Annexe A: CURRICULUM VITAE

Annexe B: WAVES AND OPERATIONAL OCEANOGRAPHY :TOWARD A COHERENT DESCRIPTION OF THE UPPER OCEAN

par Fabrice Ardhuin, Alastair D. Jenkins, Danièle Hauser, Ad Reniers, et Bertrand Chapron.
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Annexe C: INTERACTION OF OCEAN WAVES AND CURRENTS : HOW DIFFERENT APPROACHES MAY BE RECONCILED.

par Alastair D. Jenkins et Fabrice Ardhuin
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Annexe D: DÉRIVE À LA SURFACE DE L'OCÉAN SOUS L'EFFET DES VAGUES

par Fabrice Ardhuin, François-Régis Martin-Lauzer, Bertrand Chapron,
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Annexe E: MELLOR'S (2003) EQUATIONS AND GLM THEORY

The present section are the appendices of a corrected version of a paper rejected by the Journal of Fluid Mechanics. The main body of the paper is incorporated into chapter 2 of the present thesis. The paper was rejected supposedly because it was too focused on parameterizations, with a too large emphasis on parameterizations. In fact the paper was also not-so-well written, and had a few mistakes, which are corrected here.

A. Effects of surface pressure fluctuations

1. Wind-wave growth

Waves are generated by pressure and tangential stress variations on the scale of the wavelength. We solve here the problem with the usual cartesian coordinate system before transforming the solution to sigma coordinates. The variation of tangential stresses is neglected (see Lamb 1932 p. 629, Jenkins 1992). Atmospheric pressure at the surface can be described as,

$$p^a(\mathbf{x},t) = \widehat{p}^a(\mathbf{x},t) + \sum_{\mathbf{k},\sigma_p} P^a_{\mathbf{k},\sigma_p} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}-\sigma_p t}. \tag{E.1}$$

with $P^a_{\mathbf{k},\sigma_p}$ the Fourier component of the air pressure at the surface, with wavenumber $\mathbf{k}$ and angular frequency σ_p. Following the general procedure for solving second-order differential equations, the wave field can now be obtained by adding the general solution in absence of forcing and a particular solution of the wave equations that satisfies this forcing. Neglecting terms that are second order in ε_1, the surface equation for the wave potential ϕ is

$$\frac{\partial \phi}{\partial t} = -g\zeta - \frac{1}{\rho}p_a, \quad \text{at} \quad z=\zeta, \tag{E.2}$$

which can be combined with the kinematic boundary condition to give,

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = -\frac{1}{\rho}\frac{\partial p_a}{\partial t} \quad \text{at} \quad z=0. \tag{E.3}$$

A particular solution ϕ_p, that also must satisfy the Laplace equation and the bottom boundary condition, is given by,

$$\phi^p = \sum_{\mathbf{k}} \frac{\cosh\left[k(\,z+D\,)\right]}{\cosh(kD)} \Phi^p_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{E.4}$$

where $\Phi^p_{\mathbf{k}}(t)$ is the solution of (E.3). This solution can be written as a resonant term plus some bound terms, with resonance obtained for $\sigma_p = \sigma \equiv gk\tanh(kH)$

$$\Phi^p_{\mathbf{k}}(t) = \mathrm{i}\sum_{\sigma_p \neq \pm\sigma} \frac{\sigma_p P^a_{\mathbf{k},\sigma_p}}{\rho\left[\sigma^2-\sigma_p^2\right]}\mathrm{e}^{-\mathrm{i}\sigma_p t} - \sum_{s_1} t\frac{P^a_{\mathbf{k},s\sigma}}{2\rho}\mathrm{e}^{-\mathrm{i}s_1\sigma t} \tag{E.5}$$

Pressure in the water is given by the (linearized) Bernoulli equation that we may write

$$p = -\rho g z - \rho\frac{\partial \phi}{\partial t} + O\left(a_0 g\varepsilon_1\right), \tag{E.6}$$

with the non-hydrostatic part p^p given by taking the derivative of (E.4),

$$p^p = \sum_{\mathbf{k}} \frac{\cosh\left[k(\,z+D\,)\right]}{\cosh(kD)} P^p_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{E.7}$$

with

$$P^p_{\mathbf{k}}(t) = -\sum_{\sigma_p \neq \pm\sigma} \frac{\sigma_p^2 P^a_{\mathbf{k},\sigma_p}}{\left[\sigma^2 - \sigma_p^2\right]} \mathrm{e}^{-\mathrm{i}\sigma_p t} + \sum_{s_1} \frac{P^a_{\mathbf{k},s_1\sigma}}{2} e^{-is_1\sigma t} - \mathrm{i}\sigma t \sum_{s_1} s \frac{P^a_{\mathbf{k},s_1\sigma}}{2} \mathrm{e}^{-\mathrm{i}s_1\sigma t} \quad \text{(E.8)}$$

so that the pressure response under the water is not entirely in phase with the pressure forcing, which was mistakenly suggested by Mellor (2003). Although it is an apparent paradox that the wave pressure p^p at the surface is not equal to the atmospheric pressure, the difference is explained by the partial adjustment of the surface elevation and the resulting hydrostatic pressure : Again, p^p is the non-hydrostatic pressure only.

