Prediction of Occurrence of Breaking Waves in Deep Water

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ABSTRACT

A method to evaluate the frequency of occurrence of breaking waves in deep water is developed based on the joint probability distribution of wave excursion and associated time interval for a non-narrow-band random process. Wave breaking that takes place along an excursion crossing the zero-line as well as that which occurs along an excursion above the zero-line is considered. The breaking criterion is obtained from measurements of irregular waves generated in the tank. It is found that the functional relationship between wave height and period at the time of breaking of the irregular waves is by and large different from that known for regular waves. Comparisons between computed and observed frequencies of occurrence of wave breaking made for four different sea conditions generated in the tank show reasonably good agreement. The effect of sea severity on the frequency of occurrence of breaking waves is obtained by carrying out numerical computations using a family of wave spectra. The probability of occurrence of breaking waves depends to a great extent on the shape of the wave spectrum. The probability increases significantly with increase in the fourth moment of the spectrum irrespective of sea severity.

1. Introduction

The phenomenon of breaking waves in the ocean occurs whenever a momentarily high crest reaches an unstable condition. It occurs intermittently, and the frequency of occurrence depends on the severity of the sea.

The significance of information on breaking waves cannot be overemphasized. First of all, breaking waves are always associated with steep waves that occur in a given sea; therefore, consideration of breaking waves cannot be ignored in the statistical prediction of wave height. Unfortunately, currently available statistical prediction methods do not consider the concept of breaking waves. Hence, the largest wave height expected to occur in a specified period of time (the extreme wave height) in a given sea, when predicted by using the current prediction methods, is unreasonably high and is clearly beyond the breaking wave limit. This situation can be rectified by introducing the concept of breaking waves in the prediction of wave height.

Secondly, breaking waves exert by far the largest wave-induced force (often in the form of impacts) on marine systems, which may cause serious safety problems and structural damage of the systems. Hence, it is highly desirable to consider the frequency of occurrence and severity of breaking waves in evaluating hydrodynamic forces for the design of marine systems.

The area addressed by the present study is that concerned with information on the frequency of occurrence of breaking waves in deep water. Breaking waves defined here are not those whose tops are blowing off by the wind but those whose breaking is associated with steepness; more precisely, breaking that results from a steepness for which the wave height exceeds 14.2 percent of the Stokes (1880) limiting wavelength (Michell 1893). These wave phenomena are extremely complicated and the times of their occurrences are usually unpredictable. Hence, they may be most usefully evaluated through the probabilistic approach.

Only a few studies have been carried out on the frequency of occurrence of breaking waves in deep water. Nath and Ramsey (1976) assumed that the wave height H and its period squared T^2 were statistically independent and that both follow the Rayleigh probability law. On the other hand, Houmb and Overvik (1976) evaluated the probability of breaking using the joint probability function of wave height and period applicable for a narrow-band random process.

Although the assumption of the narrow-band random process was used in both of the above studies, the wave records that include very steep waves whose breaking is imminent indicate that the assumption may be highly unrealistic. Hence, in the present study, a method to evaluate the frequency of breaking waves is developed based on the joint probability distribution of wave height and period for a non-narrow-band random process. The method of approach is outlined in the following:

In general, for waves with a non-narrow-band spectrum, it may be well to consider breaking that occurs under two different situations: one that takes place along an excursion above the zero-line, the other that occurs along an excursion crossing the zero-line. This situation is shown in an explanatory sketch in Fig. 1. In the figure, A and B represent the positive maxima and minima, respectively, of a non-narrow-band random process, while C and D represent the negative minima and maxima, respectively. Note that possible occurrences of breaking are expected on the excursion CA as well as BA. These excursions may be called Type I and Type II, respectively. Hence, in order to evaluate the probability of occurrence of breaking, it is necessary to consider the joint probability distribution of the excursion and its associated time interval between two peaks as well as the frequency of occurrence of each type of excursion.

Computations of the probability of occurrence of breaking are made in various sea severities using a family of wave spectra developed from an analysis of wave data measured in the North Atlantic. From the results of the computations, the limiting sea severity is obtained below which no wave breaking is expected to occur. It is also found that the probability of occurrence of breaking increases significantly with increase in the fourth moment of the wave spectrum irrespective of sea severity.

2. Wave-breaking criterion

Prior to developing a formulation for evaluating the frequency of occurrence of wave breaking in irregular seas, it may be well to discuss the criterion of the wave breaking phenomenon that will be used in the prediction.

The criterion for breaking of regular waves in deep water has been addressed from various viewpoints. These include those of Stokes (1880), Mitchell (1893), Longuet-Higgins (1963, 1969, 1974, 1976), Cokelet (1977), etc. The most commonly known breaking criterion is that the wave height should exceed 14.2 percent of the Stokes limiting wavelength which is 20 percent greater than that of ordinary sinusoidal waves of the same frequency. That is, the breaking criterion is given by,

$$H \geqslant 0.142 L_*, \qquad (1)$$

where

- H wave height
- L_* Stokes limiting wave length $= 1.2g/(2\pi f^2)$
- f frequency in Hz.

