tag{14}$$

Hence, eq.(13) yields

$$-\frac{U^2}{g}(\varphi,_x\Big|_{x=x^I} - \varphi,_x\Big|_{x=x^U}) + U(h_2 - h_1) + \int\limits_{\partial D_I} \varphi,_x dz = 0 \;\Rightarrow \text{(using eq. 14)},$$

$$-\frac{U^2}{g}\Big(A_{-1}^{(2)} + k_0^{(2)}(-A_0^{(2)}\sin(k_0^{(2)}x^I) + B_0^{(2)}\cos(k_0^{(2)}x^I)) +$$

$$+\sum_{n=1}^{\infty} k_n^{(2)}\left(A_n^{(2)}\exp(k_n^{(2)}(x^I - a)) - B_n^{(2)}\exp(-k_n^{(2)}x^I)\right)\Big) + U(h_2 - h_1) +$$

$$+A_{-1}^{(2)}h_2 + \frac{U^2}{g}\Big(k_0^{(2)}(-A_0^{(2)}\sin(k_0^{(2)}x^I) + B_0^{(2)}\cos(k_0^{(2)}x^I)) +$$

$$+\sum_{n=1}^{\infty} k_n^{(2)}\left(A_n^{(2)}\exp(k_n^{(2)}(x^I - a)) - B_n^{(2)}\exp(-k_n^{(2)}x^I)\right)\Big) = 0, \tag{15}$$

where $\int\limits_{-h_2}^{0} Z_n(z)dz = \dfrac{U^2}{g}$ (obtained from dispersion relation (eq. (23a) §2.2), has been used. After little algebra the relation (15) yields (defining $A_{-1}^{(2)} \equiv \nu$):

$$\nu(h_2 - \frac{U^2}{g}) + U(h_2 - h_1) = 0 \;\Rightarrow\; \nu = U\frac{h_1 - h_2}{h_2 - \dfrac{U^2}{g}} \tag{16a}$$

Defining the shoaling ratio $s = h_2 / h_1$, which is taken the values $s < 1$ if the flow is from deep to shallow water and $s > 1$ if the flow is from shallow to deep water, and denoting that $Fr_1 = U/\sqrt{gh_1}$, the above relation takes the form

$$\nu(s;U) = \frac{U(1-s)}{s - Fr_1^2}. \tag{16b}$$

According to the above analysis, the relation (16) is a general result, which is independent from the position of the artificial boundary ∂D_I .

3.2.3 Weak formulation of matching -B.V.P. – Composition of the solution matrix.

In this subsection we proceed to derivation of the linear system, which arises from the matching boundary value problem P_{MM}. We recall that the matching conditions are:

$$\varphi^{(1)} = \varphi^{(2)} \qquad -h_2 < z < 0, \quad x = 0, \tag{17a}$$

$$\frac{\partial \varphi^{(1)}}{\partial x^{(1)}} = \begin{cases} \dfrac{\partial \varphi^{(2)}}{\partial x^{(2)}}, & -h_2 < z < 0, \quad x = 0 \\ -U \cdot n_x, & -h_1 < z < -h_2, \quad x = 0 \end{cases}, \tag{17b}$$

$$\varphi^{(2)} = \varphi^{(3)} \qquad -h_2 < z < 0, \quad x = a, \tag{17c}$$

$$\frac{\partial \varphi^{(3)}}{\partial x^{(3)}} = \begin{cases} \dfrac{\partial \varphi^{(2)}}{\partial x^{(2)}}, & -h_2 < z < 0, \quad x = a \\ -U \cdot n_x, & -h_3 < z < -h_2, \quad x = a \end{cases}, \tag{17d}$$

where, $\varphi^{(i)}(x,z)\ i = 1,2,3$ is the disturbance velocity potential, given by the expansions (5), (6), and (7) . Let us define the functions $f_1(z)$ and $f_2(z)$ to be equal with the right side of equations (17b), and (17d) respectively, so the above condition may be rewrite to the following form

$$\varphi^{(1)} - \varphi^{(2)} = 0 \qquad -h_2 < z < 0, \quad x = 0, \tag{17΄a}$$

$$\frac{\partial \varphi^{(1)}}{\partial x^{(1)}} - f_1(z) = 0 \qquad -h_1 < z < 0, \quad x = 0. \tag{17΄b}$$

$$\varphi^{(2)} - \varphi^{(3)} = 0 \qquad -h_2 < z < 0, \quad x = a, \tag{17΄c}$$

$$\frac{\partial \varphi^{(3)}}{\partial x^{(3)}} - f_2(z) = 0 \qquad -h_3 < z < 0, \quad x = a. \tag{17΄d}$$

According to the theorem of § 2.2 the set of eigenfunctions $Z_n^{(1)}(z)$, $Z_n^{(2)}(z)$ and $Z_n^{(3)}(z)$ constitute a *Riesz basis* on $L^2(-h_1,0)$, $L^2(-h_2,0)$ and $L^2(-h_3,0)$, respectively. Using this definition, the above system of equations is equivalent on *weak sense* to the following:

$$\left\langle \varphi^{(1)} - \varphi^{(2)}, Z_m^{(2)}(z) \right\rangle = 0, \quad x = 0 \qquad \forall m \tag{18a}$$

$$\left\langle \frac{\partial \varphi^{(1)}}{\partial x^{(1)}} - f_1(z), Z_m^{(1)}(z) \right\rangle = 0, \quad x = 0 \qquad \forall m \tag{18b}$$

$$\left\langle \varphi^{(2)} - \varphi^{(3)}, Z_m^{(2)}(z) \right\rangle = 0, \quad x = a \quad \forall m \tag{18c}$$

$$\left\langle \frac{\partial \varphi^{(3)}}{\partial x^{(3)}} - f_2(z), Z_m^{(3)}(z) \right\rangle = 0, \quad x = a \quad \forall m \tag{18d}$$

where $\langle \cdot,\cdot \rangle$ is the L^2- inner product .

Without loss of generality, we introduce the derivation of linear system, which constitutes the above equations, in the case where the flow is subcritical in the region D. Similar procedure is following in case of supercritical flow. We denote here, that in the expansion of $\varphi^{(1)}$ (Eq. (5a)), the terms that imply a disturbance on upstream region may be obtained, such that the number of equations to be equal with the number of unknowns. After the composition of the system's matrix, the radiation conditions shall be imposed. Truncating the series of expansions (5), (6) and (7) to a finite number of terms (modes), and denoting by N the number of evanescent modes retained, with above definitions we have:

$$\int_{-h_2}^{0} (\varphi^{(1)} - \varphi^{(2)}) \cdot Z_m^{(2)}(z) dz = 0 \Leftrightarrow$$

$$\Leftrightarrow A_0^{(1)} \int_{-h_2}^{0} Z_0^{(1)}(z) Z_m^{(2)}(z) dz + \sum_{n=1}^{N} C_n^{(1)} \int_{-h_2}^{0} Z_n^{(1)}(z) Z_m^{(2)}(z) dz - B_{-1}^{(2)} \int_{-h_2}^{0} Z_m^{(2)}(z) dz -$$

$$-A_0^{(2)} \int_{-h_2}^{0} Z_0^{(2)}(z) Z_m^{(2)}(z) dz - \sum_{n=1}^{N} (A_n^{(2)} \exp(-k_n^{(2)} a) + B_n^{(2)}) \int_{-h_2}^{0} Z_n^{(2)}(z) Z_m^{(2)}(z) dz = 0,$$

$$m = -1, 0, 1, ..., N, N+1. \tag{19a}$$

$$\int_{-h_2}^{0} (\varphi,_x^{(1)} - \varphi_x^{(2)}) \cdot Z_m^{(1)}(z) dz + \int_{-h_1}^{-h_2} (\varphi,_x^{(1)} + U) \cdot Z_m^{(1)}(z) dz = 0 \Leftrightarrow$$

$$\Leftrightarrow \int_{-h_1}^{0} \varphi,_x^{(1)} Z_m^{(1)}(z) dz - \int_{-h_2}^{0} \varphi,_x^{(2)} Z_m^{(1)}(z) dz = -U \int_{-h_1}^{-h_2} Z_m^{(1)}(z) dz \Leftrightarrow$$

$$\Leftrightarrow k_0^{(1)} B_0^{(1)} \int_{-h_1}^{0} Z_0^{(1)}(z) Z_m^{(1)}(z) dz + \sum_{n=1}^{N} k_n^{(1)} C_n^{(1)} \int_{-h_1}^{0} Z_n^{(1)}(z) Z_m^{(1)}(z) dz - A_{-1}^{(2)} \int_{-h_2}^{0} Z_m^{(1)}(z) dz -$$

$$-k_0^{(2)} B_0^{(2)} \int_{-h_2}^{0} Z_0^{(2)}(z) Z_m^{(1)}(z) dz - \sum_{n=1}^{N} k_n^{(2)} (A_n^{(2)} \exp(-k_n^{(2)} a) - B_n^{(2)}) \int_{-h_2}^{0} Z_n^{(2)}(z) Z_m^{(1)}(z) dz =$$

$$= -U \int_{-h_1}^{-h_2} Z_m^{(1)}(z) dz, \qquad m = -1, 0, 1, ..., N, N. \tag{19b}$$

$$\int_{-h_2}^{0} (\varphi^{(2)} - \varphi^{(3)}) \cdot Z_m^{(2)}(z) dz = 0 \Leftrightarrow$$

$$\Leftrightarrow A_{-1}^{(2)} a \int_{-h_2}^{0} Z_m^{(2)}(z) dz + B_{-1}^{(2)} \int_{-h_2}^{0} Z_m^{(2)}(z) dz + (A_0^{(2)} \cos(k_0^{(2)} a) + B_0^{(2)} \sin(k_0^{(2)} a)) \int_{-h_2}^{0} Z_0^{(2)}(z) Z_m^{(2)}(z) dz +$$

$$+ \sum_{n=1}^{N} (A_n^{(2)} + B_n^{(2)} \exp(-k_n^{(2)} a)) \int_{-h_2}^{0} Z_n^{(2)}(z) Z_m^{(2)}(z) dz - A_{-1}^{(3)} a \int_{-h_2}^{0} Z_m^{(2)}(z) dz - B_{-1}^{(3)} \int_{-h_2}^{0} Z_m^{(2)}(z) dz -$$

$$-(A_0^{(3)} \cos(k_0^{(3)} a) + B_0^{(3)} \sin(k_0^{(3)} a)) \int_{-h_2}^{0} Z_0^{(3)}(z) Z_m^{(2)}(z) dz - \sum_{n=1}^{N} C_n^{(3)} \int_{-h_2}^{0} Z_n^{(3)}(z) Z_m^{(2)}(z) dz = 0,$$

$$m = -1, 0, 1, ..., N. \qquad (20a)$$

$$\int_{-h_2}^{0} (\varphi,_x^{(2)} - \varphi_x^{(3)}) \cdot Z_m^{(3)}(z) dz - \int_{-h_3}^{-h_2} (\varphi,_x^{(3)} + U) \cdot Z_m^{(3)}(z) dz = 0 \Leftrightarrow$$

$$\Leftrightarrow \int_{-h_2}^{0} \varphi,_x^{(2)} Z_m^{(3)}(z) dz - \int_{-h_3}^{0} \varphi,_x^{(3)} Z_m^{(3)}(z) dz = U \int_{-h_3}^{-h_2} Z_m^{(3)}(z) dz \Leftrightarrow$$

$$\Leftrightarrow A_{-1}^{(2)} \int_{-h_2}^{0} Z_m^{(3)}(z) dz + k_0^{(2)} (-A_0^{(2)} \sin(k_0^{(2)} a) + B_0^{(2)} \cos(k_0^{(2)} a)) \int_{-h_2}^{0} Z_0^{(2)}(z) Z_m^{(3)}(z) dz +$$

$$+ \sum_{n=1}^{N} k_n^{(2)} (A_n^{(2)} - B_n^{(2)} \exp(-k_n^{(2)} a)) \int_{-h_2}^{0} Z_n^{(2)}(z) Z_m^{(3)}(z) dz - A_{-1}^{(3)} \int_{-h_3}^{0} Z_m^{(3)}(z) dz -$$

$$-k_0^{(3)} (-A_0^{(3)} \sin(k_0^{(3)} a) + B_0^{(3)} \cos(k_0^{(3)} a)) \int_{-h_3}^{0} Z_0^{(3)}(z) Z_m^{(3)}(z) dz + \sum_{n=1}^{N} k_n^{(3)} C_n^{(3)} \int_{-h_3}^{0} Z_n^{(3)}(z) Z_m^{(3)}(z) dz =$$

$$= U \int_{-h_3}^{-h_2} Z_m^{(3)}(z) dz, \qquad m = -1, 0, 1, ..., N, N+1. \qquad (20b)$$

The number of the unknown coefficients of expansions is $\# 2(2N+5)$, and the total number of equations from coupling the equations (19) and (20) is $\# 2(2N+5)$.

3.2.4 Numerical results and discussion.

In this paragraph a detailed presentation of the numerical results obtained, according to the present method. The numerical calculations were performed for a finite step (or trenche) of variable height and for a wide range of upstream Froude numbers when the flow is subcritical or supercritical.

The physical investigation of the influence of the size of obstruction and of Froude number – the physical parameters of the problem – to the numerical solutions of the problem is introduced, in order to define the range of its values for which the linear solution is valid. The numerical accuracy of the problem is interpreted to satisfaction of the matching conditions at the artificial interfaces, i.e. the continuity of the pressure field and the continuity of the velocity field. This is concerned with the appropriate number of evanescent modes N, such that the week solution of the problem P_{MM} to converge to the exact solution in the L^2 – sense.

1. Subcritical flows $(Fr < 1)$

Numerical solutions have been obtained for a wide range of the physical parameters. However, in order to illustrate the results obtained we first investigate in detail the situations when the depth is constant far upstream and far downstream and the flow obstructed by a finite step. The depth is $1m$, and the finite step heights h_s are $0.1m$, 0.2 m, $0.3m$, at upstream $Fr = 0.5$. The length of the finite step is $\ell = 0.6m$.

The figures 3.23 shows, as one would expect, that the level of the free surface falls as it approaches the finite step, while downstream of the finite step a periodic steady wave motion is predicted. The amplitude of these waves depends upon both finite step height and Froude number whereas the wavelength depends only upon Froude number as shown in figure 3.23 (see also fig.3.24).

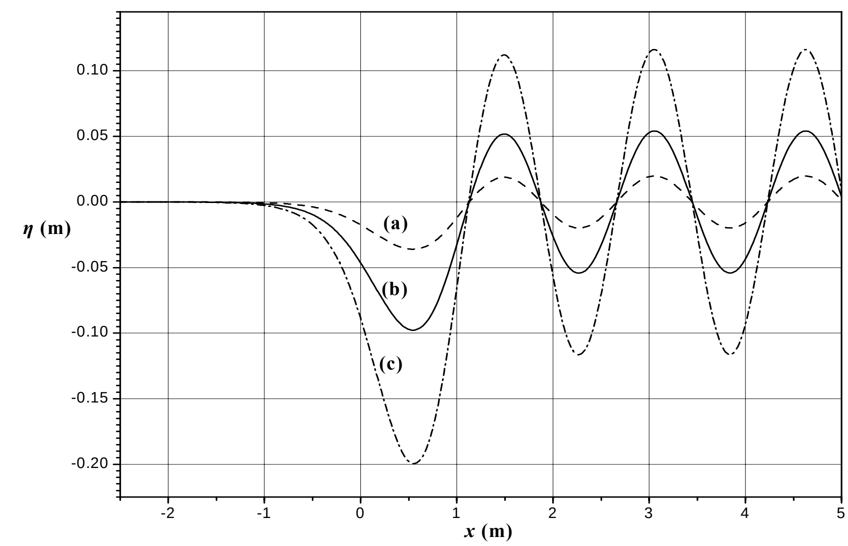

FIGURE 3.23. The free-surface elevation of present model for $Fr = 0.5$, for three values of finite step height: (a) h_s = 0.1, (b) h_s = 0.2, (c) h_s = 0.3. The length of the finite step is 0.6 m and the location of its center is on x=0.3.

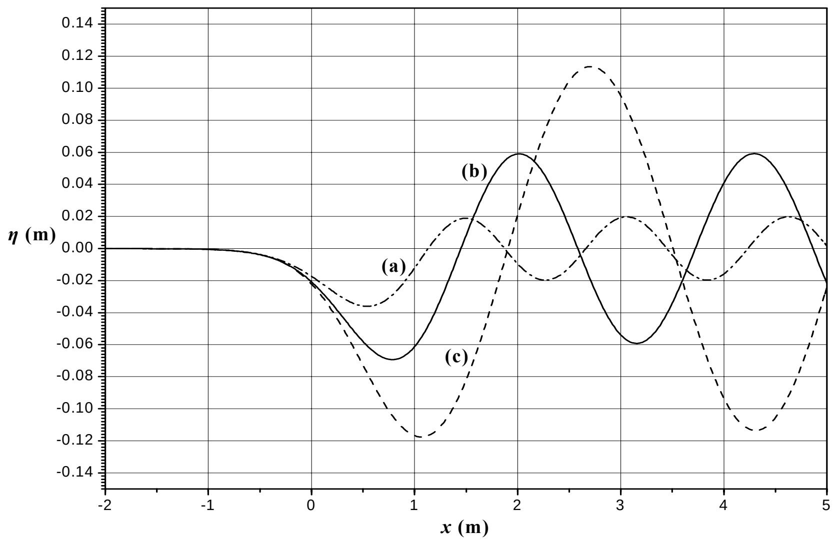

FIGURE 3.24. The free-surface elevation of present model for three values of Froude number: (a) $Fr = 0.5$, (b) $Fr = 0.6$, (c) $Fr = 0.7$. The height and length of the finite step are $0.1m$ and 0.6 *m*, respectively. The location of its center is on *x*=0.3.

In figure 3.25 the variation of the blockage parameter C against the Froude number, for various heights of finite step is presented.