To obtain the surface elevation amplitudes $Z^p_{\mathbf{k}}(t)$ at first order, we subtract (E.2) from (E.6) at $z = 0$,

$$\zeta = \frac{1}{\rho g}[p^p|_{z=0} - p^a].$$

Hence

$$Z^p_{\mathbf{k}}(t) = -\sum_{\sigma_p \neq \pm\sigma} \left[\frac{\sigma_p^2}{\sigma^2 - \sigma_p^2} + 1\right] \frac{P^a_{\mathbf{k},s\sigma}}{\rho g} \mathrm{e}^{-\mathrm{i}\sigma_p t} - \sum_{s_1} \frac{P^a_{\mathbf{k},s_1\sigma}}{2\rho g} \mathrm{e}^{-\mathrm{i}s_1\sigma t} - \mathrm{i}\sigma t \sum_{s_1} s_1 \frac{P^a_{\mathbf{k},s_1\sigma}}{2\rho g} \mathrm{e}^{-\mathrm{i}s_1\sigma t} \quad \text{(E.9)}$$

Atmospheric pressure is generally influenced by the waves, say, to first order, proportional with a complex coefficient $\beta_C = (-\beta_R - \mathrm{i}\beta_I)$ to the elevation,

$$P^a_{\mathbf{k},\sigma_p} = \delta(\sigma_p, s_1\sigma)\, \rho g \beta_C Z^{s_1}_{\mathbf{k}} \quad \text{(E.10)}$$

where $\delta(x, y)$ equals 0 unless $x = y$, and with β_I positive for growing waves, and β_R positive also due to the Bernoulli equation in the air : for winds faster than the waves, the flow accelerates over the wave crests due to streamline convergence, and thus the pressure decreases.

Thus the wave energy will be augmented at first order in β_I by the following term, E^p,

$$E^p(\mathbf{k}) = t\sigma\beta_I E_2(\mathbf{k}) \quad \text{(E.11)}$$

This equation is only valid for short time scales since we have assumed a constant spectrum, it is thus better written as a time derivative (over long times), following the method of Hasselman (1962),

$$\frac{\partial E(\mathbf{k})}{\partial t} = \sigma\beta_I E_2(\mathbf{k}) = S_{\mathrm{in}}(\mathbf{k}) \quad \text{(E.12)}$$

2. Bound waves and momentum equation

We have thus computed waves that are induced by air pressure fluctuations. These waves are characterized by ζ^p, p^p, ϕ^p. They have a free wave structure propagating at the speed of the air pressure perturbation : the polarization relations between all variables are identical to those of free waves, except for one extra term in the pressure and elevation represented by the second terms in (E.8) and (E.9), respectively.

The bound wave terms (p^{pb}, ζ^{pb}) can be written as

$$p^{pb} = \sum_{\mathbf{k},s_1} -Z^{pb,s_1}_{\mathbf{k}}(t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - s_1\sigma t)} \quad \text{(E.13)}$$

$$\zeta^{pb} = \sum_{\mathbf{k},s_1} Z^{pb,s_1}_{\mathbf{k}}(t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - s_1\sigma t)} \quad \text{(E.14)}$$

with

$$Z^{pb,s_1}_{\mathbf{k}}(t) = -\frac{\beta_C}{2} Z^{s_1}_{\mathbf{k}} = \frac{\beta_R + \mathrm{i}\beta_I}{2} Z^{s_1}_{\mathbf{k}}. \quad \text{(E.15)}$$

The β_I component of the pressure fluctuations, in quadrature with the free wave elevation, clearly drives bound waves with a surface elevation in quadrature ahead of the free waves.

The β_R component of the pressure fluctuations, in anti-phase with the elevation, tends to increase the wave height since the resulting surface elevation is in phase with the free waves.

It is striking that this 'bound wave' has no corresponding velocity and a pressure opposite to the corresponding pressure if it were a free wave. Indeed the associated velocity is part of the rate of change of the free wave velocity. It should be noted that integration of the vertical velocity does yield the vertical displacements of the bound wave. In terms of the coordinate transform (2.47), keeping $s(\varsigma = 0) = \zeta$ requires a modification of s due to the bound wave. In order to change s we can add to it a term of amplitude $S_{\mathbf{k}}^{pb,s_1}$, with a vertical profile given by bound terms in the vertical displacement, that happen to have the same vertical profile as the free waves,

$$S_{\mathbf{k}}^{pb,s_1}(t) = -\beta_C S_{\mathbf{k}}^{s_1}. \tag{E.16}$$

This part of the change of variable s induces extra terms in the equations of motion, including a vertical velocity $\widetilde{w}^{pb}$, which now has a component in quadrature with the velocity and pressure. Considering only the solution driven by the pressure component in quadrature with the elevation, and evaluating all modified terms in the equations of motion, one gets exactly the same term as in Mellor's (2003) equation (51a), that is,

$$-\frac{\partial}{\partial\varsigma}\overline{\left(\widetilde{s}_\alpha^{pb}\widetilde{p} + \widetilde{s}_\alpha\widetilde{p}^{pb} + \widetilde{w}^{pb}\widetilde{u}_\alpha\right)} = \frac{\partial F_{SS}F_{CC}}{\partial\varsigma}\rho g\int_{\mathbf{k}}\beta_I k_\alpha E(\mathbf{k})\mathrm{d}\mathbf{k}, \tag{E.17}$$

that we may rewrite as

$$T_\alpha^{\mathrm{in}} = \overline{\widetilde{p}_{w\varsigma}\frac{\partial\widetilde{\zeta}}{\partial x_\alpha}}\frac{\partial F_{SS}F_{CC}}{\partial\varsigma}. \tag{E.18}$$

Indeed, if one considers the hypothetical case of a uniform wave field with no current and no dissipation we see that the wind to wave momentum flux is distributed over depth in the same way as the Stokes drift.