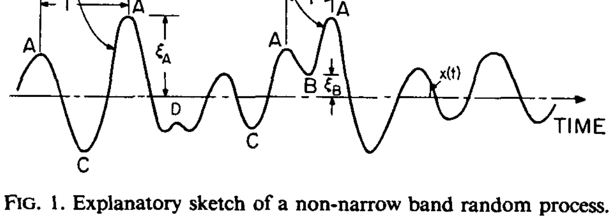

FIG. 1. Explanatory sketch of a non-narrow band random process.

Hence, in terms of wave period, wave breaking occurs when the wave height–period relationship becomes,

$$H \geqslant 0.027 gT^2. \qquad (2)$$

Dean (1968) gives a value of 0.033 for the magnitude of the constant in Eq. (2) based on his computational results using the stream function representation of nonlinear waves. On the other hand, the results of laboratory tests carried out by Van Dorn and Pazan (1975) show that all steep regular waves generated in deep water break at some lower wave height than that given in Eq. (2).

Since there are no specific criteria for breaking of irregular waves, the condition given in Eq. (2) was first used in this study for evaluating the frequency of wave breaking occurrences. However, the results of tests on the frequency of breaking of irregular waves generated in the tank have shown that the observed number of breakings during a specified time period is several times greater than that theoretically computed based on the criterion given in Eq. (2). In order to clarify this difference, an experimental study was carried out on the wave height–period relationship when irregular waves break. It was found that the breaking criterion for irregular waves is substantially different from that known for regular waves. The details of the tests on the breaking criterion applicable for irregular waves are discussed below.

Several series of irregular waves with different wave spectra were generated in the 40-m tank in the Coastal and Oceanographic Engineering Laboratory. Waves were generated by using pre-programmed tapes which yield relatively severe waves leading to breaking. A capacitance-type wave height probe was installed at a fixed point in the tank, and the height and period of incident waves that were expected to break right at that location were measured. Waves that were already broken before they reached the probe were excluded in establishing the breaking criterion. Examples of incident waves for which breaking was imminent are shown in Fig. 2. Included also in the figure are the wave height H, and period T, defined for establishing the criterion.

The wave height–period relationship was obtained from measurements of over 40 incident waves about to break and the results are plotted in Fig. 3. As can be seen in the figure, the functional relationship between wave height and period at the time of breaking of irregular waves is by and large different from that known for regular waves. By drawing the average line of the observed results plotted in Fig. 3, the following relationship is obtained as the criterion for breaking of irregular waves:

$$H \geqslant 0.020 gT^2. \qquad (3)$$

Let us express the criterion given in Eq. (3) in dimensionless form. As stated in the previous section,

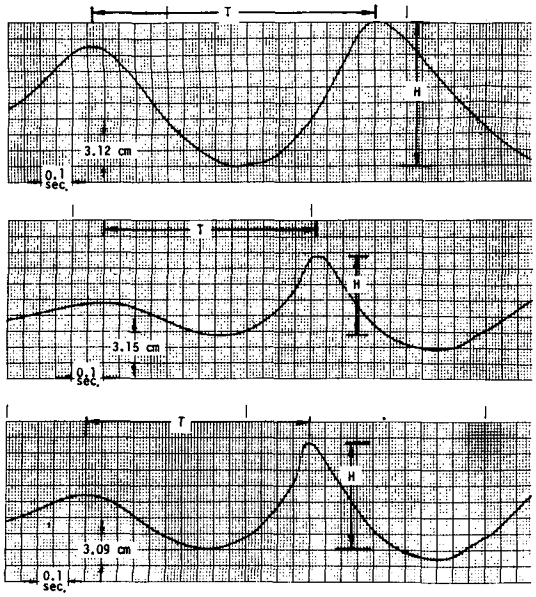

FIG. 2. Examples of irregular waves imminent to breaking.

two different types of breaking are considered in the present study. For this, consider H in Eq. (3) as the Type I as well as Type II excursions, denoted by ζ, and T as the time interval. Then, ζ and T are nondimensionalized by dividing by $m_0^{1/2}$ (where, m_0 is the area under the wave spectrum), and $\bar{T}_m$, where $\bar{T}_m$ is the average time interval between positive maxima, respectively. This results in the following breaking criterion in dimensionless form:

$$\nu \geqslant \alpha\left(\frac{\bar{T}_m^2}{m_0^{1/2}}\right)\lambda^2, \quad (4)$$

where

- ν dimensionless excursion $(=\zeta/m_0{}^{1/2})$
- λ dimensionless time interval between positive maxima $(=T/\bar{T}_m)$
- ζ excursion
- T time interval between positive maxima
- $\bar{T}_m$ average time interval between successive positive maxima A $(=4\pi\{(1-\epsilon^2)^{1/2}/[1+(1-\epsilon^2)^{1/2}]\}(m_0/m_2)^{1/2})$
- ϵ band-width parameter of the spectrum $(=\{1-[m_2^2/(m_0m_4)]\}^{1/2})$
- m_j the jth moment of the wave spectral density function
- α 0.196 m s^{-1}.