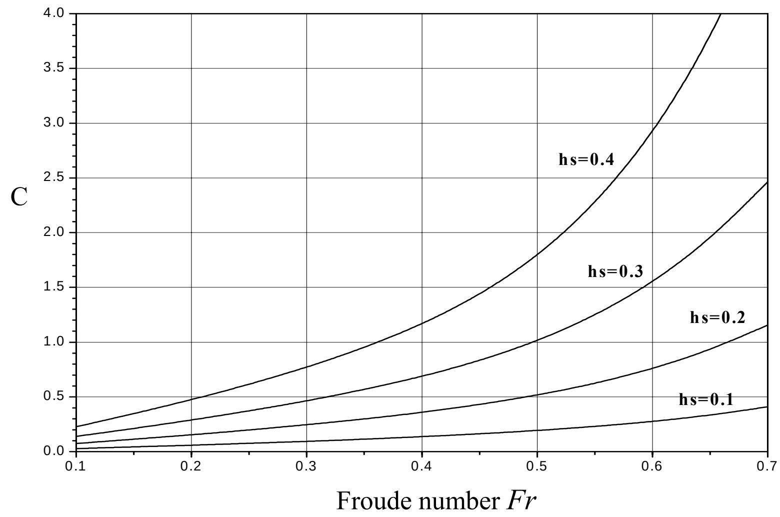

FIGURE 3.25. The dependence of blockage parameter C upon the Froude number *Fr* and upon the height of the finite step h_s .

In figure 3.26a the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model. The flow obstructed by a finite step of height $h_s = 0.3$ and length $\ell = 1m$ at Froude number $Fr = 0.5$. This figure shows, as one would expect, that the disturbance potential field becomes infinite at the singular points $(x,z) = (0,-h_2)$ and $(x,z) = (a,-h_2)$.

In figures 3.26.b-c and 3.26.d-e the continuity of the disturbance potential field φ (C^0- continuity) and the continuity of the disturbance velocity field $\varphi_{,x}$ (C^1- continuity), at $x = 0$ and $x = a$, respectively, have been plotted. These figures are obtained by retaining 51 evanescent modes ($N = 51$).

In figure 3.27a the equipotential lines of the disturbance potential field $\varphi(x, z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model. The flow obstructed by a trench of depth h_s = 1.4 and length $\ell = 1m$ at Froude number $Fr = 0.6$.

In figures 3.28-3.29 the equipotential lines of the potential field $\varphi(x, z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, in case where the far depths is different.

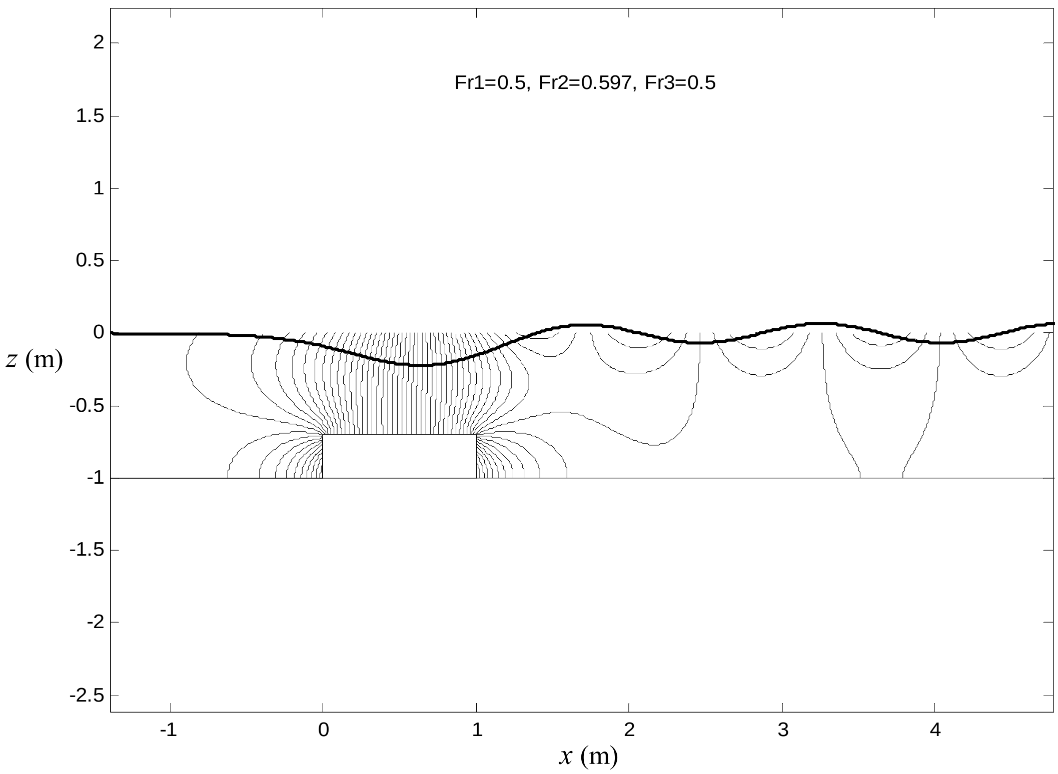

FIGURE 3.26a. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model.

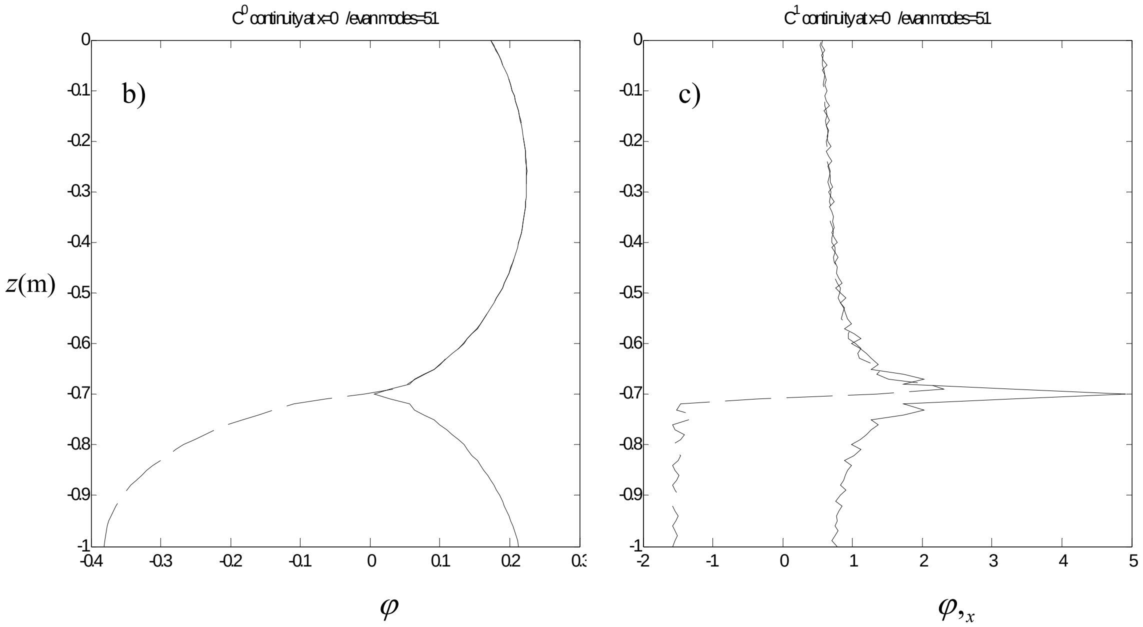

FIGURE 3.26b-c. C^0- continuity b) and C^1- continuity c) of the disturbance potential at $x = 0$, for Froude number Fr=0.5. The height and length of the finite step are 0.3*m* and 1*m*, respectively. Number of evanescent modes N = 51.

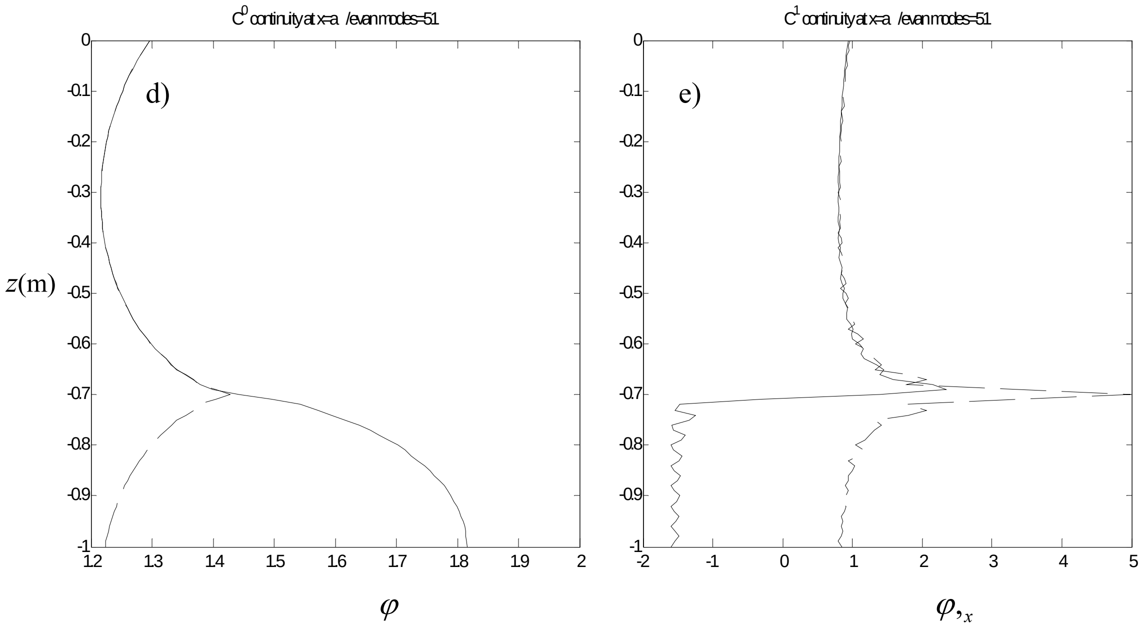

FIGURE 3.26d-e. C^0 - continuity d) and C^1 - continuity e) of the disturbance potential at $x = a$, for Froude number *Fr*=0.5. The height and length of the finite step are 0.3*m* and 1*m*, respectively. Number of evanescent modes *N* = 51.

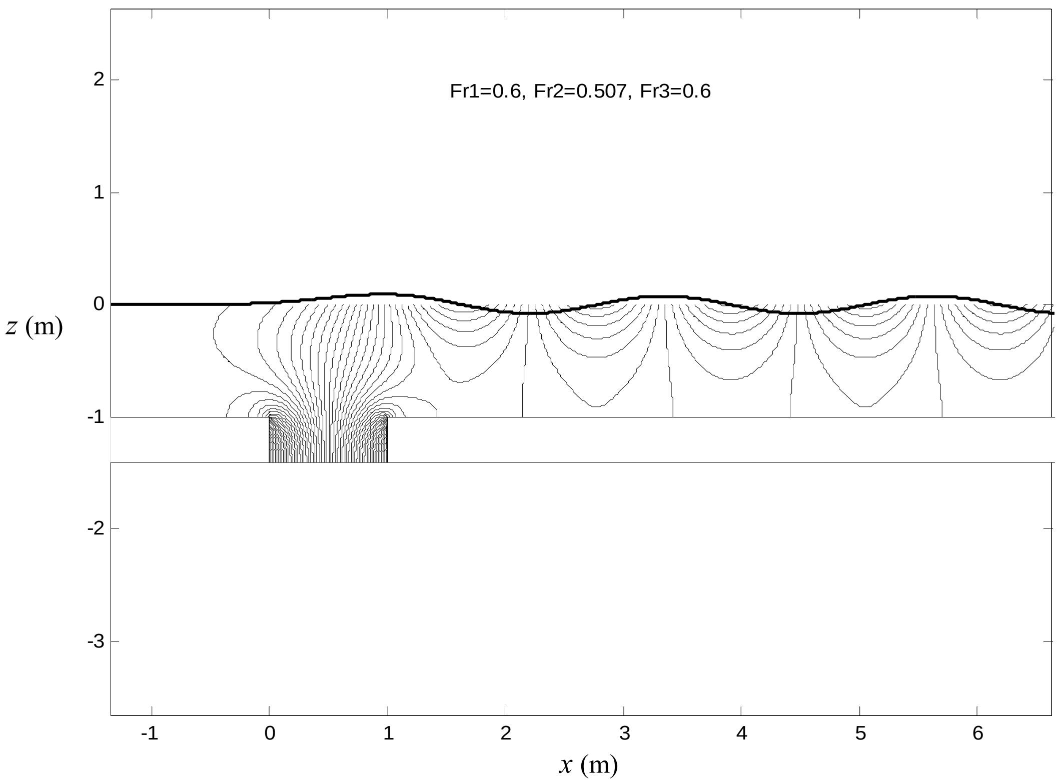

FIGURE 3.27a. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model, for Froude number *Fr*=0.6. The depth at the trench is 1.4*m* and its length is 1*m*.

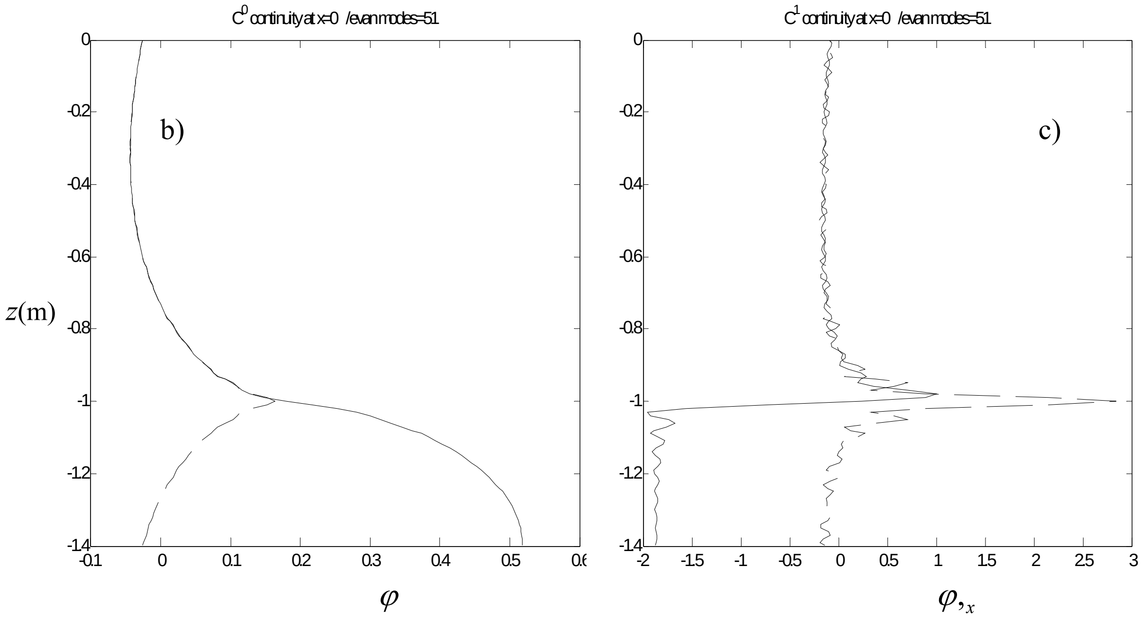

FIGURE 3.27b-c. C^0 - continuity b) and C^1 - continuity c) of the disturbance potential at $x = 0$, for Froude number Fr_1=0.6. The depth at the trench is 1.4m and its length is 1m.

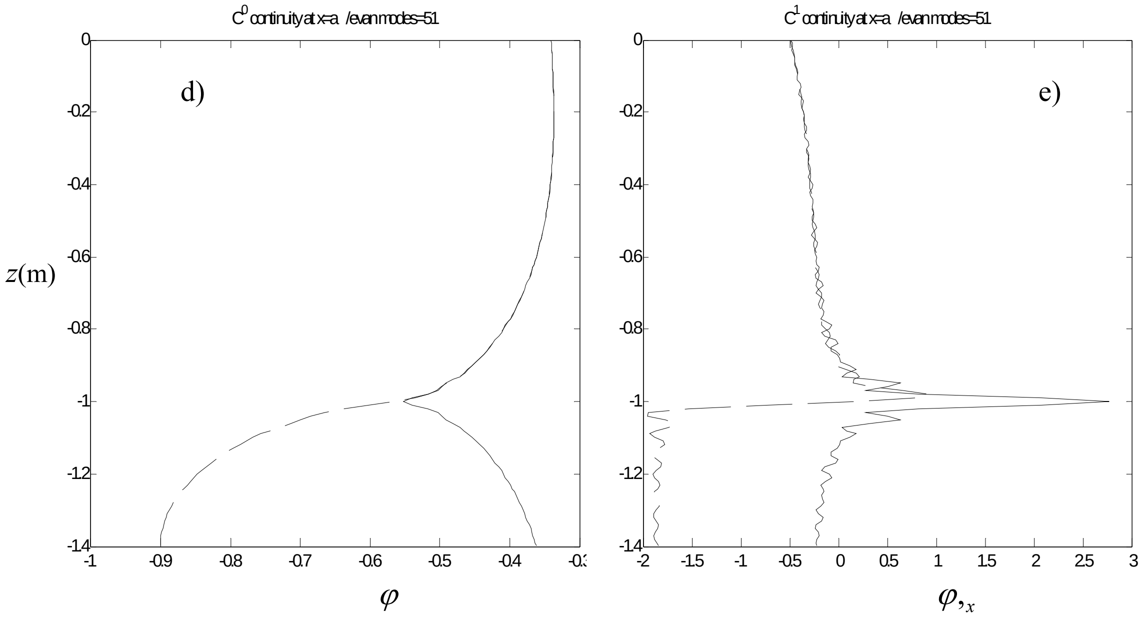

FIGURE 3.27d-e. C^0 - continuity d) and C^1 - continuity e) of the disturbance potential at $x = a$, for Froude number Fr=0.5. The depth at the trench is 1.4m and its length is 1m.