The part of the pressure that is in anti-phase with the surface elevation modifies slightly the term

$$\overline{\widetilde{p}\,\widetilde{s}_\alpha}, \tag{E.19}$$

in the momentum equation, which is already a second order correction. We may therefore neglect this effect, and obtain Mellor's (2003) momentum equation with the effect of random waves accurate to second order in ε_1.

B. Transformation of the GLM equation to sigma-coordinates

For simplicity equations are derived considering a single wave train of wavenumber vector $\mathbf{k}$ and intrinsic frequency σ. The various physical quantities that oscillate with the waves are, at first order in the wave slope and zeroth order in the bottom slope and wave amplitude gradients, given by the free wave polarization relation (e.g. Mei 1989),

$$p = \rho g a F_{CC}\mathcal{R}(\mathrm{e}^{\mathrm{i}\psi}) \tag{E.20}$$

$$u_\alpha = a\sigma\frac{k_\alpha}{k}F_{CS}\mathcal{R}(\mathrm{e}^{\mathrm{i}\psi}) \tag{E.21}$$

$$\xi_\alpha = a\frac{k_\alpha}{k}F_{CS}\mathcal{R}(\mathrm{i}\mathrm{e}^{\mathrm{i}\psi}) \tag{E.22}$$

$$w = -a\sigma F_{SS}\mathcal{R}(\mathrm{i}\mathrm{e}^{\mathrm{i}\psi}) \tag{E.23}$$

$$\xi_3 = aF_{SS}\mathcal{R}(\mathrm{e}^{\mathrm{i}\psi}), \tag{E.24}$$

with a the local amplitude, $\mathcal{R}$ is the real part, and ψ the local phase, such that $\mathbf{k} = \boldsymbol{\nabla}\psi$ and $\omega = \sigma + \mathbf{k}\cdot\widehat{\mathbf{u}}_A = -\partial\psi/\partial t$, and

$$F_{CC} = \frac{\cosh\left[kD(\,1+\varsigma/D\,)\right]}{\cosh\left(kD\right)}. \tag{E.25}$$

Over a gently sloping bottom these expressions should be corrected to include the effect of the vertical velocity at the bottom. Here we only write out the local slope correction, other terms are given by Ardhuin and Herbers (2002) with the erratum that their equation (D6) is obviously wrong and should be,

$$\Phi_{3,\mathbf{k}}^{\mathrm{si},s} = -\mathrm{i}\mathbf{k}\cdot\boldsymbol{\nabla}H\,\Phi_{1,\mathbf{k}}^{s}, \tag{E.26}$$

which gives,

$$
\begin{aligned}
p &= \rho g a \mathcal{R}\left[\left(F_{CC} - \mathrm{i}F_{SC}\frac{\mathbf{k}\cdot\boldsymbol{\nabla}H}{k}\right)\mathrm{e}^{\mathrm{i}\psi}\right] + ... && \text{(E.27)}\\
u_\alpha &= a\sigma\mathcal{R}\left[\left(F_{CS} - \mathrm{i}F_{SS}\frac{k_\alpha\mathbf{k}\cdot\boldsymbol{\nabla}H}{k^2}\right)\mathrm{e}^{\mathrm{i}\psi}\right] + ... && \text{(E.28)}\\
\xi_\alpha &= a\mathcal{R}\left[\left(\mathrm{i}F_{CS} + F_{SS}\frac{k_\alpha\mathbf{k}\cdot\boldsymbol{\nabla}H}{k^2}\right)\mathrm{e}^{\mathrm{i}\psi}\right] + ... && \text{(E.29)}\\
w &= a\sigma\mathcal{R}\left[\left(-\mathrm{i}F_{SS} - F_{CS}\frac{\mathbf{k}\cdot\boldsymbol{\nabla}H}{k}\right)\mathrm{e}^{\mathrm{i}\psi}\right] + ... && \text{(E.30)}\\
\xi_3 &= a\mathcal{R}\left[\left(F_{SS} - \mathrm{i}F_{CS}\frac{\mathbf{k}\cdot\boldsymbol{\nabla}h}{k}\right)\mathrm{e}^{\mathrm{i}\psi}\right] + ..., && \text{(E.31)}
\end{aligned}
$$

where the missing terms refer to effects of horizontal gradients, that are easily derived from the velocity potential amplitudes $\Phi_{3,\mathbf{k}}^{\mathrm{coz2},s}$, $\Phi_{3,\mathbf{k}}^{\mathrm{coz},s}$, and $\Phi_{3,\mathbf{k}}^{\mathrm{siz},s}$ (Ardhuin and Herbers 2002, eq. D3, D4, D5), so that Laplace's equation may be satisfied exactly. These missing terms are first order in ε_2, just like the bottom slope terms that have been added here. It should be noted that further correction due to the vertical current shear are given by McWilliams et al. (2004, eq. A3).

Results from random waves are obtained by replacing the surface elevation variance $\mathrm{var}_\zeta = a^2/2$ by the spectral energy density. All second order quantities are simply the sums of the following monochromatic solution for all wavenumber vectors. Another option is to use a narrow spectrum approximation and resolve explicitly the variations in wave properties over the scale of wave groups.

We shall apply results obtained by Andrews and McIntyre (1978a, 1978b), replacing their equations (3.2) and (4.1) by Reynolds-Averaged Navier Stokes equations (RANS), which means that their dissipative forces X represent both viscous forces and turbulent Reynolds stresses. We shall retain all wave effects up to second order in the wave slope and first order in the wave. The resulting equations are therefore second order Generalized Lagrangian Mean RANS equation, abbreviated as GLM2-RANS.

1. Mass conservation

The Jacobian J of the GLM coordinate transformation (from Eulerian coordinates) can be shown to be equal to 1 plus a second order quantity. Using the 3D wave action A (see Andrews and McIntyre 1978b), one has,

$$
\begin{aligned}
J &= 1 + J_2 + O(\varepsilon_1^3) && \text{(E.32)}\\
J_2 &= -\frac{kA}{\sigma} = -k^2\mathrm{var}_\zeta\frac{\cosh\left[2k(z+h)\right]}{\sinh^2(kD)}, && \text{(E.33)}
\end{aligned}
$$

where var_ζ is the surface elevation variance due to the waves. Because there is no mean stretching of the horizontal coordinates, a vertical distance $\mathrm{d}z' = J\mathrm{d}z$ in GLM corresponds to a Cartesian distance $\mathrm{d}z$. As $J < 1$ over the entire water column on has $\mathrm{d}z' > \mathrm{d}z$. Thus the vertical GLM position is everywhere larger than the mean Eulerian elevation of the same water particles. In a sense this is because at any time there are more particles per horizontal length of crest than of trough (McIntyre 1988).

Integrating over depth we define

$$
s^G(x,z,t) = -\int_{-h}^{z} J_2(z')\mathrm{d}z' = k\mathrm{var}_\zeta\frac{\sinh\left[2k(z+h)\right]}{2\sinh^2(kD)}. \quad \text{(E.34)}
$$

Using the second order expression for $\overline{\zeta}^L$ (e.g. Jenkins & Ardhuin 2004)

$$
\overline{\zeta}^L = \overline{\zeta} + \mathrm{var}_\zeta\frac{k}{\tanh kD}, \quad \text{(E.35)}
$$

we can see that

$$
\int_{-h}^{\overline{\zeta}} J\mathrm{d}z = \overline{\zeta}^L + h + \overline{\zeta} - s^G(0) = D, \quad \text{(E.36)}
$$

which is a further verfication of the vertical stretching induced by GLM.

By analogy to 2.47 we thus define

$$s = \varsigma D + s^G + \widehat{\zeta} \tag{E.37}$$

for which we can use the chain rules given by Mellor (2003) to go from (x_α, z, t) to $(x^\star_\alpha, \varsigma, t^\star)$, i.e. for any variable ϕ

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi^\star}{\partial t^\star} - \frac{s_{,t}}{s_{,\varsigma}}\frac{\partial \phi^\star}{\partial \varsigma} \tag{E.38}$$

$$\frac{\partial \phi}{\partial x_\alpha} = \frac{\partial \phi^\star}{\partial x^\star_\alpha} - \frac{s_{,\alpha}}{s_{\varsigma}}\frac{\partial \phi^\star}{\partial \varsigma} \tag{E.39}$$

$$\frac{\partial \phi}{\partial z} = \frac{1}{s_{,\varsigma}}\frac{\partial \phi^\star}{\partial \varsigma} \tag{E.40}$$

with $s_{,t}$, $s_{,\varsigma}$ and $s_{,\alpha}$ the partial derivatives of s with respect to t, ς and x_α, respectively. In our case we have the remarkable identity

$$s_{,\varsigma} J = D. \tag{E.41}$$

Dropping the star superscripts just like Mellor (2003) going from his equation (14) to (22), we can transform the GLM mass conservation equation

$$\frac{\partial \rho_w J}{\partial t} + \frac{\partial \rho_w J \overline{u}^L_\alpha}{\partial x_\alpha} + \frac{\partial J \overline{w}^L}{\partial z} = 0 \tag{E.42}$$

to the following

$$\frac{\partial \rho_w \widehat{\zeta}}{\partial t} + \frac{\partial D \rho_w U_\alpha}{\partial x_\alpha} + \frac{\partial \rho_w W}{\partial \varsigma} = 0, \tag{E.43}$$

by defining

$$W = \left[J\overline{w}^L - J\left(\overline{u}^L_\alpha s_\alpha + s_t\right)\right]^\star, \tag{E.44}$$

and $(\overline{u}^L_\alpha)^\star = U_\alpha$, that is, the Lagrangian drift velocity in Mellor's coordinate is indeed the transformed GLM velocity. For constant ρ_w (E.42) is clearly Mellor's equation (51).