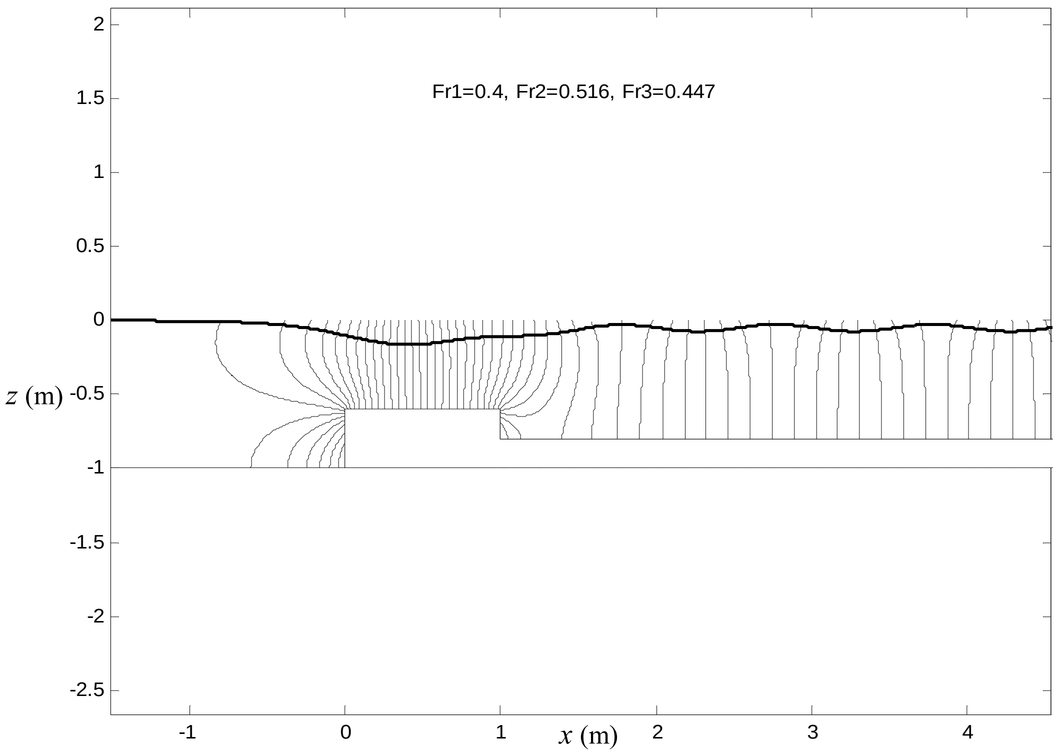

FIGURE 3.28. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model, for upstream Froude number Fr=0.4. The depths are h_1=1, h_2=0.6, h_3=0.8.

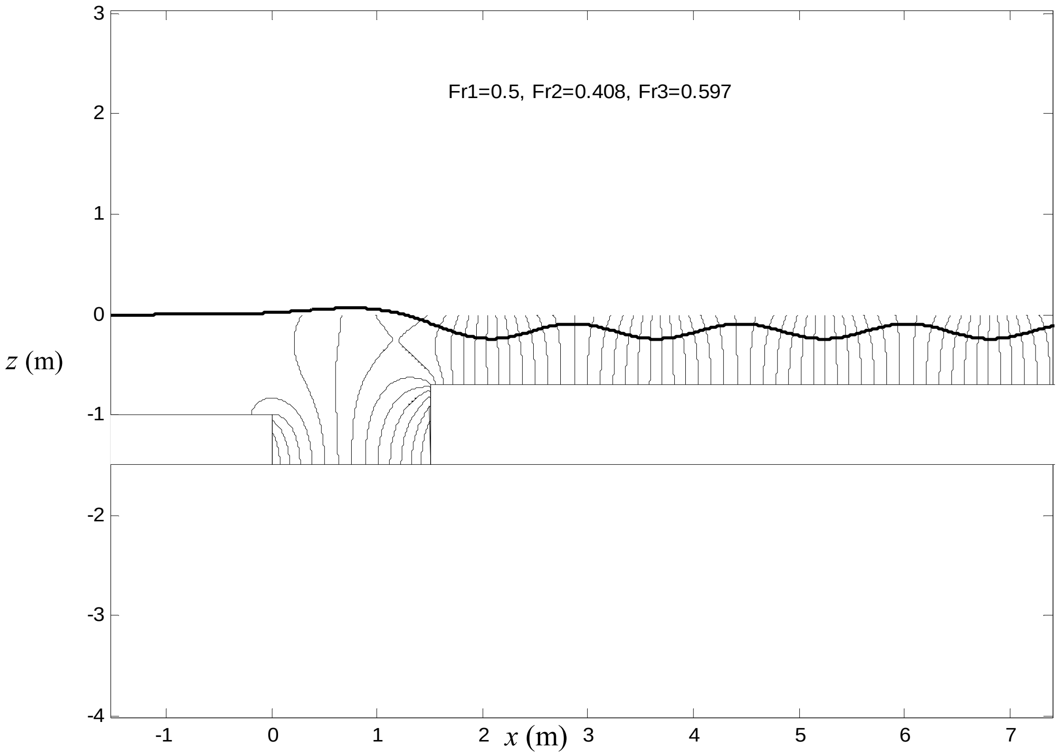

FIGURE 3.29. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model, for upstream Froude number Fr=0.5. The depths are h_1=1, h_2=1.5, h_3=0.7.

2. *Supercritical flows* $(Fr > 1)$

Numerical solutions have been obtained for a wide range of the physical parameters. However, in order to illustrate the results obtained we first investigate in detail the situations when the depth is constant far upstream and far downstream and the flow obstructed by a finite step. The depth is $1m$, and the finite step heights h_s are $0.1m$, $0.2m$, $0.3m$, $Fr = 2$. The length of the finite step is $\ell = 1m$.

The figures 3.30 shows, as one would expect, that the level of the free surface rises monotonically as it approaches the finite step, while downstream of the finite decreases monotonically and becoming asymptotically flat.

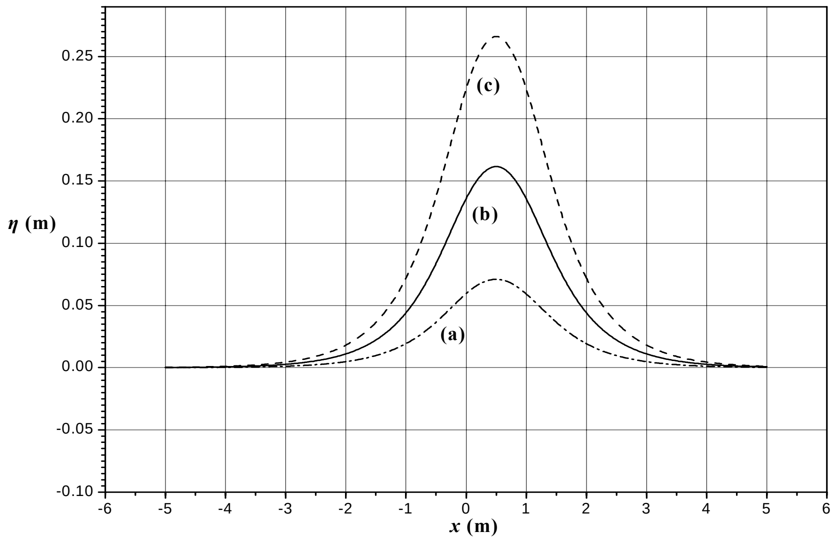

FIGURE 3.30. The free-surface elevation of present model for $Fr = 2$, for three values of finite step height: (a) $h_s = 0.1$, (b) $h_s = 0.2$, (c) $h_s = 0.3$. The length of the finite step is 1 m and the location of its center is on x=0.5.

In figure 3.31 the variation of the blockage parameter C against the Froude number, for various heights of finite step is presented.

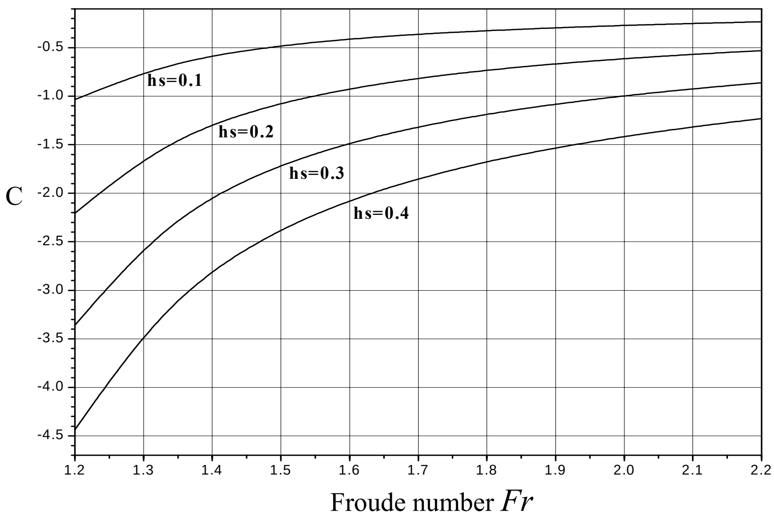

FIGURE 3.31. The dependence of blockage parameter C upon the Froude number Fr and upon the height of the finite step h_s.

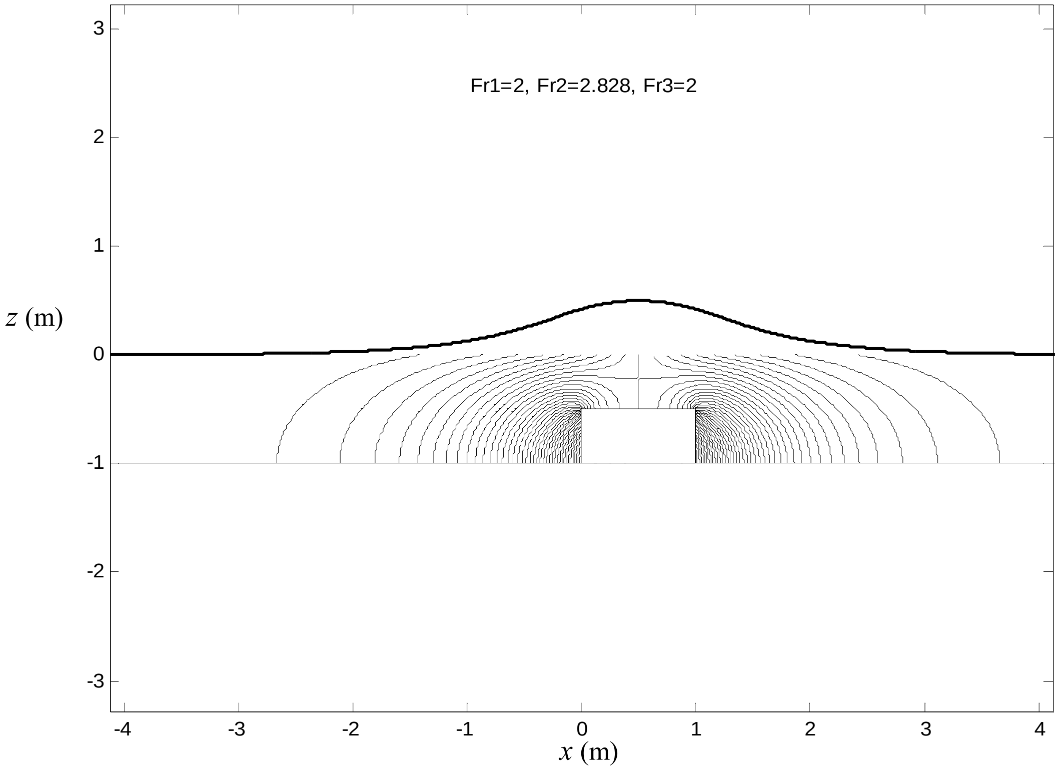

FIGURE 3.32. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model, for Froude number Fr=2. The depths are h_1=1, h_2=0.5, h_3=1.

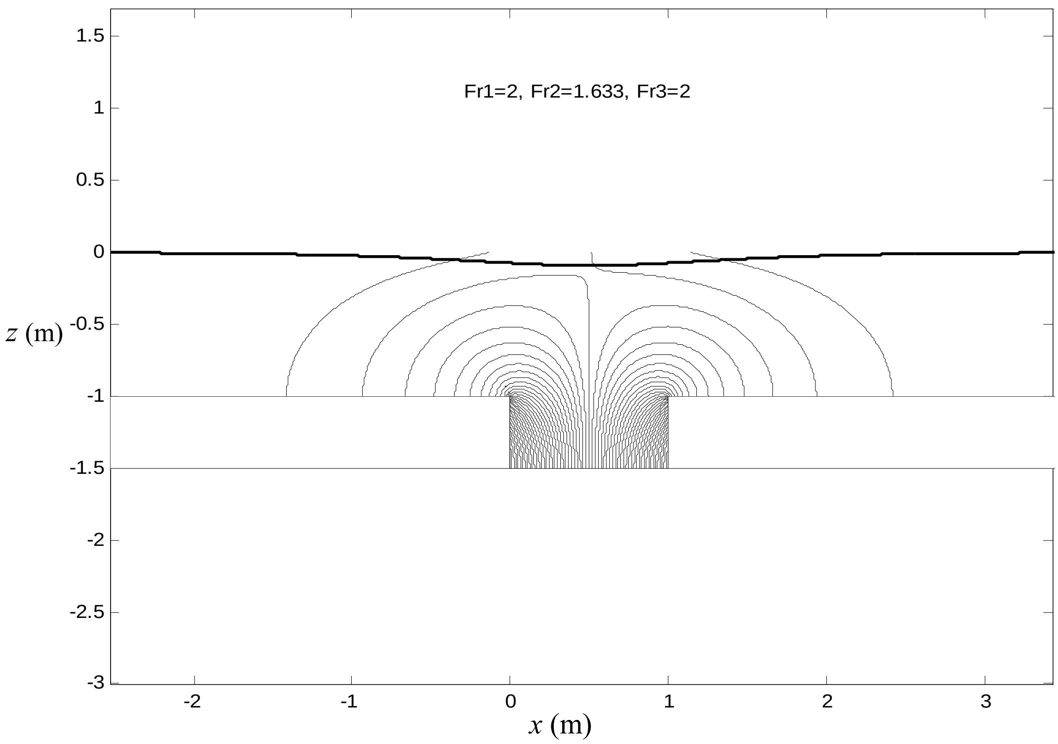

FIGURE 3.32. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model, for Froude number Fr=2. The depths are h_1=1, h_2=1.5, h_3=1.

Chapter Four
Steady Free-Surface Flows over an arbitrary topography in the presence of submerged bodies

4. Steady Free-Surface flows over an arbitrary topography in the presence of submerged bodies.

4.1 Differential formulation of the problem.

The steady two-dimensional free-surface flow of a stream, of an inviscid incompressible and irrotational fluid, which is obstructed by variable topography on the bottom with the presence of submerged body, is considered. The wave field is excited by an incident uniform flow with direction normal to the bottom contours, and velocity *U*. The studied free-surface flow consists of a water layer D_{3D} bounded above by the free surface $\partial D_{F,3D}$ and below by a rigid bottom $\partial D_{\Pi,3D}$. It is assumed that the bottom slope exhibits an arbitrary one-dimensional variation in a subdomain of finite length, i.e. the bathymetry is characterized by parallel, straight bottom contours lying between two regions of constant but different depth, $h = h_1$ (region of incidence) and $h = h_3$ (region of transmission); see figure 4.1.

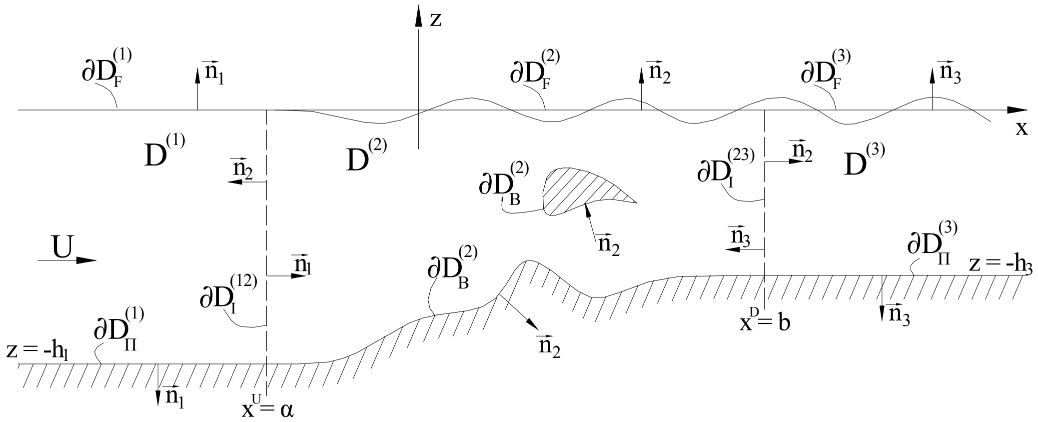

FIGURE 4.1. Domain decomposition and basic notation

Before proceeding to the formulation of the problem, we shall introduce some geometrical notation. A Cartesian coordinate system is introduced, with its origin on the mean water level (in the variable bathymetry region), the *z*-axis pointing upwards and the *y*-axis being parallel to the bottom contours. See figure 4.1.
The liquid domain D_{3D} will be represented by $D_{3D} = D \times R$, where D is the (two-dimensional) intersection of D_{3D} by a vertical plane perpendicular to the bottom contours, and $R = (-\infty, +\infty)$, is a copy of the real line:
$D_{3D} = \{(x, y, z):\ (x, y) \in R^2, -h(x) < z < 0\}$, $D = \{(x, z):\ x \in R, -h(x) < z < 0\}$. The function $h(x)$, appearing in the above definitions, represents the local depth, measured from the mean water level. It is considered to be a twice continuously differentiable function defined on the real axis R, such that

$$h(x) = \begin{cases} h_1, & x \le a \\ h_2(x), & a < x \le b \\ h_3, & x > b \end{cases} . \qquad (1)$$

The liquid domain D_{3D} is decomposed in three subdomains $D_{3D}^{(i)} = D^{(i)} \times R,\ i = 1,2,3,$ defined us follows: $D_{3D}^{(1)}$ is the constant-depth upstream subdomain characterized by $x < a$, $D_{3D}^{(3)}$ is the constant-depth downstream subdomain characterized by $x > b$, and $D_{3D}^{(2)}$ is the variable bathymetry subdomain lying between $D_{3D}^{(1)}$ and $D_{3D}^{(3)}$. Without loss of generality, we assume that $h_1 > h_3$. The decomposition is also applied to the boundaries $\partial D_{F,3D} = \partial D_F \times R$ and $\partial D_{\Pi,3D} = \partial D_\Pi \times R$. The lines ∂D_F and ∂D_Π are decomposed in three pieces each, for example, $\partial D_F = \partial D_F^{(1)} \cup \partial D_F^{(2)} \cup \partial D_F^{(3)}$, where $\partial D_F^{(1)}$ belongs to the boundary of $D^{(1)}$, and similarly for ∂D_Π. Finally, we define the artificial vertical interfaces $\partial D_{I,3D}^{(12)} = \partial D_I^{(12)} \times R$ and $\partial D_{I,3D}^{(23)} = \partial D_I^{(23)} \times R$, which is the common vertical boundaries of subdomains $D_{3D}^{(1)}$ and $D_{3D}^{(2)}$, and $D_{3D}^{(2)}$ and $D_{3D}^{(3)}$, respectively. Clearly, $\partial D_I^{(12)}$ and $\partial D_I^{(23)}$ are vertical segments (between the bottom and the mean water level) at $x = a$ and $x = b$, respectively. See figure 4.1.

We consider that the motion arising from disturbances created by the obstruction of the uniform flow by the variable bathymetry, have the velocity potential

$$\Phi(x,z) = Ux + \varphi(x,z), \qquad (x,z) \in D. \tag{2}$$

Assuming that the disturbance velocity potential $\varphi(x,z)$ and the velocity of the stream are small enough, the linearized equations of the *Neumann-Kelvin* problem can be used (cf. §2.1). By the decomposition of the liquid domain $D = D^{(1)} \cup D^{(2)} \cup D^{(3)}$, the studied problem should be formulated with the aid of the general representation of the disturbance velocity potential $\varphi(x,z)$ in the semi-infinite strips $D^{(1)}$ and $D^{(3)}$ (see e.g. § 3.1.1), with respect to the studied cases:

(a) <u>Upstream</u> ($\varphi \in D^{(1)}$)

a1. *Subcritical case* ($Fr_1 < 1$):

$$\varphi^{(1)}(x,z) = \sum_{n=1}^{\infty} C_n^{(1)} \exp(k_n^{(1)}(x-a)) Z_n^{(1)}(z), \quad ((x,z) \in D^{(1)}), \tag{3a}$$

a2. *Supercritical case* ($Fr_1 > 1$):

$$\varphi^{(1)}(x,z) = \sum_{n=0}^{\infty} C_n^{(1)} \exp(k_n^{(1)}(x-a)) Z_n^{(1)}(z), \quad ((x,z) \in D^{(1)}), \tag{3b}$$

(b) <u>Downstream</u> ($\varphi \in D^{(3)}$)

b1. *Subcritical case* ($Fr_3 < 1$):

$$\varphi^{(3)}(x,z) = A_{-1}^{(3)} x + B_{-1}^{(3)} + (A_0^{(3)} \cos(k_0^{(3)} x) + B_0^{(3)} \sin(k_0^{(3)} x)) Z_0^{(3)}(z) + \\ + \sum_{n=1}^{\infty} C_n^{(3)} \exp(-k_n^{(3)}(x-b)) Z_n^{(3)}(z), \quad ((x,z) \in D^{(3)}), \tag{4a}$$

b2. *Supercritical case* ($Fr_3 > 1$):

$$\varphi^{(3)}(x,z) = A_{-1}^{(3)}x + B_{-1}^{(3)} + \sum_{n=0}^{\infty} C_n^{(3)} \exp(-k_n^{(3)}(x-b))Z_n^{(3)}(z)\,, \quad ((x,z)\in D^{(3)})\,, \tag{4b}$$

where $A_{-1}^{(3)}, B_{-1}^{(3)}$, $A_0^{(3)}, B_0^{(3)}$ and $C_n^{(i)}$, $n \in N_f$ or $n \in N_t$, $i = 1,3$, are real constants.

The sets of index N_f , N_t are defined as follows:

- *Subrcritical case* ($Fr < 1$): $N_f = \{n:\ n = 1,2,...\}$
- *Supercritical case* ($Fr > 1$): $N_t = \{n:\ n = 0,1,2,...\}$

In the expansions (3) and (4) the sets of numbers $\{k_n^{(1)}, n \in N_f\} \cap \{0\}$, $\{k_0^{(3)}, k_n^{(3)}, n \in N_f\} \cap \{0\}$, $\{k_n^{(i)}, n \in N_t\} \cap \{0\}$ $i = 1,3$, and the sets of vertical functions $\{Z_n^{(1)}(z), n \in N_f\} \cap \{1\}$, $\{Z_0^{(3)}(z), Z_n^{(3)}(z), n \in N_f\} \cap \{1\}$, $\{Z_n^{(i)}(z), n \in N_t\} \cap \{1\}$ $i = 1,3$ are the eigenvalues and the corresponding eigenfunctions of the vertical eigenvalue problems $VE(-h_i, 0)$, $i = 1,3$ obtained by separation of variables in the half strips $D^{(1)}$ and $D^{(3)}$ (cf. § 2.2). The eigenvalues are given as the roots of the relations

$$k_n^{(i)} h_i = \frac{1}{Fr_i^2} \tan(k_n^{(i)} h_i) \quad (n \in N_t \text{ or } n \in N_f\ , i = 1,3), \tag{5a}$$

$$k_0^{(3)} h_2 = \frac{1}{Fr_3^2} \tanh(k_0^{(3)} h_2)\,, \quad \text{where} \quad Fr_i = \frac{U}{\sqrt{g h_i}} \quad i = 1,3\ ,$$

and the eigenfunctions are given by

$$Z_0^{(3)}(z) = \frac{\cosh\left(k_0^{(3)}(z+h_2)\right)}{\cosh(k_0^{(3)} h_2)}\,, \quad Z_n^{(i)}(z) = \frac{\cos\left(k_n^{(i)}(z+h_i)\right)}{\cos(k_n^{(i)} h_i)}\,, \quad (n \in N_t \text{ or } n \in N_f\ , i = 1,2). \tag{5b}$$

The correctness (completeness) of the expansions (3) and (4) follows by the theorem, which is introduced in § 2.2.

We remark that in the general representations of disturbance potential (eq. (4)), the coefficient B_{-1} represents a part of the potential difference between far upstream and downstream, due to the fluid acceleration and deceleration by the obstruction, and is essentially the so-called *blockage parameter* which is discussed by Newman (1969) in connection with channel flow. Without loss of generality we define as essential condition upstream that $B_{-1}^{(1)} = 0$, i.e. the radiation condition upstream is

$$\varphi(x,z) \to 0 \ \text{ as } x \to -\infty\,. \tag{6a}$$

and the radiation condition downstream is

$$\left|\nabla\varphi\right| < \infty \quad \text{as } x \to +\infty\,. \tag{6b}$$

Given the upstream velocity *U*, the half strip potentials $\varphi^{(1)}$ and $\varphi^{(3)}$ are uniquely determined by the means of the real coefficients, $\left\{C_n^{(1)}\right\}$ $n \in N_t$ or $n \in N_f$, and $A_{-1}^{(3)}$, $B_{-1}^{(3)}$, $A_0^{(3)}, B_0^{(3)}$, $\left\{C_n^{(3)}\right\}$ $n \in N_f$ or $\left\{C_n^{(3)}\right\}$ $n \in N_t$, respectively; see (3) and (4). Bearing this in mind, we shall occasionally use the notation:

$$\varphi^{(1)} = \varphi^{(1)}(x,z;\ \left\{C_n^{(1)}\right\}_{n\in N}),\ \ \varphi^{(3)} = \varphi^{(3)}(x,z;\ A_{-1}^{(3)}, B_{-1}^{(3)}, \left\{C_n^{(3)}\right\}_{n\in N})$$

where, the set of index N is defined with respect to the studied cases as follows: $N = \{0\} \cap N_f$ if $Fr < 1$ and $N = N_t$ if $Fr > 1$.

By exploiting the representations (3) and (4), of disturbance velocity potential the problem can be formulated as a transmission boundary-value problem in the bounded domain $D^{(2)}$, as follows:

<u>*PROBLEM*</u> $P_T(D^{(2)}, \varphi^{(2)}, A_{-1}^{(3)}, B_{-1}^{(3)}, \{C_n^{(i)}\}_{n\in N})$. *Given the upstream velocity U, and the representations (3) and (4) of the disturbance velocity potential in the semi-infinite strips* $D^{(1)}$ *and* $D^{(3)}$, *find the coefficients* $\left\{C_n^{(1)}\right\}_{n\in N}$ *and* $A_{-1}^{(3)}$, $B_{-1}^{(3)}$, $\left\{C_n^{(3)}\right\}_{n\in N}$, *and the function* $\varphi^{(2)}(x,z)$, *defined in* $D^{(2)}$, *satisfying the following system of equations, boundary and matching conditions:*

$$\nabla^2 \varphi^{(2)} = 0 \qquad (x,z) \in D^{(2)}, \tag{7a}$$

$$\varphi,_{xx}^{(2)} + \frac{g}{U^2}\varphi,_z^{(2)} = 0 \qquad (x,z) \in \partial D_F^{(2)}, \tag{7b}$$

$$\frac{\partial \varphi^{(2)}}{\partial \vec{n}^{(2)}} = -U n_x,\ (x,z) \in \partial D_B^{(2)},\ \ \frac{\partial \varphi^{(2)}}{\partial \vec{n}^{(2)}} = 0,\ (x,z) \in \partial D_\Pi^{(2)}, \tag{7c,d}$$

$$\varphi^{(2)} = \varphi^{(1)},\ \ \frac{\partial \varphi^{(2)}}{\partial \vec{n}^{(2)}} = -\frac{\partial \varphi^{(1)}}{\partial \vec{n}^{(1)}}\ (x,z) \in \partial D_I^{(12)}, \tag{7e,f}$$

$$\varphi^{(2)} = \varphi^{(3)},\ \ \frac{\partial \varphi^{(2)}}{\partial \vec{n}^{(2)}} = -\frac{\partial \varphi^{(3)}}{\partial \vec{n}^{(3)}} \quad (x,z) \in \partial D_I^{(23)}, \tag{7g,h}$$

$$\varphi^{(2)} = \varphi^{(1)},\quad \varphi,_x^{(2)} = \varphi,_x^{(1)} \qquad (x,z) = (x^U, 0), \tag{7i,j}$$

$$\varphi^{(2)} = \varphi^{(3)},\quad \varphi,_x^{(2)} = \varphi,_x^{(3)} \qquad (x,z) = (x^D, 0),\ (7k,l)$$

where $\vec{n}^{(i)} = (n_x^{(i)}, n_z^{(i)})$ *is the unit normal vector to the boundary* $\partial D^{(i)}$ *directed to the exterior of* $D^{(i)}$ $i = 1,2,3$.

4.2 Variational formulation of the problem.

Consider the functional:

$$\mathscr{F}(\varphi^{(2)},\left\{C_n^{(1)}\right\}_{n\in N},A_{-1}^{(3)},B_{-1}^{(3)},\left\{C_n^{(3)}\right\}_{n\in N})=$$

$$=\frac{1}{2}\int_{D^{(2)}}(\nabla\varphi^{(2)})^2dV-\frac{1}{2}\mu\int_{\partial D_F^{(2)}}(\varphi,_x^{(2)})^2dS+\int_{\partial D_B^{(2)}}Un_x\varphi^{(2)}dS+$$

$$+\int_{\partial D_I^{(12)}}(\varphi^{(2)}-\frac{1}{2}\varphi^{(1)}(\{C_n^{(1)}\}_{n\in N}))\frac{\partial\varphi^{(1)}(\{C_n^{(1)}\}_{n\in N})}{\partial\vec{n}^{(1)}}dS+$$

$$+\int_{\partial D_I^{(23)}}(\varphi^{(2)}-\frac{1}{2}\varphi^{(3)}(A_{-1}^{(3)},B_{-1}^{(3)},\{C_n^{(3)}\}_{n\in N}))\frac{\partial\varphi^{(3)}(A_{-1}^{(3)},B_{-1}^{(3)},\{C_n^{(3)}\}_{n\in N})}{\partial\vec{n}^{(3)}}dS+$$

$$+\mu\left[(\frac{1}{2}\varphi^{(1)}-\varphi^{(2)})\varphi,_x^{(1)}\right]_{(x,z)=(x^U,0)}+\mu\left[(\varphi^{(2)}-\frac{1}{2}\varphi^{(3)})\varphi,_x^{(3)}\right]_{(x,z)=(x^D,0)}, \tag{8}$$

where $\mu=U^2/g$.

The Variational formulation of the problem $P_T(D^{(2)},\varphi^{(2)},A_{-1}^{(3)},B_{-1}^{(3)},\{C_n^{(i)}\}_{n\in N})$ is now stated as follows:

THEOREM 1 [*The Variational principle*]. *The function* $\varphi^{(2)}(x,z)$, $(x,z)\in D^{(2)}$ *and the coefficients* $\left\{C_n^{(1)}\right\}$ *and* $A_{-1}^{(3)},B_{-1}^{(3)},\left\{C_n^{(3)}\right\}$, $(n\in N)$ *constitute a solution of the problem* $P_T(D^{(2)},\varphi^{(2)},A_{-1}^{(3)},B_{-1}^{(3)},\{C_n^{(i)}\}_{n\in N})$ *if and only if they render the functional* $\mathscr{F}$, *equation (8), stationary, i.e.*

$$\delta\mathscr{F}(\varphi^{(2)},\left\{C_n^{(1)}\right\}_{n\in N},A_{-1}^{(3)},B_{-1}^{(3)},\left\{C_n^{(3)}\right\}_{n\in N})=0. \tag{9}$$

For the proof of the above theorem the following lemmas will be used:

LEMMA 1 *Consider the functions* $\varphi(x,z)$, $g(x,z)$ *which are the solutions of the Neumann-Kelvin problem in the semi-infinite upstream region* $D^{(1)}=\left\{(x,z):\ -\infty<x\le x^U,\ -h_1<z<0\right\}$. *These functions satisfy the following relation:*

$$\int_{\partial D_I^{(12)}}(\varphi\partial_n g-g\partial_n\varphi)dS=\mu(\varphi g,_x-g\varphi,_x)\Big|_{(x,z)=(x^U,0)}, \tag{10}$$

where $\mu=U^2/g$.

Proof. The solutions $\varphi(x,z)$, $g(x,z)$ both satisfy the same boundary conditions and Laplace's equation throughout the fluid region $D^{(1)}$ (cf. problem P_M, §3.1.1), see figure 4.2a. Hence, applying the Green's formula we are obtained:

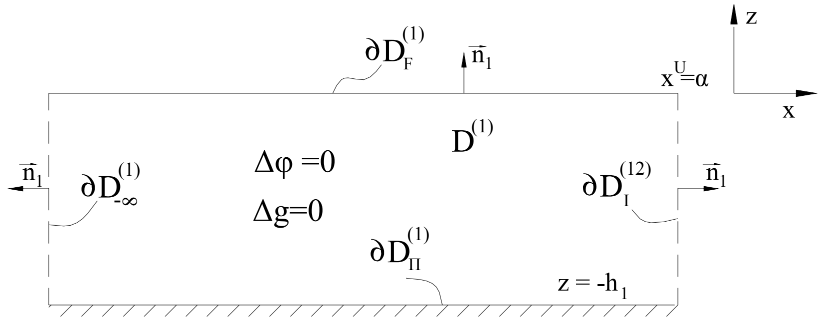

FIGURE 4.2a

$$\int_{D^{(1)}} (\varphi\Delta g - g\Delta\varphi)dV = \int_{\partial D^{(1)}} (\varphi\partial_n g - g\partial_n\varphi)dS \Rightarrow$$

$$\Rightarrow \int_{\partial D^{(1)}} (\varphi\partial_n g - g\partial_n\varphi)dS = 0\,. \tag{11a}$$

The integrals at the boundaries $\partial D_\Pi^{(1)}$ and $\partial D_{-\infty}^{(1)}$ are vanished, and this follows from the bottom boundary condition and from the fact that asymptotically as $x \to -\infty$ $|\nabla\varphi| \to 0$ and $|\nabla g| \to 0$, respectively. Using this and that the functions φ , g are satisfy the free surface boundary condition, we have:

$$\int_{\partial D_I^{(12)}} (\varphi\partial_n g - g\partial_n\varphi)dS = -\int_{\partial D_F^{(1)}} (\varphi\partial_n g - g\partial_n\varphi)dS = \mu\int_{\partial D_F^{(1)}} (\varphi g,_{xx} - g\varphi,_{xx})dS =$$

$$= \mu \int_{x=-\infty}^{x=x^U} (\varphi g,_x - g\varphi,_x),_x dS = \mu(\varphi g,_x - g\varphi,_x)\Big|_{(x,z)=(x^U,0)} . \blacksquare \tag{11b}$$

LEMMA 2 *Consider the functions* $\varphi(x,z)$, $g(x,z)$ *which are the solutions of the Neumann-Kelvin problem in the semi-infinite downstream region* $D^{(3)} = \{(x,z):\ x^D \le x < \infty,\ -h_3 < z < 0\}$. *These functions satisfy the following relation:*

$$\int_{\partial D_I^{(23)}} (\varphi\partial_n g - g\partial_n\varphi)dS = -\mu(\varphi g,_x - g\varphi,_x)\Big|_{(x,z)=(x^D,0)}, \tag{12}$$

where $\mu = U^2/g$.

Proof. The solutions $\varphi(x,z)$, $g(x,z)$ both satisfy the same boundary conditions and Laplace's equation throughout the fluid region $D^{(1)}$ (cf. problem P_M, §3.1.1), see figure 4.2b. So applying the Green's formula we are obtained:

$$\int_{D^{(3)}} (\varphi\Delta g - g\Delta\varphi)dV = \int_{\partial D^{(3)}} (\varphi\partial_n g - g\partial_n\varphi)dS \Rightarrow$$

$$\Rightarrow \int_{\partial D^{(3)}} (\varphi\partial_n g - g\partial_n\varphi)dS = 0\,. \tag{13a}$$

The integral at the boundary $\partial D_\Pi^{(3)}$ is vanished, as φ , g are satisfy the bottom b.c.