2. 3D momentum in Mellor's coordinates

We start here from the 'alternative form' of the GLM equations, (Andrews and McIntyre 1978, equation 8.7a), considering only the vertical component of the Earth's rotation,

$$\overline{D}^L \overline{u}^L_\alpha + \epsilon_{\alpha 3 \beta} f_3 \overline{u}^L_\beta + \frac{1}{\rho_w J}\frac{\partial \overline{p}^L}{\partial x_\alpha} + \overline{X}^L_\alpha = \frac{1}{\rho_w J}\left(\frac{\partial R_{\alpha\beta}}{\partial x_\beta} + \frac{\partial R_{\alpha 3}}{\partial z}\right). \tag{E.45}$$

First looking at the Lagrangian-mean pressure we have to second order in ε_1,

$$\overline{p}^L = \overline{p} + \overline{\xi_j \frac{\partial p'}{\partial x_j}}. \tag{E.46}$$

The mean vertical momentum equation for steady mean motions gives,

$$-\frac{\partial \overline{p}}{\partial z} = \rho_w g + \frac{\partial \overline{w^2}}{\partial z} + \frac{\partial \overline{u_\alpha w}}{\partial x_\alpha}. \tag{E.47}$$

Over a gently sloping bottom $u_\alpha w$ is third order in ε_1 and (E.47) integrates vertically to

$$\overline{p} = \overline{p^H} - \rho_w \sigma^2 F_{SS} F_{SS} \mathrm{var}_\zeta, \tag{E.48}$$

where $\overline{p^H}$ is the mean hydrostatic pressure and the second term is the wave-induced Eulerian mean pressure.

Assuming that we have only free waves, the Stokes correction in (E.46) can be rewritten as

$$\overline{\xi_j \frac{\partial p'}{\partial x_j}} = \rho_w \mathrm{var}_\zeta \left(\sigma^2 F_{SS} F_{SS} - gk F_{CC} F_{CS}\right), \tag{E.49}$$

so that

$$\overline{p}^L = \overline{p^H} - \rho_w g \mathrm{var}_\zeta k F_{CC} F_{CS}. \tag{E.50}$$

Finally the radiation stress is defined by Andrews and McIntyre (1978a, equation 8.6) as

$$\begin{aligned} R_{\alpha j} &= \overline{p^\xi (1-J)}\delta_{\alpha j} + \overline{p^\xi \frac{\partial \xi_i}{\partial x_\alpha} K_{ij}}, \\ &= \overline{p^\xi (1-J)}\delta_{\alpha j} + B_{\alpha j} \end{aligned} \tag{E.51}$$

where p^ξ is the pressure at the displaced position and K_{ij} are the cofactors of the coordinate transform matrix $\mathbf{x} \rightarrow (\mathbf{x} + \xi)$, and $B_{\alpha j}$ is the flux in direction j of wave pseudo-momentum in direction opposite to x_α (Andrews and McIntyre 1978b, equation 2.7b and 2.17 see also Jenkins & Ardhuin 2004), which is, at second order in the wave slope,

$$B_{\alpha j} = \overline{p^\xi \xi_{j,\alpha}} + \overline{p^\xi \left(\xi_{j,\alpha}\xi_{m,m} - \xi_{j,m}\xi_{m,\alpha}\right)}. \tag{E.52}$$

where the subscripts with commas denote partial derivatives, i.e. $\phi_{,i} = \partial\phi/\partial x_i$.

This was rewritten by Andrews and McIntyre (1978a, eq. 8.11) as,

$$B_{\alpha j} = \overline{p' \xi_{j,\alpha}} + Z_{\alpha j} + N_{\alpha j} + O(\varepsilon_1^3), \tag{E.53}$$

with

$$Z_{\alpha j} = -\overline{p}^L \left(\overline{\xi_m \xi_{j,m}}\right)_{,\alpha} + \frac{1}{2}\left[\overline{p}^L \left(\overline{\xi_m \xi_j}\right)_{,\alpha}\right]_{,m}, \tag{E.54}$$

and a tensor $N_{\alpha j}$ that can be neglected when considering the divergence $\partial B_{\alpha j}/\partial x_j$ because for any displacement field, $\partial N_{\alpha j}/\partial x_j = 0$. Thus we shall use

$$B'_{\alpha j} = \overline{p' \xi_{j,\alpha}} + Z_{\alpha j}. \tag{E.55}$$

The various terms in $B'_{\alpha\beta}$ can be approximated to second order in ε_1 and zeroth order in ε_2 since the horizontal divergence of $B_{\alpha\beta}$ makes the resulting momentum term first order in ε_2. We can thus neglect $Z_{\alpha\beta}$, and, using (E.34) we find

$$R'_{\alpha\beta} = R'_{\alpha\beta} + \overline{p'\xi_{\beta,\alpha}} = R''_{\alpha\beta} - E\frac{k_\alpha k_\beta}{k} F_{CC} F_{CS}, \tag{E.56}$$

with $E = \rho_w g \mathrm{var}_\zeta$, and

$$R''_{\alpha\beta} = \overline{p}^H \frac{s^G_{,\varsigma}}{D}\delta_{\alpha\beta}. \tag{E.57}$$