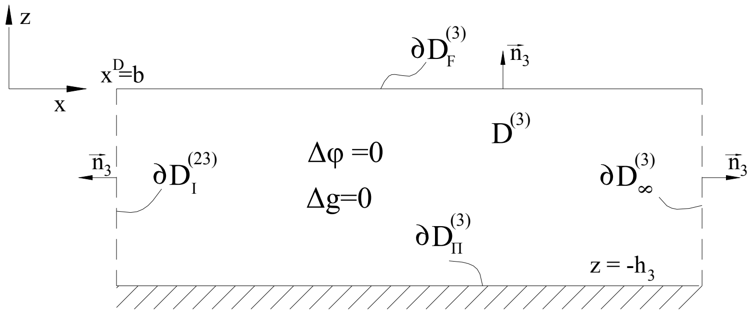

FIGURE 4.2b

Asymptotically,

$$\text{as } x\to\infty \Rightarrow \begin{cases} \varphi \to A_{-1}^{(3)}x + B_{-1}^{(3)} + A\exp(jk_0^{(3)}x) \Rightarrow \varphi,_x \to A_{-1}^{(3)} + jk_0^{(3)}A\exp(jk_0^{(3)}x) \\ g \to A_{-1}^{(3)}x + B_{-1}^{(3)} + B\exp(jk_0^{(3)}x) \Rightarrow g,_x \to A_{-1}^{(3)} + jk_0^{(3)}B\exp(jk_0^{(3)}x) \end{cases},$$

hence the integral on the boundary $\partial D_\infty^{(3)}$ is vanished:

$$\int_{\partial D_\infty^{(3)}} (\varphi\partial_n g - g\partial_n\varphi)dS = \int_{\partial D_\infty^{(3)}} (\varphi jk_0^{(3)} g - gjk_0^{(3)}\varphi)dS = 0\,.$$

According to the above definitions as well as that the functions φ, g are satisfy the free surface boundary condition, we have:

$$\int_{\partial D_I^{(23)}} (\varphi\partial_n g - g\partial_n\varphi)dS = -\int_{\partial D_F^{(3)}} (\varphi\partial_n g - g\partial_n\varphi)dS = \mu\int_{\partial D_F^{(3)}} (\varphi g,_{xx} - g\varphi,_{xx})dS =$$

$$= \mu\int_{x=x^D}^{\infty} (\varphi g,_x - g\varphi,_x),_x dS = -\mu(\varphi g,_x - g\varphi,_x)\Big|_{(x,z)=(x^D,0)} . \blacksquare \tag{13b}$$

Proof of the Theorem 1.

By calculating the first variation $\delta\mathscr{F}$ of the functional we obtain:

$$\begin{aligned}
\delta\mathscr{F} &= \mathscr{F}(\varphi+\delta\varphi) - \mathscr{F}(\varphi) = \\
&= \int_{D^{(2)}} \nabla\varphi^{(2)}\nabla\delta\varphi^{(2)}dV - \mu\int_{\partial D_F^{(2)}} \varphi,_x^{(2)} \frac{\partial}{\partial x}(\delta\varphi^{(2)})dS + \int_{\partial D_B^{(2)}} Un_x\delta\varphi^{(2)}dS + \\
&+ \int_{\partial D_I^{(12)}} \delta\varphi^{(2)}\frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}}dS + \int_{\partial D_I^{(12)}} (\varphi^{(2)} - \varphi^{(1)})\cdot\delta(\frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}})dS + \\
&+ \frac{1}{2}\mu\left[\varphi^{(1)}\delta\varphi,_x^{(1)} - \delta\varphi^{(1)}\varphi,_x^{(1)}\right]_{(x^U,0)} + \int_{\partial D_I^{(23)}} \delta\varphi^{(2)}\frac{\partial\varphi^{(3)}}{\partial\vec{n}^{(3)}}dS + \\
&+ \int_{\partial D_I^{(23)}} (\varphi^{(2)} - \varphi^{(3)})\cdot\delta(\frac{\partial\varphi^{(3)}}{\partial\vec{n}^{(3)}})dS - \frac{1}{2}\mu\left[\varphi^{(3)}\delta\varphi,_x^{(3)} - \delta\varphi^{(3)}\varphi,_x^{(3)}\right]_{(x^D,0)} + \\
&+ \mu\left[\frac{1}{2}\varphi,_x^{(1)}\delta\varphi^{(1)} + \frac{1}{2}\varphi^{(1)}\delta\varphi,_x^{(1)} - \varphi,_x^{(1)}\delta\varphi^{(2)} - \varphi^{(2)}\delta\varphi,_x^{(1)}\right]_{(x^U,0)} + \\
&+ \mu\left[\varphi,_x^{(3)}\delta\varphi^{(2)} + \varphi^{(2)}\delta\varphi,_x^{(3)} - \frac{1}{2}\varphi,_x^{(3)}\delta\varphi^{(3)} - \frac{1}{2}\varphi^{(3)}\delta\varphi,_x^{(3)}\right]_{(x^D,0)} \Leftrightarrow
\end{aligned}$$

$$\delta\mathscr{F} = \int_{D^{(2)}} \nabla\varphi^{(2)}\nabla\delta\varphi^{(2)}dV - \mu\int_{\partial D_F^{(2)}} \varphi,_x^{(2)} \frac{\partial}{\partial x}(\delta\varphi^{(2)})dS + \int_{\partial D_B^{(2)}} Un_x\delta\varphi^{(2)}dS +$$

$$+\int_{\partial D_I^{(12)}} \delta\varphi^{(2)} \frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}}dS + \int_{\partial D_I^{(12)}} (\varphi^{(2)} - \varphi^{(1)})\cdot\delta(\frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}})dS + \int_{\partial D_I^{(23)}} \delta\varphi^{(2)} \frac{\partial\varphi^{(3)}}{\partial\vec{n}^{(3)}}dS$$

$$+\int_{\partial D_I^{(23)}} (\varphi^{(2)} - \varphi^{(3)})\cdot\delta(\frac{\partial\varphi^{(3)}}{\partial\vec{n}^{(3)}})dS + \mu\left[(\varphi^{(1)} - \varphi^{(2)})\delta\varphi,_x^{(1)}\right]_{(x^U,0)} - \mu\left[\varphi_x^{(1)}\delta\varphi^{(2)}\right]_{(x^U,0)}$$

$$+\mu\left[(\varphi^{(2)} - \varphi^{(3)})\delta\varphi,_x^{(3)}\right]_{(x^D,0)} + \mu\left[\varphi_x^{(3)}\delta\varphi^{(2)}\right]_{(x^D,0)}. \tag{14}$$

Applying Green's formula to the first integral and partial integration to the second integral of relation (14), the variational equation (9) takes the following form:

$$\delta\mathscr{F} = 0 \Leftrightarrow$$

$$-\int_{D^{(2)}} \left(\nabla^2\varphi^{(2)}\right)\delta\varphi^{(2)}dV + \int_{\partial D_\Pi^{(2)}} \left(\frac{\partial\varphi^{(2)}}{\partial\vec{n}^{(2)}}\right)\delta\varphi^{(2)}dS + \int_{\partial D_F^{(2)}} \left(\frac{\partial\varphi^{(2)}}{\partial\vec{n}^{(2)}} + \mu\varphi,_{xx}^{(2)}\right)\delta\varphi^{(2)}dS$$

$$+\int_{\partial D_B^{(2)}} \left(\frac{\partial\varphi^{(2)}}{\partial\vec{n}^{(2)}} + Un_x\right)\delta\varphi^{(2)}dS + \int_{\partial D_I^{(12)}} (\varphi^{(2)} - \varphi^{(1)})\cdot\delta(\frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}})dS + \int_{\partial D_I^{(12)}} \left(\frac{\partial\varphi^{(2)}}{\partial\vec{n}^{(2)}} + \frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}}\right)\delta\varphi^{(2)}dS$$

$$+\int_{\partial D_I^{(23)}} (\varphi^{(2)} - \varphi^{(3)})\cdot\delta(\frac{\partial\varphi^{(3)}}{\partial\vec{n}^{(3)}})dS + \int_{\partial D_I^{(23)}} \left(\frac{\partial\varphi^{(2)}}{\partial\vec{n}^{(2)}} + \frac{\partial\varphi^{(3)}}{\partial\vec{n}^{(3)}}\right)\delta\varphi^{(2)}dS + \mu\left[(\varphi^{(1)} - \varphi^{(2)})\delta\varphi,_x^{(1)}\right]_{(x^U,0)}$$

$$+\mu\left[(\varphi,_x^{(2)} - \varphi,_x^{(1)})\delta\varphi^{(2)}\right]_{(x^U,0)} + \mu\left[(\varphi^{(2)} - \varphi^{(3)})\delta\varphi,_x^{(3)}\right]_{(x^D,0)} + \mu\left[(\varphi,_x^{(3)} - \varphi,_x^{(2)})\delta\varphi^{(2)}\right]_{(x^D,0)} = 0. \tag{15}$$

The two functions $\partial\varphi^{(i)}/\partial n^{(i)}$, $i = 1,3$, appearing in (15), are considered to be represented by means of their series expansions obtained by differentiating (3) and (4). Consequently, the variations $\delta(\partial\varphi^{(1)}/\partial n^{(1)})$ and $\delta(\partial\varphi^{(3)}/\partial n^{(3)})$ are, eventually, expressed in terms of the variations of the coefficients $\left\{\delta C_n^{(1)}\right\}_{n\in N}$ and $\delta A_{-1}^{(3)}, \delta B_{-1}^{(3)}, \left\{\delta C_n^{(3)}\right\}_{n\in N}$, respectively.

The proof of the equivalence of the variational equation (15) and the problem $P_T(D^{(2)}, \varphi^{(2)}, A_{-1}^{(3)}, B_{-1}^{(3)}, \{C_n^{(i)}\}_{n\in N})$ is completed by using standard arguments of calculus of variations (see e.g. Gelfant & Fomin 1963).

In order for $\delta\mathscr{F} = 0$ for arbitrary variations of the potential field $\delta\varphi^{(2)}$ in $D^{(2)}$, $\delta\varphi^{(2)}$ on $\partial D^{(2)}$ and of the coefficients of the upstream and downstream expansions $\delta C_n^{(1)}$, $\delta A_{-1}^{(3)}$, $\delta B_{-1}^{(3)}$, $\delta C_n^{(3)}$ ($n \in N$), -in conjunction with the fact that the systems $\left\{Z_n^{(i)}(z)\right\}_{n\in N}$ are complete in the intervals $(-h_i, 0)$ $i = 1,3$, respectively- it is both necessary and sufficient that Laplace's equation to be satisfied as the *Euler-Lagrange* equation, while all the other boundary conditions of the problem must be satisfied as *natural conditions*. Thus the stationary of $\mathscr{F}$ is equivalent to the problem P_T. ■

Apart from its theoretical interest, the usefulness of the above variational principle hinges on the fact that it leaves us the freedom to choose any particular representation for the unknown potential $\varphi^{(2)}$ in $D^{(2)}$. In this way, a variety of possible algorithms for the numerical solution of the problem can be constructed.

4.3 The Finite Element approximation.

As the basis of the present numerical procedure, Galerkin's method will be used. In order to use the standard Galerkin method in the finite subdomain $D^{(2)}$, we will apply the Green's theorem and partial integration to lower the continuity requirements for the disturbance potential $\varphi^{(2)}$. Moreover, applying Green's theorem and partial integration in the finite subdomain $D^{(2)}$, the variational equation (15) takes the following form, which is the *weak formulation* of the problem P_T (cf. §4.1).

$$\int_{D^{(2)}} \nabla\varphi^{(2)}\nabla\delta\varphi^{(2)} dV - \mu\int_{\partial D_F^{(2)}} \varphi,_x^{(2)} \frac{\partial}{\partial x}(\delta\varphi^{(2)}) dS + \int_{\partial D_B^{(2)}} U n_x \delta\varphi^{(2)} dS +$$

$$+\int_{\partial D_I^{(12)}} \delta\varphi^{(2)} \frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}} dS + \int_{\partial D_I^{(12)}} (\varphi^{(2)} - \varphi^{(1)}) \cdot \delta\left(\frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}}\right) dS + \int_{\partial D_I^{(23)}} \delta\varphi^{(2)} \frac{\partial\varphi^{(3)}}{\partial\vec{n}^{(3)}} dS$$

$$+\int_{\partial D_I^{(23)}} (\varphi^{(2)} - \varphi^{(3)}) \cdot \delta\left(\frac{\partial\varphi^{(3)}}{\partial\vec{n}^{(3)}}\right) dS + \mu\left[(\varphi^{(1)} - \varphi^{(2)})\delta\varphi,_x^{(1)}\right]_{(x^U,0)} - \mu\left[\varphi_x^{(1)}\delta\varphi^{(2)}\right]_{(x^U,0)}$$

$$+\mu\left[(\varphi^{(2)} - \varphi^{(3)})\delta\varphi,_x^{(3)}\right]_{(x^D,0)} + \mu\left[\varphi_x^{(3)}\delta\varphi^{(2)}\right]_{(x^D,0)} = 0 . \tag{16}$$

The Galerkin method is based on the following representation of the unknown potential $\varphi^{(2)}$:

$$\varphi^{(2)}(x,z) = \sum_{j=1}^{+\infty} \hat{\varphi}_j^{(2)} \beta_j(x,z) \tag{17}$$

where $\beta_j(x,z)$, $j = 1,2,3,...$ are linear independent functions satisfying $span\left\{\beta_j\right\}_{j=1}^{+\infty} = V$, and V the appropriate function space where the problem on $\varphi^{(2)}$ is posed.

4.3.1 The F.E.M. system of equations.

Let us consider the variational equation (16), assuming that the unknown potential $\varphi^{(2)}$ is based on the above representation (17). The use of a different (equivalent) set of degrees of freedom of the system in the variational equation (16) leads to a different (equivalent) set of equations for the same problem P_T. Without loss of generality the system of equations for the subcritical case $(Fr < 1)$ is obtained.

Hence, by assuming that all variation except $\delta\varphi^{(2)}(x,z)$ in $D^{(2)} \bigcup \partial D_F^{(2)} \bigcup \partial D_B^{(2)}$ are kept zero, the variational equation (16) becomes:

$$\int_{D^{(2)}} \nabla\varphi^{(2)}\nabla\delta\varphi^{(2)}dV - \mu\int_{\partial D_F^{(2)}} \varphi,_x^{(2)} \frac{\partial}{\partial x}(\delta\varphi^{(2)})dS + \int_{\partial D_B^{(2)}} Un_x\delta\varphi^{(2)}dS = 0\,. \tag{18a}$$

By introducing in the above equation the representation (17) for $\varphi^{(2)}$, we obtain the following variational equation:

$$\sum_{i=1}^{+\infty}\delta\hat{\varphi}_i^{(2)}\left\{\sum_{j=1}^{+\infty}\hat{\varphi}_j^{(2)}\left(\int_{D^{(2)}}\nabla\beta_i\nabla\beta_j dV - \mu\int_{\partial D_F^{(2)}}\frac{\partial\beta_i}{\partial x}\frac{\partial\beta_j}{\partial x}dS\right) + \int_{\partial D_B^{(2)}} Un_x\beta_i dS\right\} = 0\,. \tag{18b}$$

Since the variations $\delta\varphi^{(2)}(x,z) = \sum_{i=1}^{+\infty}\delta\hat{\varphi}_i^{(2)}\beta_i(x,z)$ is arbitrary, it follows that the equation (18b) holds, only if the term in brackets is zero, i.e. the variational equation is equivalent to the following infinite system of linear equations:

$$\sum_{j=1}^{+\infty}\hat{\varphi}_j^{(2)}\left(\int_{D^{(2)}}\nabla\beta_i\nabla\beta_j dV - \mu\int_{\partial D_F^{(2)}}\frac{\partial\beta_i}{\partial x}\frac{\partial\beta_j}{\partial x}dS\right) = -\int_{\partial D_B^{(2)}} Un_x\beta_i dS,\ i = 1,2,3,... \tag{18c}$$

We denote that the term appearing in the right-hand side of the above system of equations is the forcing of the system.

We now continue by the implementation of the matching conditions on the upstream interface $\partial D_I^{(12)}$ and on the point $(x^U,0)$. We recall that these conditions are the following:

$$\int_{\partial D_I^{(12)}}\delta\varphi^{(2)}\frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}}dS,\ \ \int_{\partial D_I^{(12)}}(\varphi^{(2)}-\varphi^{(1)})\cdot\delta(\frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}})dS,$$

$$\mu\left[(\varphi^{(1)}-\varphi^{(2)})\delta\varphi,_x^{(1)}\right]_{(x^U,0)},\ \ -\mu\left[\varphi_x^{(1)}\delta\varphi^{(2)}\right]_{(x^U,0)}.$$

By assuming that all variation except $\delta\varphi^{(2)}$ in $\partial D_I^{(12)}$ are kept zero the variational equation (16) becomes:

$$\int_{\partial D_I^{(12)}}\delta\varphi^{(2)}\frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}}dS = 0\,. \tag{19a}$$

Expressing the normal derivative $\partial\varphi^{(1)}\big/\partial\vec{n}^{(1)}$ of the disturbance potential by means of the termwise differentiated series (3a), and using the representation (17) for the potential $\varphi^{(2)}$, equation (19a) takes the form:

$$\sum_{i=1}^{+\infty}\delta\hat{\varphi}_i^{(2)}\left\{\sum_{n=1}^{+\infty}k_n^{(1)}C_n^{(1)}\int_{-h_1}^{0}Z_n^{(1)}(z)\beta_i(x^U,z)dz\right\}=0\,. \tag{19b}$$

Since the variations $\delta\varphi^{(2)}$ is arbitrary, the equation (19b) is equivalent to the following infinite system of linear equations:

$$\sum_{n=1}^{+\infty}k_n^{(1)}C_n^{(1)}\int_{-h_1}^{0}Z_n^{(1)}(z)\beta_i(x^U,z)dz=0,\quad i=1,2,3... \tag{19c}$$

Similarly, by assuming that all variation except $\delta(\dfrac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}})$ in $\partial D_I^{(12)}$ are kept zero the variational equation (16) becomes:

$$\int_{\partial D_I^{(12)}}(\varphi^{(2)}-\varphi^{(1)})\cdot\delta(\frac{\partial\varphi^{(1)}}{\partial\vec{n}^{(1)}})dS\,. \tag{20a}$$

By substituting equations (3a) and (17) into equation (20a), we obtain the equation:

$$\sum_{m=1}^{+\infty}\delta C_m^{(1)}\left\{k_m^{(1)}\sum_{j=1}^{+\infty}\hat{\varphi}_j^{(2)}\int_{-h_1}^{0}Z_m^{(1)}(z)\beta_j(x^U,z)dz-k_m^{(1)}\sum_{n=1}^{+\infty}C_n^{(1)}\int_{-h_1}^{0}Z_n^{(1)}(z)Z_m^{(1)}(z)dz\right\}=0\,. \tag{20b}$$

Since the variations $\delta C_m^{(1)}$ is arbitrary, the equation (20b) is equivalent to the following set of linear equations at $x=x^U$:

$$k_m^{(1)}\sum_{j=1}^{+\infty}\hat{\varphi}_j^{(2)}\int_{-h_1}^{0}Z_m^{(1)}(z)\beta_j(x^U,z)dz-k_m^{(1)}\sum_{n=1}^{+\infty}C_n^{(1)}\int_{-h_1}^{0}Z_n^{(1)}(z)Z_m^{(1)}(z)dz=0,m=1,2,3,... \tag{20c}$$

Further, by assuming that all variation except $\delta\varphi^{(2)}$ in the point $(x^U,0)$ are kept zero the variational equation (16) becomes:

$$-\mu\left[\varphi_x^{(1)}\delta\varphi^{(2)}\right]_{(x^U,0)}=0\,. \tag{21a}$$

By substituting equations (3a) and (17) into equation (21a), and using that $Z_n^{(1)}(0)=1$, we obtain the equation:

$$\delta\hat{\varphi}_{P_1}^{(2)}\left(-\mu\sum_{n=1}^{+\infty}k_n^{(1)}C_n^{(1)}\beta_{P_1}(x^U,0)\right)=0\,, \tag{21b}$$

where $\delta\hat{\varphi}_{P_1}^{(2)}$ denotes the arbitrary variation of $\varphi^{(2)}$ at the point $(x^U,0)$. Thus, for arbitrary variation of $\delta\hat{\varphi}_{P_1}^{(2)}$, we obtain the following equation:

$$-\mu\sum_{n=1}^{+\infty}k_n^{(1)}C_n^{(1)}\beta_{P_1}(x^U,0)=0\,,\quad i=P_1\,. \tag{21c}$$

Similarly, by assuming that all variation except $\delta\varphi,_x^{(1)}$ in the point $(x^U,0)$ are kept zero the variational equation (16) becomes:

$$\mu\left[(\varphi^{(1)}-\varphi^{(2)})\delta\varphi,_x^{(1)}\right]_{(x^U,0)}=0\,. \tag{22a}$$

By substituting equations (3a) and (17) into equation (22a), and using that $Z_n^{(1)}(0)=1$, we obtain the equation:

$$\sum_{m=1}^{+\infty}\delta C_m^{(1)}\left\{\mu\left(k_m^{(1)}\sum_{n=1}^{+\infty}C_n^{(1)}-k_m^{(1)}\hat{\varphi}_{\mathrm{P}_1}^{(2)}\beta_{\mathrm{P}_1}(x^U,0)\right)\right\}=0\,. \tag{22b}$$

Thus, for arbitrary variations of $\delta C_m^{(1)}$, we obtain the following set of equations:

$$\mu\left(k_m^{(1)}\sum_{n=1}^{+\infty}C_n^{(1)}-k_m^{(1)}\hat{\varphi}_{\mathrm{P}_1}^{(2)}\beta_{\mathrm{P}_1}(x^U,0)\right)=0,\quad m=1,2,3,... \tag{22c}$$

Working similarly with the terms of variational equation (16) defined on downstream interface $\partial D_I^{(23)}$ and on $(x^D,0)$, we derive the following set of conditions at $x=x^D$:

$$-A_{-1}^{(3)}\int_{-h_3}^{0}\beta_i(x^D,z)dz-k_0^{(3)}\left(-A_0^{(3)}\sin(k_0^{(3)}x^D)+B_0^{(3)}\cos(k_0^{(3)}x^D)\right)\int_{-h_3}^{0}Z_0^{(3)}(z)\beta_i(x^D,z)dz+$$
$$+\sum_{n=1}^{+\infty}k_n^{(3)}C_n^{(3)}\int_{-h_3}^{0}Z_n^{(3)}(z)\beta_i(x^D,z)dz=0,\quad i=1,2,3... \tag{23}$$

$$c_m(A_{-1}^{(3)}x^D+B_{-1}^{(3)})k_m^{(3)}\int_{-h_3}^{0}Z_m^{(3)}(z)dz+c_mk_m^{(3)}\left(A_0^{(3)}\cos(k_0^{(3)}x^D)+B_0^{(3)}\sin(k_0^{(3)}x^D)\right)\int_{-h_3}^{0}Z_0^{(3)}(z)Z_m^{(3)}(z)dz+$$
$$+c_mk_m^{(3)}\sum_{n=1}^{+\infty}C_n^{(3)}\int_{-h_3}^{0}Z_n^{(3)}(z)Z_m^{(3)}(z)dz-\sum_{j=1}^{+\infty}\hat{\varphi}_{\mathrm{j}}^{(2)}\left\{\ell_mk_m^{(3)}\int_{-h_3}^{0}\beta_i(x^D,z)Z_m^{(3)}(z)dz\right\}=0,m=-1,0,1,2...$$

where $c_m=-1$, for $m=1,2,...$, $c_m=1$, for $m=-1,0$ and $\ell_m=\left(-\sin(k_0^{(3)}x^D)+\cos(k_0^{(3)}x^D)\right)$, for $m=0$, $\ell_m=1$, for $m=-1,1,2...$ and also we define $k_{-1}^{(3)}=1$. (24)

$$\mu\left(A_{-1}^{(3)}+k_0^{(3)}\left(-A_0^{(3)}\sin(k_0^{(3)}x^D)+B_0^{(3)}\cos(k_0^{(3)}x^D)\right)-\sum_{n=1}^{+\infty}k_n^{(3)}C_n^{(3)}\right)\beta_{\mathrm{P}_2}(x^D,0)=0\,,$$
$$i=\mathrm{P}_2\,. \tag{25}$$

$$\mu\left\{\left(c_m(A_{-1}^{(3)}x^D+B_{-1}^{(3)})k_m^{(3)}+c_mk_m^{(3)}\left(A_0^{(3)}\cos(k_0^{(3)}x^D)+B_0^{(3)}\sin(k_0^{(3)}x^D)\right)+c_mk_m^{(3)}\sum_{n=1}^{+\infty}C_n^{(3)}\right)-\right.$$
$$\left.-\hat{\varphi}_{\mathrm{P}_2}^{(2)}\left(1+k_0^{(3)}\left(-\sin(k_0^{(3)}x^D)+\cos(k_0^{(3)}x^D)\right)-k_m^{(3)}\right)\beta_{\mathrm{P}_2}(x^D,0)\right\}=0\,,\ m=-1,0,1,2... \tag{26}$$

Recapitulating the above results we can state the following theorem:

THEOREM 2 [*The F.E.M. system*]. *The variational equation* (9) *and, thus, the problem* $P_T(D^{(2)},\varphi^{(2)},A_{-1}^{(3)},B_{-1}^{(3)},\{C_n^{(i)}\}_{n\in N})$, *are equivalent to the following system of equations, boundary and matching conditions:*

$$\sum_{j=1}^{+\infty}\hat{\varphi}_j^{(2)}\left(\int_{D^{(2)}}\nabla\beta_i\nabla\beta_j dV-\mu\int_{\partial D_F^{(2)}}\frac{\partial\beta_i}{\partial x}\frac{\partial\beta_j}{\partial x}dS\right)=-\int_{\partial D_B^{(2)}}Un_x\beta_i dS,\ i=1,2,3,... \tag{27a}$$

$$\sum_{n=1}^{+\infty}k_n^{(1)}C_n^{(1)}\int_{-h_1}^{0}Z_n^{(1)}(z)\beta_i(x^U,z)dz=0,\quad i=1,2,3... \tag{27b}$$

$$k_m^{(1)}\sum_{j=1}^{+\infty}\hat{\varphi}_j^{(2)}\int_{-h_1}^{0}Z_m^{(1)}(z)\beta_j(x^U,z)dz-k_m^{(1)}\sum_{n=1}^{+\infty}C_n^{(1)}\int_{-h_1}^{0}Z_n^{(1)}(z)Z_m^{(1)}(z)dz=0,m=1,2,3,... \tag{27c}$$

$$-A_{-1}^{(3)}\int_{-h_3}^{0}\beta_i(x^D,z)dz-k_0^{(3)}\left(-A_0^{(3)}\sin(k_0^{(3)}x^D)+B_0^{(3)}\cos(k_0^{(3)}x^D)\right)\int_{-h_3}^{0}Z_0^{(3)}(z)\beta_i(x^D,z)dz+$$
$$+\sum_{n=1}^{+\infty}k_n^{(3)}C_n^{(3)}\int_{-h_3}^{0}Z_n^{(3)}(z)\beta_i(x^D,z)dz=0,\quad i=1,2,3... \tag{27d}$$

$$c_m(A_{-1}^{(3)}x^D+B_{-1}^{(3)})k_m^{(3)}\int_{-h_3}^{0}Z_m^{(3)}(z)dz+c_mk_m^{(3)}\left(A_0^{(3)}\cos(k_0^{(3)}x^D)+B_0^{(3)}\sin(k_0^{(3)}x^D)\right)\int_{-h_3}^{0}Z_0^{(3)}(z)Z_m^{(3)}(z)dz+$$
$$+c_mk_m^{(3)}\sum_{n=1}^{+\infty}C_n^{(3)}\int_{-h_3}^{0}Z_n^{(3)}(z)Z_m^{(3)}(z)dz-\sum_{j=1}^{+\infty}\hat{\varphi}_j^{(2)}\left\{\ell_mk_m^{(3)}\int_{-h_3}^{0}\beta_i(x^D,z)Z_m^{(3)}(z)dz\right\}=0,$$
$$m=-1,0,1,2.. \tag{27e}$$

$$-\mu\sum_{n=1}^{+\infty}k_n^{(1)}C_n^{(1)}\beta_{P_1}(x^U,0)=0,\quad i=P_1. \tag{27f}$$

$$\mu\left(k_m^{(1)}\sum_{n=1}^{+\infty}C_n^{(1)}-k_m^{(1)}\hat{\varphi}_{P_1}^{(2)}\beta_{P_1}(x^U,0)\right)=0,\quad m=1,2,3,... \tag{27g}$$

$$\mu\left(A_{-1}^{(3)}+k_0^{(3)}\left(-A_0^{(3)}\sin(k_0^{(3)}x^D)+B_0^{(3)}\cos(k_0^{(3)}x^D)\right)-\sum_{n=1}^{+\infty}k_n^{(3)}C_n^{(3)}\right)\beta_{P_2}(x^D,0)=0,$$
$$i=P_2. \tag{27h}$$

$$\mu\Bigg\{\left(c_m(A_{-1}^{(3)}x^D+B_{-1}^{(3)})k_m^{(3)}+c_mk_m^{(3)}\left(A_0^{(3)}\cos(k_0^{(3)}x^D)+B_0^{(3)}\sin(k_0^{(3)}x^D)\right)+c_mk_m^{(3)}\sum_{n=1}^{+\infty}C_n^{(3)}\right)-$$
$$-\hat{\varphi}_{P_2}^{(2)}\left(1+k_0^{(3)}\left(-\sin(k_0^{(3)}x^D)+\cos(k_0^{(3)}x^D)\right)-k_m^{(3)}\right)\beta_{P_2}(x^D,0)\Bigg\}=0,\ m=-1,0,1,2... \tag{27i}$$

4.3.2 Construction of the solution matrix.

In this subsection, a discrete scheme for the numerical solution of the F.E.M. system of equations (27), is introduced.

Truncating the series (3), (4) of the disturbance potential in the semi-infinite strips $D^{(1)}$ and $D^{(3)}$, to a finite number of terms (modes), –denoting by N_m the number of evanescent modes retained– and the series (17) of the disturbance potential in the finite subdomain $D^{(2)}$, to a finite number of basis functions –denoting by N_{ne} the number of basis functions in the Galerkin method retained– the infinite system of equations (27) reduced to a finite system. Hence, the following approximation of the disturbance potential in $D^{(2)}$ is obtained:

$$\varphi^{(2)}(x,z)=\sum_{j=1}^{N_{ne}}\hat{\varphi}_j^{(2)}\beta_j(x,z)\,. \qquad (28)$$

The region $D^{(2)}$ is divided into triangles to form a network with N_{ne} nodes. Let us introduced $\Im_h$ be a triangulation of $D^{(2)}$ with triangles τ, whose interior belongs to $D^{(2)}$. We define the finite element space:

$$V_h=\left\{\beta(x,z)\in C(\overline{D}^{(2)}),\ \beta\big|_\tau\in \mathrm{P}_1,\forall\,\tau\in\Im_h\right\}, \qquad (29)$$

where, by $\left\{\beta_1,\beta_2,...,\beta_{N_{ne}}\right\}$ we denote its usual finite element basis, where the bases functions β_j –polynomials of first degree (P_1)– and the nodes α_i, $i,j=1,2,...,N_{ne}$, of the triangulation $\Im_h$ are such that $\beta_j(a_i)=\delta_{ij}$, with δ_{ij} as the Kronecker symbol.

According to the above definitions the F.E.M. system of equations can be expressed in the following matrix form:

Eq.(27a) $\Rightarrow [KS]\left\{\hat{\varphi}_j^{(2)}\right\}=\{Fs\}$.

The matrix $[KS]$ is sparse and has dimension $N_{ne}\times N_{ne}$.

The vector $\{Fs\}^T=\left\{\{0\}^T,\{fs\}^T\right\}$ $(1\times N_{ne})$, is the forcing of the system arising by localized disturbances (i.e. submerged bodies, bottom topography). If the total number of nodes on $\partial D_B^{(2)}$ is N_b, the dimension of the forcing sub-vector $\{fs\}$ is $N_b\times 1$.

Eq.(27b) $\Rightarrow [KU1]\left\{C_n^{(1)}\right\}=\{0\}$.

The matrix $[KU1]=\big[[Ku1],[0]\big]$ has dimension $\left(N_m+2\right)\times N_{ne}$. If the total number of nodes on $\partial D_I^{(12)}$ is N_U, the dimension of the sub-matrix $[Ku1]$ is $\left(N_m+2\right)\times N_U$.

Eq.(27c) $\Rightarrow [KU2]\left\{\hat{\varphi}_j^{(2)}\right\}=\{0\}$ and $[KU]\left\{C_n^{(1)}\right\}=\{0\}$.

The matrix $[KU]$ has dimension $\left(N_m+2\right)\times N_m$ and the matrix $[KU2]=\big[[Ku2],[0]\big]$ has dimension $N_{ne}\times N_m$. The dimension of the sub-matrix $[Ku2]$ is $N_U\times N_m$.

Eq.(27d) $\Rightarrow [KD1]\left\{C_n^{(3)}\right\}=\{0\}$.

The matrix $\left[KD1\right]=\left[\left[Kd1\right],\left[0\right]\right]$ has dimension $\left(N_m+2\right)\times N_{ne}$. If the total number of nodes on $\partial D_I^{(23)}$ is N_D, the dimension of the sub-matrix $[Kd1]$ is $\left(N_m+2\right)\times N_D$.

Eq.(27e) $\Rightarrow \left[KD2\right]\left\{\hat{\varphi}_j^{(2)}\right\}=\left\{0\right\}$ and $\left[KD\right]\left\{C_n^{(3)}\right\}=\left\{0\right\}$.

The matrix $\left[KD\right]$ has dimension $(N_m+2)\times(N_m+4)$ and the matrix $\left[KD2\right]=\left[\left[Kd2\right],\left[0\right]\right]$ has dimension $N_{ne}\times\left(N_m+4\right)$. The dimension of the sub-matrix $[Kd2]$ is $N_D\times\left(N_m+4\right)$.

The equations (27f)-(27i) are added to the corresponding positions of the above matrices, i.e. eq.(27f) is added in P_1 row of $[Ku1]$.

Assembling the above equations we obtain :

$$\left[KK\right]\left\{\hat{\varphi}\right\}=\left\{FS\right\} \tag{30}$$

where the transpose of the unknown column vector $\left\{\hat{\varphi}\right\}$ is arranged as follows

$$\left\{\hat{\varphi}\right\}^T=\left\{\left\{C_1^{(1)},...,C_{N_m}^{(1)}\right\}^T,\left\{\hat{\varphi}_1^{(2)},...,\hat{\varphi}_{Nne}^{(2)}\right\}^T,\left\{A_{-1}^{(3)},B_{-1}^{(3)},A_0^{(3)},B_0^{(3)},C_1^{(3)},...,C_{N_m}^{(3)}\right\}^T\right\}, \tag{31}$$

and the forcing vector $\left\{FS\right\}$ has non zero entries only for the nodes on $\partial D_B^{(2)}$.

The global stiffness matrix $\left[KK\right]$ has the following form:

$$\left[\begin{array}{c|c|c} \left[KU\right]_{((Nm+2)\times Nm)} & \left[KU1\right]_{((Nm+2)\times Nne)} & \left[0\right]_{((Nm+2)\times(Nm+4))} \\ \hline \left[KU2\right]_{(Nne\times Nm)} & \left[KS\right]_{(Nne\times Nne)} & \left[KD2\right]_{(Nne\times(Nm+4))} \\ \hline \left[0\right]_{((Nm+2)\times Nm)} & \left[KD1\right]_{((Nm+2)\times Nne)} & \left[KD\right]_{((Nm+2)\times(Nm+4))} \end{array}\right]. \tag{32}$$

The dimension of the global matrix $\left[KK\right]$ is $\left(2\cdot\left(N_m+2\right)+N_{ne}\right)\times\left(2\cdot\left(N_m+2\right)+N_{ne}\right)$, and the number of unknowns is $\#2\cdot\left(N_m+2\right)+N_{ne}$.

4.4 Numerical results and discussion.

In this paragraph a detailed presentation of the numerical results obtained, according to the hybrid Finite Element Method. The numerical calculations were performed for various topographies (and/or submerged bodies), and for a wide range of upstream Froude numbers when the flow is subcritical or supercritical. The source code was written in Matlab® v.6.1. The triangulation based on the Delaunay algorithm (see fig. 4.13) and the mesh size is determined from the shape of geometry.

The physical investigation of the influence of the size of obstruction and of Froude number – the physical parameters of the problem – to the numerical solutions of the problem is introduced, in order to define the range of its values for which the linear solution is valid. The numerical accuracy of the problem is interpreted to satisfaction of the matching conditions at the artificial interfaces, i.e. the continuity of the pressure field and the continuity of the velocity field.

4.4.1 Flow over shoaling bottom.

The case of a smooth underwater shoaling is introduced. The environment is characterized by the following depth function

$$h(x)=\begin{cases} h_1 = 1m \ \ (x<a=0) \\ \dfrac{1}{2}(h_1+h_3)-\dfrac{1}{2}(h_1+h_3)\tanh\left(4\pi\left(\dfrac{x}{b}-\dfrac{1}{2}\right)\right) \\ h_3 = 0.8m \ \ (x>b=4) \end{cases} \tag{1}$$

1. *Subcritical flows* $(Fr<1)$

In figure 4.3a the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model, for $Fr=0.6$. In figures 4.3b-c and 4.3.d-e the continuity of the disturbance potential field φ (C^0- continuity) and the continuity of the disturbance velocity field $\varphi,_x$ (C^1- continuity), at $x=a(=0)$ and $x=b$, respectively, have been plotted. One observes that the free surface elevation has similar form with analytical solution obtained in §3.1.4. (i.e the level of the free surface falls as it approaches the shoaling bottom while downstream a periodic steady wave motion is predicted).