The vertical flux $R_{\alpha 3}$ is much more delicate to deal with because its vertical divergence is of the same order in ε_2 as the flux itself, due to the large vertical gradient in all the vertical profiles. Thus $R_{\alpha 3}$ must be estimated to first order in ε_2, including all the gradients of the wave amplitude and functions of k and D. A direct calculation of $R_{\alpha 3}$ from (E.51) yields, to second order in ε_1,

$$R_{\alpha 3} = B_{\alpha 3} = \overline{p^\xi \xi_{3,\alpha}} + \left[\overline{p}\overline{[\xi_{1,\alpha}\xi_{3,1} - \xi_{2,\alpha}\xi_{3,2} + \xi_{3\alpha}\left(\xi_{2,2} + \xi_{1,1}\right)]}\right] \tag{E.58}$$

$$= \overline{p^\xi \xi_{3,\alpha}} + \overline{p}^H \left[\overline{\xi_{3,\alpha}\xi_{\beta,\beta}} - \overline{\xi_{3,\beta}\xi_{\beta,\alpha}}\right]. \tag{E.59}$$

To zeroth order in ε_2, only the first term remains in the expression of $R_{\alpha 3}$. $\overline{p^\xi \xi_{3,\alpha}}$ is identical to Mellor's (2003) $\widetilde{p s_\alpha}$, clearly expressing the correlation of pressure and streamline slope on the wave streamlines (i.e. at the displaced position). The vertical integral of that term over a flat bottom is zero in the absence of forcing (e.g. by the wind as seen in Appendix A).

However, to first order in ε_2, and because we have already used the zero-divergence property of $N_{\alpha j}$ in order to compute $R'_{\alpha\beta}$ instead of $R_{\alpha\beta}$, it is easier to stick with the decomposition of $R_{\alpha j}$ given by (E.53),

$$\begin{aligned} R'_{\alpha 3} &= \overline{p'\xi_{3,\alpha}} + Z_{\alpha,3} \\ &= \rho_w g a \mathcal{R}\left[F_{CC} - \mathrm{i}F_{SC}\frac{\mathbf{k}\cdot\boldsymbol{\nabla}H}{k}\right]\mathcal{R}\left[a\left(F_{SS} - \mathrm{i}F_{CS}\frac{\mathbf{k}\cdot\boldsymbol{\nabla}H}{k}\right)\right]_{,\alpha} + Z_{\alpha,3} \\ &= E^{1/2}F_{CC}\left(E^{1/2}F_{SS}\right)_{,\alpha} + E\left(-F_{SC}F_{SS} + F_{CC}F_{CS}\right)\frac{k_\alpha \mathbf{k}\cdot\boldsymbol{\nabla}H}{k} + Z_{\alpha,3}, \end{aligned} \tag{E.60}$$

where

$$Z_{\alpha,3} = -\overline{p}^H \left[ka^2 F_{SS}F_{CS}\right]_{,\alpha} + \frac{1}{2}\left[\overline{p}^H\left((F_{SS})^2\frac{a^2}{2}\right)_{,\alpha}\right]_{,z}$$
$$= -\frac{\overline{p}^H}{\rho g}\left(kF_{SS}F_{CS}E\right)_{,\alpha} - \frac{1}{2}\left((F_{SS})^2 E\right)_{,\alpha}. \tag{E.61}$$

We can now also consider the contribution of bound waves due to air pressure fluctuations over the waves, as considered in § 2. These clearly contribute to $\partial R_{\alpha 3}/\partial z$, giving the extra term found in Appendix A, with the same vertical profiles as the Stokes drift,

$$T_\alpha^{\mathrm{in}}(z) = \overline{p^a\frac{\partial \zeta}{\partial x_\alpha}} kD(F_{CC}F_{CS} + F_{SC}F_{SS}). \tag{E.62}$$

We may now transform (E.45), to the new coordinates, using the GLM mass conservation equation. We first consider the Lagrangian mean derivative. Using (E.44) and (E.41) we get

$$s_{,\varsigma}\rho_w J\overline{D}^L \overline{u}_\alpha^L = s_{,\varsigma}\left[\frac{\partial}{\partial t}(\rho_w J\overline{u}_\alpha^L) + \frac{\partial}{\partial x_\beta}(\rho_w J\overline{u}_\alpha^L\overline{u}_\beta^L) + \frac{\partial}{\partial z}(\rho_w J\overline{w}^L)\right]$$
$$= \frac{\partial}{\partial t}(\rho_w DU_\alpha) + \frac{\partial}{\partial x_\beta}(\rho_w DU_\alpha U_\beta) + \frac{\partial}{\partial \varsigma}(\rho_w J\overline{w}^L U_\alpha)$$
$$-\frac{\partial s}{\partial t}\frac{\partial}{\partial \varsigma}(\rho_w JU_\alpha) - \frac{\partial s}{\partial x_\beta}\frac{\partial}{\partial \varsigma}(\rho_w JU_\alpha U_\beta) - \rho_w JU_\alpha\left(\frac{\partial^2 s}{\partial t\partial\varsigma} + U_\beta\frac{\partial^2 s}{\partial\varsigma\partial x_\beta}\right)$$
$$= \frac{\partial}{\partial t}(\rho_w DU_\alpha) + \frac{\partial}{\partial x_\beta}(\rho_w DU_\alpha U_\beta) + \frac{\partial}{\partial \varsigma}(\rho_w WU_\alpha), \tag{E.63}$$

with the * superscripts omitted on the right hand side.