2. *Supercritical flows* $(Fr>1)$

In figure 4.4 the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model, for $Fr=1.5$. One observes that the free surface elevation has similar form with analytical solution obtained in §3.1.4. (i.e. the level of the free surface rises monotonically as it approaches the shoaling bottom, the slope of the surface becoming more gradual, until far downstream is asymptotically flat).

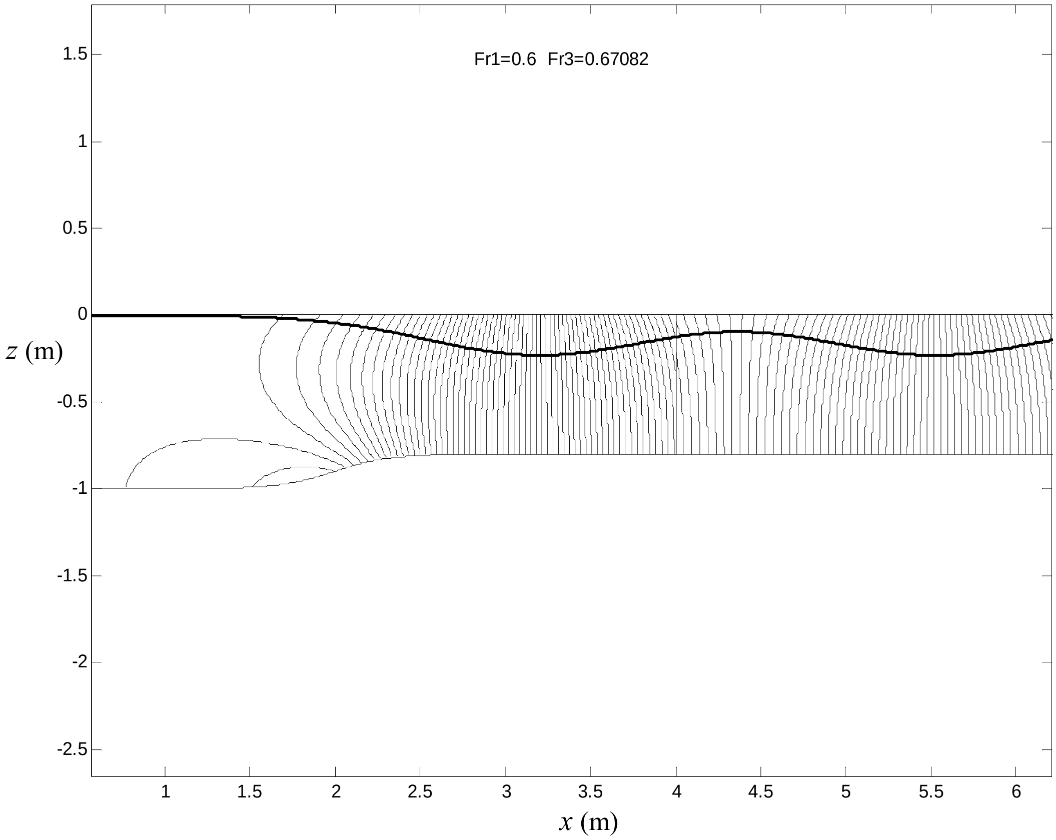

FIGURE 4.3a. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by the F.E.M, for Froude number *Fr*=0.6. The depth function is given by equation (1).

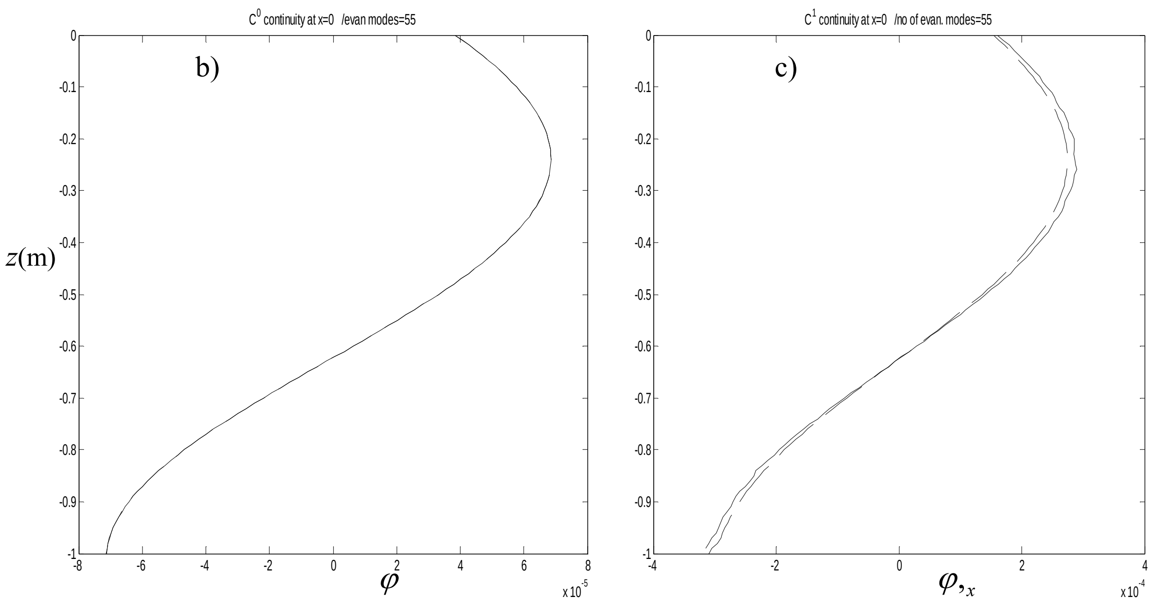

FIGURE 4.3b-c. C^0 - continuity b) and C^1 - continuity c) of the disturbance potential at $x = a$.

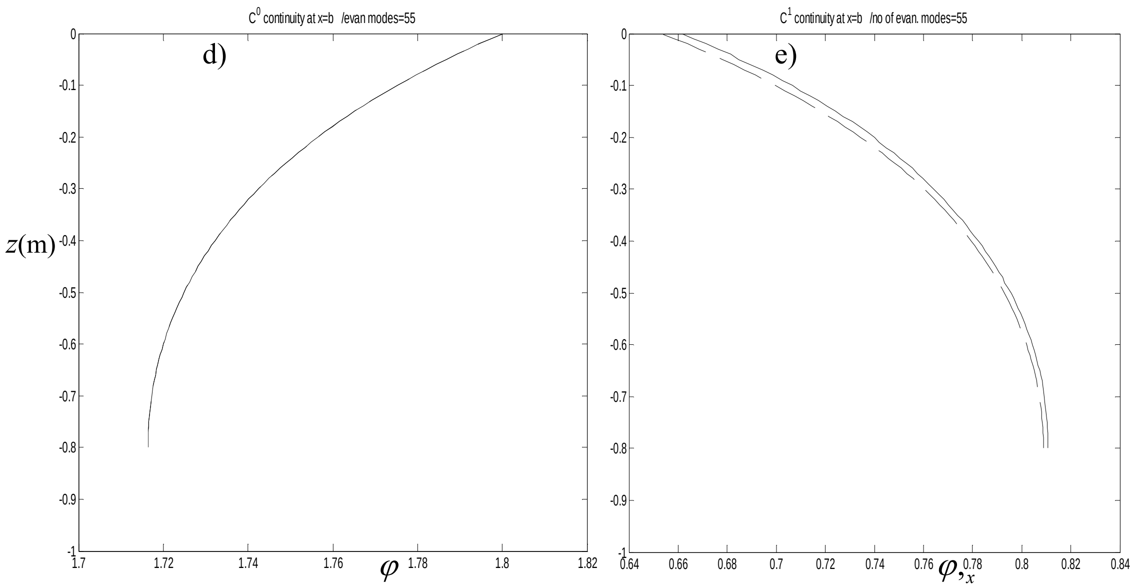

FIGURE 4.3d-e. C^0 - continuity d) and C^1 - continuity e) of the disturbance potential at $x = b$.

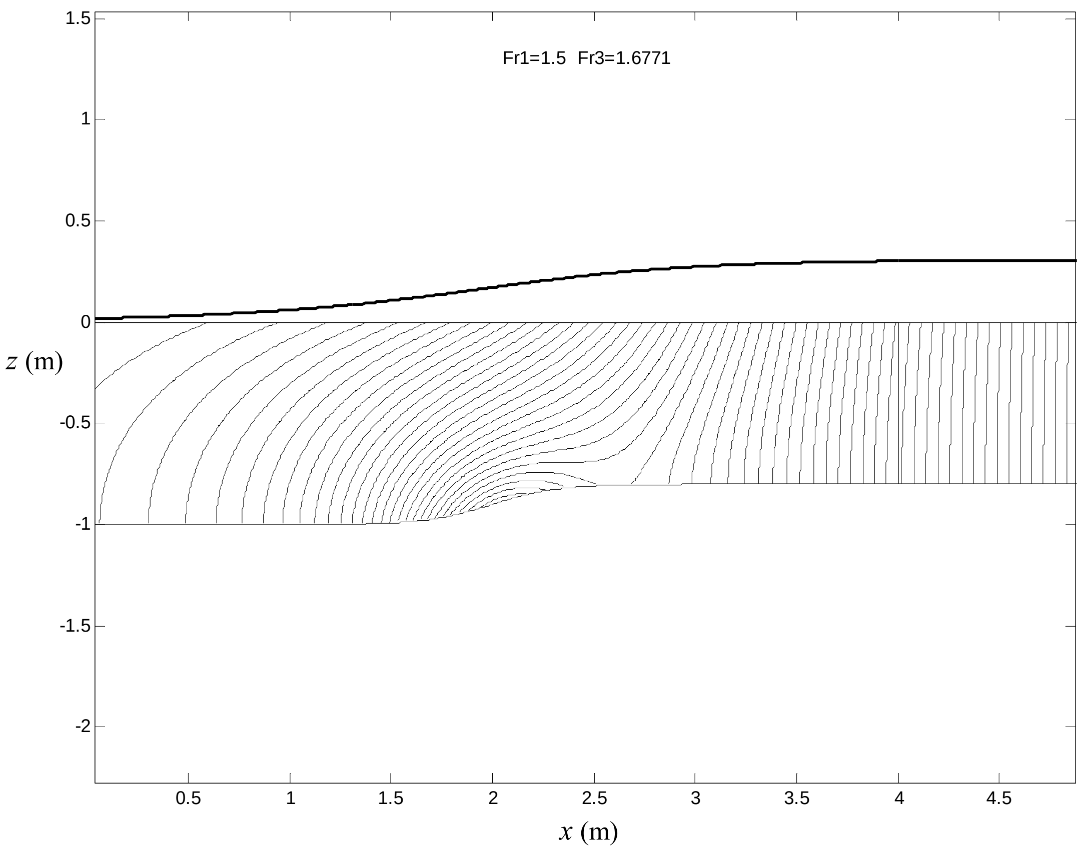

FIGURE 4.4. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by present model, for Froude number *Fr*=1.5. The depth function is given by equation (1).

4.4.2 Flow over bottom obstruction.

The case of a smooth sinusoidal bump is introduced. The bottom shape is given by the following depth function

$$h(x) = \begin{cases} -h + \dfrac{h_{bump}}{2}\left(1 + \cos(\dfrac{\pi x}{a})\right), & |x| \le a \\ -h, & |x| > a \end{cases} \tag{2}$$

1. Subcritical case $(Fr < 1)$

In figure 4.5 the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model, for $Fr = 0.5$. The height of sinusoidal bump is $h_{bump} = 0.3m$ and its length is $3m$.

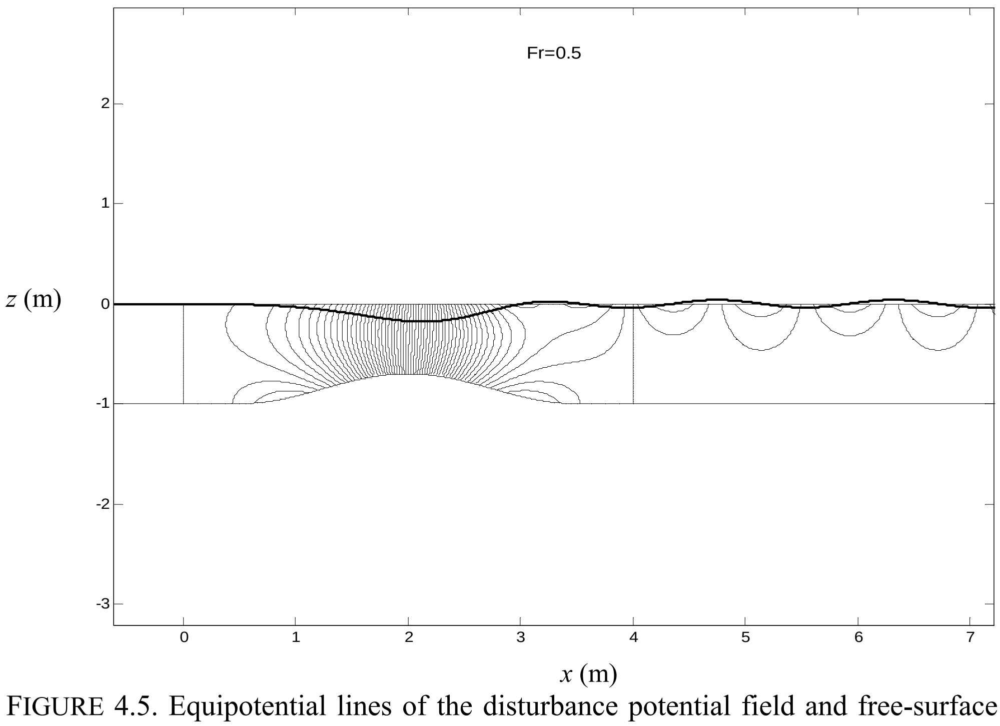

FIGURE 4.5. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by the F.E.M, for Froude number *Fr*=0.5. The depth function is given by equation (2).

2. Supercritical case $(Fr > 1)$

In figure 4.6 the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model, for $Fr = 2$. The height of sinusoidal bump is $h_{bump} = 0.3m$ and its length is $3m$.

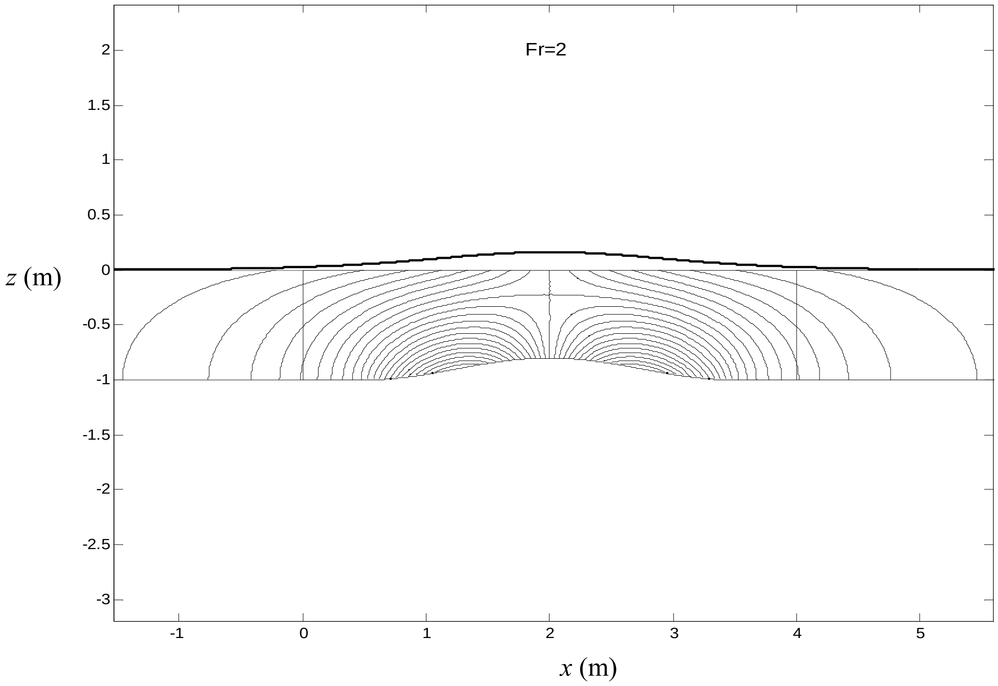

FIGURE 4.6. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by the F.E.M, for Froude number *Fr*=2. The depth function is given by equation (2).

4.4.3 Flow past a submerged body (constant depth).

The case of a submerged cylinder with a circular section is introduced. The cylinder is submerged at depth $d = \frac{h}{2}$, with radius $\rho = 0.15m$ (where h is the constant depth).

1. Subcritical case $(Fr < 1)$

In figure 4.7 the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model, for $Fr = 0.4$.

2. Supercritical case $(Fr > 1)$

In figure 4.8 the equipotential lines of the disturbance potential field $\varphi(x,z)$ have been plotted, together with the calculated free-surface elevation $\eta(x)$, as obtained by the present model, for $Fr = 2$.