Transforming the pressure gradient term using (E.50), and combining it with $R''_{\alpha\beta}$, one gets to second order in ε_1,

$$s_{,\varsigma}\left(\frac{\partial\overline{p}^L}{\partial x_\alpha} - \frac{\partial R''_{\alpha\beta}}{\partial x_\beta}\right) = \frac{\partial}{\partial x_\alpha}\left[\left(s - s^G\right)_{,\varsigma}\overline{p}^H\right] - \frac{\partial}{\partial\varsigma}\left[\left(s - s^G\right)_{,\alpha}\overline{p}^H\right]$$
$$-\frac{\partial}{\partial x_\alpha}\left(s_{,\varsigma}kEF_{CC}F_{CS}\right) + \frac{\partial}{\partial\varsigma}\left(kEF_{CC}F_{CS}\frac{\partial s}{\partial x_\alpha}\right)$$
$$= gD\widehat{\zeta}_{,\alpha} + \frac{\partial}{\partial x_\alpha}\left(D\overline{p}^H\right) - \frac{\partial}{\partial\varsigma}\left(\varsigma\overline{p}^H\frac{\partial D}{\partial x_\alpha}\right) - \frac{\partial}{\partial x_\alpha}\left(kDEF_{CC}F_{CS}\right)$$
$$+\frac{\partial}{\partial\varsigma}\left(\varsigma kEF_{CC}F_{CS}\frac{\partial D}{\partial x_\alpha}\right) \tag{E.64}$$

where $s_{,\varsigma} = D + s^G_{,\varsigma}$ has been used. Defining the buoyancy as $b = -g\left(\widehat{\rho}_w - \rho_{w0}\right)/\rho_{w0}$, with ρ_{w0} a reference water density, the first two hydrostatic terms can be expressed as (Mellor 2003)

$$\frac{\partial}{\partial x_\alpha}\left(D\overline{p}^H\right) - \frac{\partial}{\partial\varsigma}\left(\varsigma\overline{p}^H\frac{\partial D}{\partial x_\alpha}\right) = +\rho_{w0}D^2\int_\varsigma^0\left(\frac{\partial b}{\partial x_\alpha} - \varsigma\frac{\partial D}{\partial x_\alpha}\frac{\partial b}{\partial\varsigma}\right) \tag{E.65}$$

and we define the last term

$$T_\alpha^{p1} = \frac{\partial}{\partial\varsigma}\left(\varsigma kEF_{CC}F_{CS}\frac{\partial D}{\partial x_\alpha}\right) \tag{E.66}$$

Now transforming the remaining terms of the radiation stresses,

$$s_\varsigma \left[\frac{\partial}{\partial x_\beta}\left(R'_{\alpha\beta} - R''_{\alpha\beta}\right) + \frac{\partial R'_{\alpha 3}}{\partial z}\right]$$
$$= -\frac{\partial}{\partial x_\beta}\left(s_\varsigma E\frac{k_\alpha k_\beta}{k}F_{CC}F_{CS}\right) + \frac{\partial}{\partial\varsigma}\left(R'_{\alpha 3} + s_\beta E\frac{k_\alpha k_\beta}{k}F_{CC}F_{CS}\right)$$
$$= \frac{\partial}{\partial x_\beta}S^u_{\alpha\beta} + T_\alpha^{\mathrm{in}} + \frac{\partial}{\partial\varsigma}\left(\rho_w gaF_{CC}\frac{\partial\left(aF_{SS}\right)}{\partial x_\alpha} + Z_{\alpha,3}\right) + T_\alpha^{\mathrm{bottom}}, \tag{E.67}$$

with $S^u_{\alpha\beta}$ the non-isotropic part of the 3D radiation stresses,

$$S^u_{\alpha\beta} = kDE\frac{k_\alpha k_\beta}{k^2}F_{CC}F_{CS}, \tag{E.68}$$

and $S^{\mathrm{bottom}}_\alpha$ the vertical fluxes induced by the bottom slope,

$$T^{\mathrm{bottom}}_\alpha = \frac{\partial}{\partial\varsigma}\left[E\left[F_{CC}F_{CS}\left(1+\varsigma\right) - F_{SC}F_{SS}\right]\frac{k_\alpha \mathbf{k}\cdot\boldsymbol{\nabla}H}{k}\right], \tag{E.69}$$

which integrates to zero over the vertical.

Finally one can see that parts of the two terms $\partial S_{\alpha\beta}/\partial x_\beta$ and $\partial\overline{\widetilde{s}_\alpha\widetilde{p}}/\partial\varsigma$ that appear in Mellor's (2003) equation cancel. Namely,

$$\frac{\partial}{\partial\varsigma}\left[F_{SS}E^{1/2}\frac{\partial}{\partial x_\alpha}\left(E^{1/2}F_{SS}\right)\right] = -\frac{\partial}{\partial x_\alpha}\left(kDEF_{SS}F_{CS}\right). \tag{E.70}$$

Putting all the pieces of this puzzle together (E.45) can be rewritten in the new coordinate system to obtain a generalization of Mellor's (2003) equation (51a)

$$\begin{aligned}\frac{\partial}{\partial t}(\rho_w DU_\alpha) \quad &+ \quad \frac{\partial}{\partial x_\beta}(\rho_w DU_\alpha U_\beta) + \frac{\partial}{\partial\varsigma}(\rho_w WU_\alpha) + \epsilon_{\alpha 3\beta}f_3DU_\beta \\ &+ \quad D\frac{\partial}{\partial x_\alpha}(\rho_w g\widehat{\zeta} + \widehat{p}^a) + \rho_{w0}D^2\int\left(\frac{\partial b}{\partial x_\alpha} - \varsigma\frac{\partial D}{\partial x_\alpha}\frac{\partial b}{\partial\varsigma}\right) \\ &= \quad -\frac{\partial S_{\alpha\beta}}{\partial x_\beta} + T^{\mathrm{in}}_\alpha + \frac{\partial S^3_\alpha}{\partial\varsigma} + \frac{\partial Z_{\alpha,3}}{\partial\varsigma} + T^{\mathrm{bottom}}_\alpha + T^{p1}_\alpha - (\overline{X}^L_\alpha)^\star,\end{aligned} \tag{E.71}$$

with $(\overline{X}^L_\alpha)^\star$ the transformed viscous and turbulent stresses, and, with the same definition as given by Mellor (2003),

$$S_{\alpha\beta} = kDE\left[\frac{k_\alpha k_\beta}{k^2}F_{CS}F_{CC} + \delta_{\alpha\beta}\left(F_{CS}F_{CC} - F_{SS}F_{CS}\right)\right]. \tag{E.72}$$

and S^3_α corresponds to Mellor's $\overline{\widetilde{s}_\alpha\widetilde{p}}$,

$$S^3_\alpha = E\left(F_{CC} - F_{SS}\right)\frac{\partial F_{SS}}{\partial x_\alpha} + \frac{F_{SS}}{2}\left(F_{CC} - F_{SS}\right)\frac{\partial E}{\partial x_\alpha}. \tag{E.73}$$

Our equation apparently differs from Mellor's due to the term T^{p1} as well as $Z_{\alpha,3}$ and $T^{\mathrm{bottom}}_\alpha$. Although the latter term is readily interpreted physically and integrates to zero over the vertical, the other two are probably due to the missing terms in (E.27)–(E.31) that contribute to S^3_α.

Annexe F: BRAGG SCATTERING OF RANDOM SURFACE GRAVITY WAVES BY IRREGULAR SEABED TOPOGRAPHY

par Fabrice Ardhuin et Thomas H. C. Herbers.
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Annexe G: TOPOGRAPHICAL SCATTERING OF WAVES : A SPECTRAL APPROACH

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Annexe H: A HYBRID EULERIAN-LAGRANGIAN MODEL FOR SPECTRAL WAVE EVOLUTION WITH APPLICATION TO BOTTOM FRICTION ON THE CONTINENTAL SHELF

par Fabrice Ardhuin, Thomas H. C. Herbers et W. C. O'Reilly
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Annexe I: NUMERICAL AND PHYSICAL DIFFUSION : CAN WAVE PREDICTION MODELS RESOLVE DIRECTIONAL SPREAD ?

par Fabrice Ardhuin and Thomas H. C. Herbers
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Annexe J: ON THE EFFECT OF WIND AND TURBULENCE ON OCEAN SWELL

Fabrice Ardhuin et Alastair D. Jenkins
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Annexe K: OBSERVATIONS OF WAVE-GENERATED VORTEX RIPPLES ON THE NORTH CAROLINA CONTINENTAL SHELF

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Annexe L: SWELL TRANSFORMATION ACROSS THE CONTINENTAL SHELF. PART I : ATTENUATION AND DIRECTIONAL BROADENING

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Annexe M: SWELL TRANSFORMATION ACROSS THE CONTINENTAL SHELF. PART II : VALIDATION OF A SPECTRAL ENERGY BALANCE EQUATION

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Annexe N: EXTRACTION OF COASTAL OCEAN WAVE FIELDS FROM SAR IMAGES

par Fabrice Collard, Fabrice Ardhuin et Bertrand Chapron
IEEE Journal of Ocean Engineering, in press

Annexe O: DIRECT MEASUREMENTS OF OCEAN SURFACE VELOCITY FROM SPACE : INTERPRETATION AND VALIDATION

par Bertrand Chapron, Fabrice Collard et Fabrice Ardhuin
Journal of Geophysical Research, vol. 110, C07008, doi :10.1029/2004JC002809, 2005