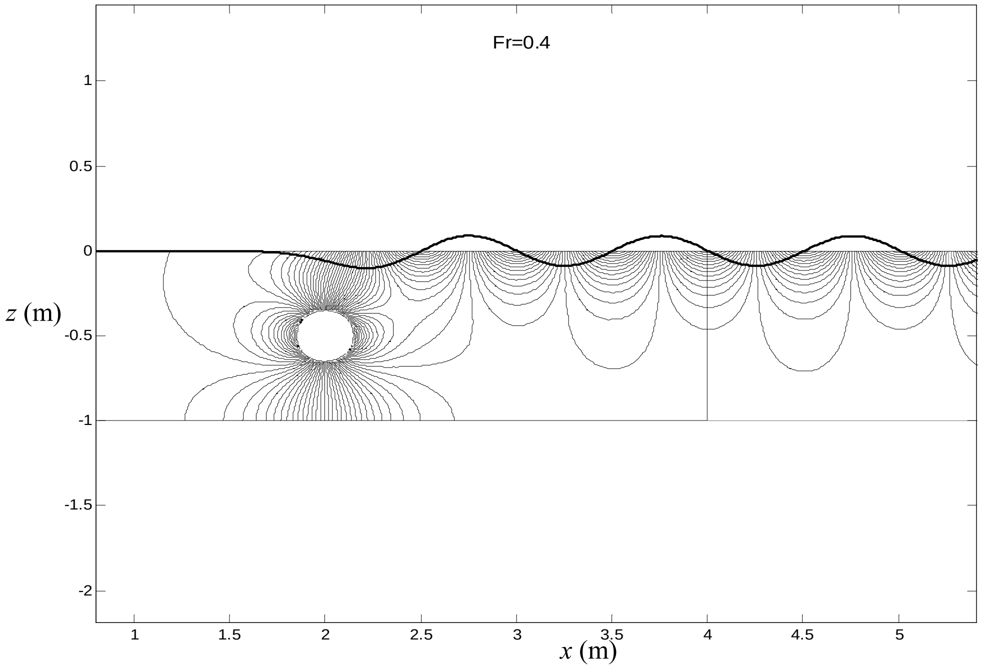

FIGURE 4.7. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by the F.E.M, for Froude number *Fr*=0.4. $\left(d = \dfrac{h}{2},\ \rho = 0.15m\right)$.

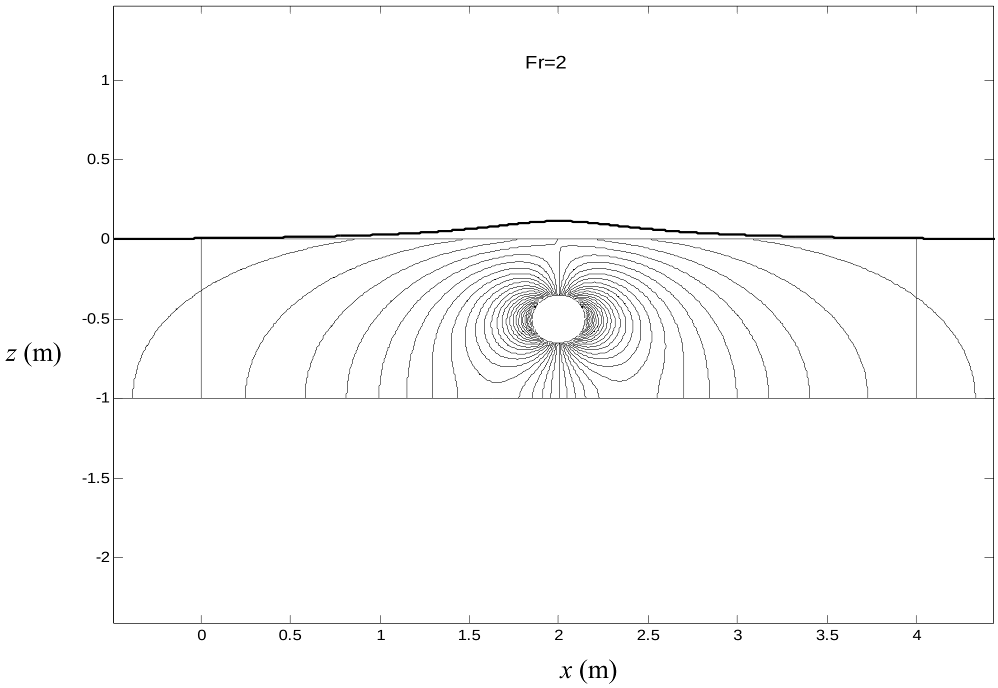

FIGURE 4.8. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by the F.E.M, for Froude number *Fr*=2. $\left(d = \dfrac{h}{2},\ \rho = 0.15m\right)$.

4.4.4 Flow over shoaling bottom with submerged body.

The case of a submerged cylinder with a circular section over a smooth underwater shoaling is introduced. The bottom shape is given by the relation (1).

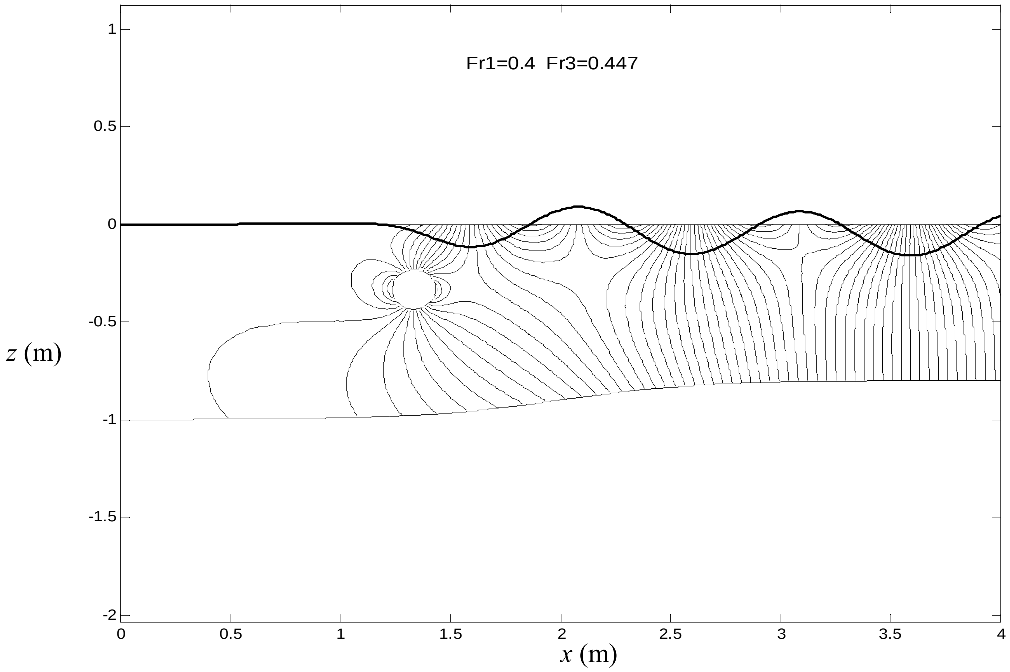

FIGURE 4.9. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by the F.E.M, for Froude number *Fr*=0.4.
($x_{cy} = 4/3$, $d = h/3$, $\rho = 0.1m$, $h_3 = 0.8$).

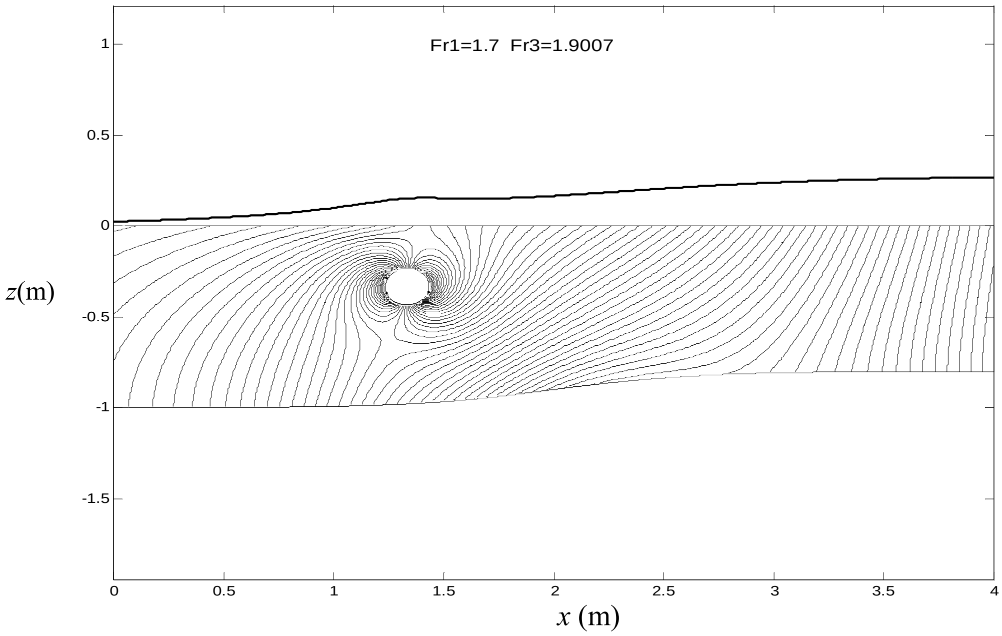

FIGURE 4.10. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by the F.E.M, for Froude number *Fr*=1.7.
($x_{cy} = 4/3$, $d = h/3$, $\rho = 0.1m$, $h_3 = 0.8$).

4.4.5 Flow over bottom obstruction with submerged body.

The case of a submerged cylinder with a circular section over a sinusoidal bump is introduced.. The bottom shape is given by the relation (2).

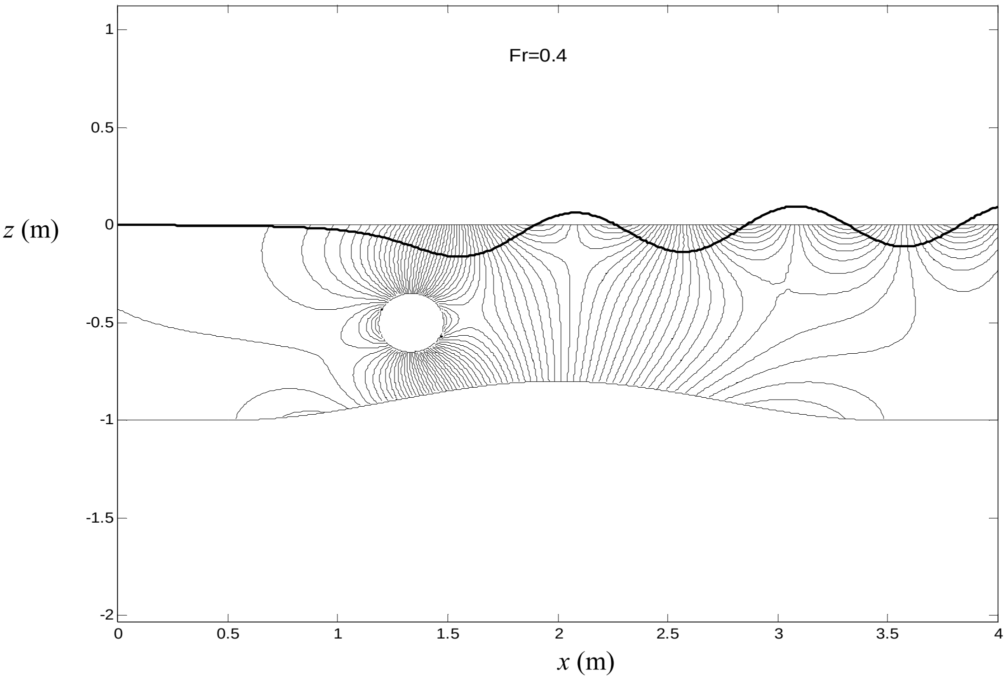

FIGURE 4.11. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by the F.E.M, for Froude number *Fr*=0.4.
($x_{cy} = 4/3$, $d = h/2$, $\rho = 0.15m$, $h_{pb} = 0.2$).

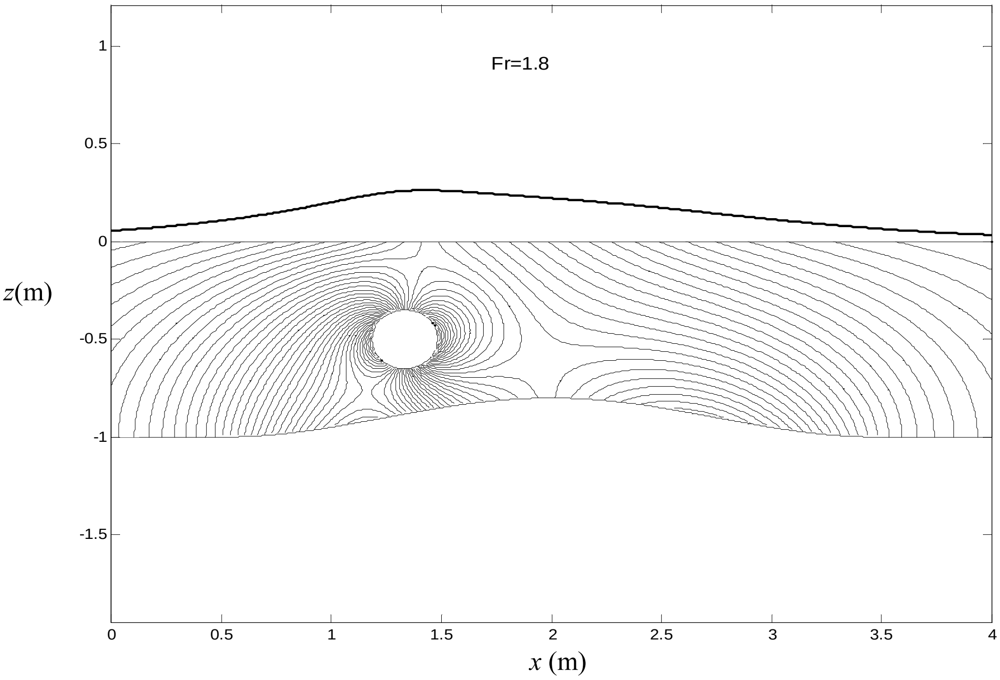

FIGURE 4.12. Equipotential lines of the disturbance potential field and free-surface elevation as obtained by the F.E.M, for Froude number *Fr*=1.8.
($x_{cy} = 4/3$ **,** $d = h/2$ **,** $\rho = 0.15m$ **,** $h_{pb} = 0.2$ **).**

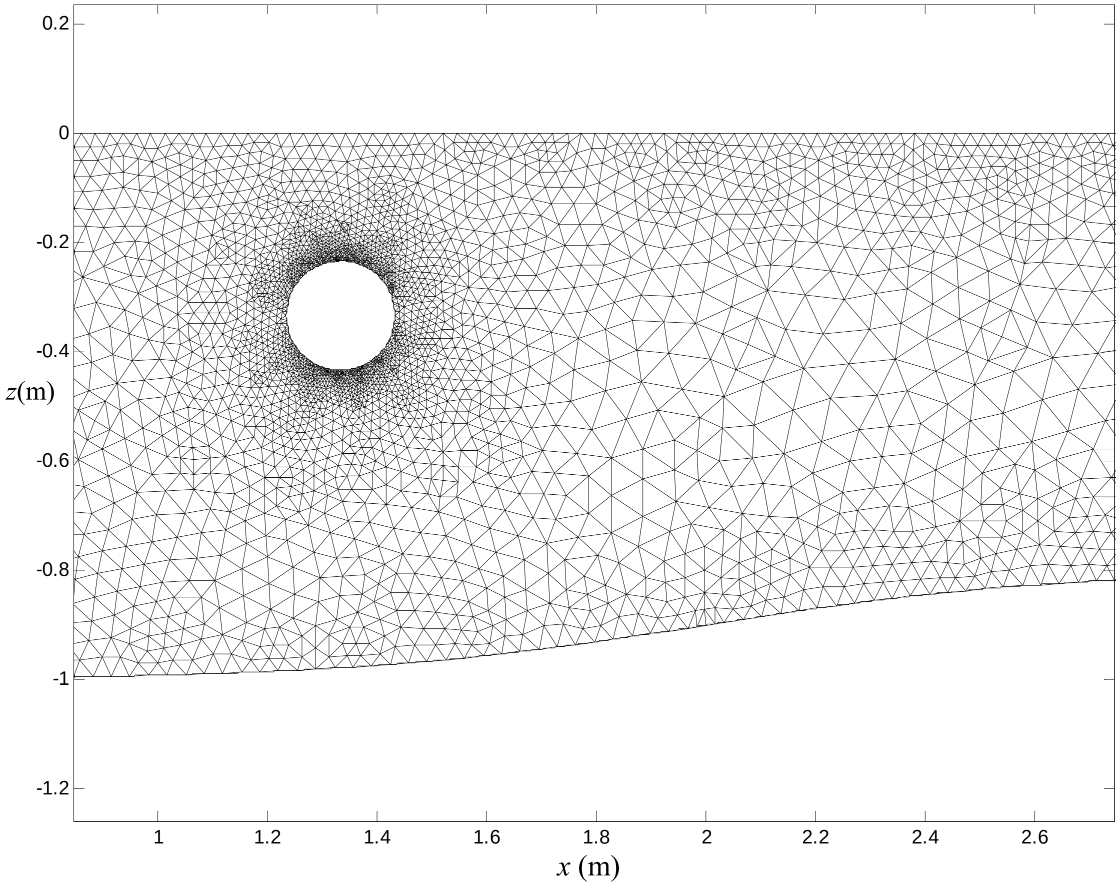

FIGURE 4.13. The constructed mesh, where the triangulation based on the Delaunay algorithm.

APPENDIX 4.A. Calculation of the Wave Drag Force.

After the solution of the problem, the velocity potential $\Phi(x,z) \in D$ is uniquely determined by the means of the coefficients of general representations at the semi-infinite strips and the potential $\Phi^{(2)}(x,z) \in D^{(2)}$. As the velocity potential is determined, the pressure p of any point of the fluid region can be calculated using the Bernoulli equation:

$$p = -\frac{1}{2}\rho\left|\nabla\Phi\right|^2 - \rho g z\ , \quad (x,z) \in D\ , \tag{1}$$

where the atmospheric pressure is taken to be zero.
The calculation of the wave drag force on a bottom topography caused by the fluid flow, as well as on a body moving with constant velocity is obtained by integrating the pressure over its surface ∂D_B:

$$D = \int\limits_{\partial D_B} p \cdot n_x dS\ , \tag{2}$$

where n_x is the normal vector in the direction of the motion, taken to be positive when pointing out of the fluid volume. Alternative, the wave drag force on a moving body may be calculated using the conservation of energy, see i.e. Wehausen & Laitone (1960, §8):

$$D = \frac{1}{2}\rho \int\limits_{-h}^{\eta(x^D)} (\varphi,_z^2 - \varphi,_x^2) dS + \frac{1}{2}\rho g \eta(x^D)\ . \tag{3}$$

The wave drag force for an ideal fluid represents the change in momentum flux due to the change in the stream produced by the obstruction, so the conversation of momentum may also be applied:

$$\int\limits_{\partial D} \rho \cdot \Phi,_x \cdot \frac{\partial \Phi}{\partial \vec{n}} \cdot dS = -\int\limits_{\partial D} p \cdot n_x dS\ , \tag{4}$$

where ∂D denotes the boundary of the fluid volume D.

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