3. Probability of occurrence of breaking waves

As was shown in the previous section, the breaking criterion consists of two random variables, the dimensionless excursion ν and time interval λ. Hence, the probability of occurrence of wave breaking can be evaluated from the joint probability density function of excursion and associated time interval.

The joint-probability density function of Type I excursion defined in the Introduction (CA in Fig. 1) and the associated time interval T is essentially the same as that for the positive maxima A and the associated time interval, which was derived by researchers at the Centre National Pour L'Exploitation des Océans (CNEXO) (Arhan *et al.*, 1976). Let us assume the wave profile $x(t)$ illustrated in Fig. 1 to be a stationary Gaussian process which has an arbitrary spectral bandwidth. It is also assumed that $x(t)$ has zero mean, and the derivatives with respect to time, $\dot{x}(t)$ and $\ddot{x}(t)$, both exist with probability one. By considering twice the magnitude of the positive maxima, the joint probability density function can be written as follows:

$$f_I(\nu, \lambda) = \frac{1}{32(2\pi)^{1/2}} \frac{[1+(1-\epsilon^2)^{1/2}]^3}{\epsilon(1-\epsilon^2)} \frac{\nu^2}{\lambda^5} \times \exp\left(-\frac{\nu^2}{8\epsilon^2\lambda^4}\left\{\frac{1}{16}\frac{[1+(1-\epsilon^2)^{1/2}]^4}{1-\epsilon^2} - \frac{1}{2}[1+(1-\epsilon^2)^{1/2}]^2\lambda^2 + \lambda^4\right\}\right), \quad (5)$$

where

$\nu = \zeta/m_0{}^{1/2}$ with $\zeta = 2\zeta_A$, $\quad 0 \leqslant \nu < \infty$

ζ_A = magnitude of the positive maxima, $0 \leqslant \lambda < \infty$.

It should be noted that Eq. (5) includes the positive maxima belonging to both Type I and Type II excursions. Hence, in evaluating the probabiity of occurrence of wave breaking associated with Type I excursions,

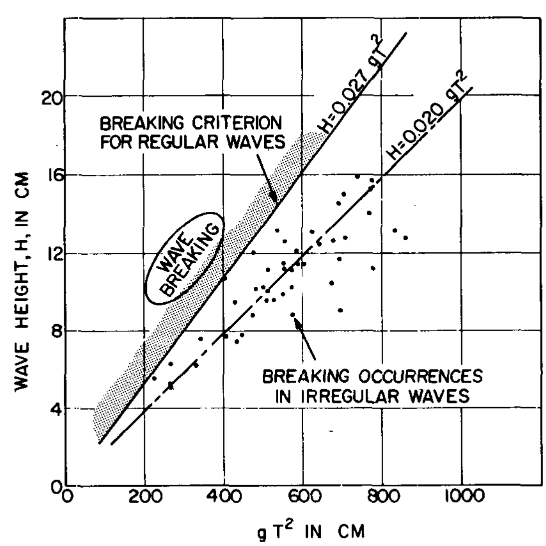

FIG. 3. Relationship between wave height and period for breaking occurrences in irregular waves.

the maxima belonging to Type II excursions have to be deleted. This can be done by taking into consideration the frequency of occurrence of each excursion. That is, the expected number of the positive maxima and minima, denoted by $\bar{N}_A$ and $\bar{N}_B$, respectively, are given by (Cartwright and Longuet–Higgins, 1956),

$$\bar{N}_A = \frac{1}{4\pi}\left[\frac{1+(1-\epsilon^2)^{1/2}}{(1-\epsilon^2)^{1/2}}\right]\left(\frac{m_2}{m_0}\right)^{1/2},$$
$$\bar{N}_B = \frac{1}{4\pi}\left[\frac{1-(1-\epsilon^2)^{1/2}}{(1-\epsilon^2)^{1/2}}\right]\left(\frac{m_2}{m_0}\right)^{1/2}. \quad (6)$$

The expected number of the excursion BA, denoted by $\bar{N}_{BA}$, is equal to $\bar{N}_B$. On the other hand, the expected number of the excursion CA, denoted by $\bar{N}_{CA}$, is equal to the difference of $\bar{N}_A$ and $\bar{N}_B$, which in turn is equal to the expected number of the positive zero-crossings. Then, the expected number of occurrences of Type I and Type II excursions, respectively, is given by,

$$\bar{N}_{CA} = \bar{N}_A - \bar{N}_B = \frac{1}{2\pi}\left(\frac{m_2}{m_0}\right)^{1/2},$$
$$\bar{N}_{BA} = \bar{N}_B = \frac{1}{4\pi}\left[\frac{1-(1-\epsilon^2)^{1/2}}{(1-\epsilon^2)^{1/2}}\right]\left(\frac{m_2}{m_0}\right)^{1/2}. \quad (7)$$

Hence, the frequency of occurrence of Type I and Type II excursions, denoted by p_I and p_{II} becomes

$$p_I = \frac{\bar{N}_{CA}}{\bar{N}_A} = \frac{2(1-\epsilon^2)^{1/2}}{1+(1-\epsilon^2)^{1/2}},$$
$$p_{II} = \frac{\bar{N}_{BA}}{\bar{N}_A} = \frac{1-(1-\epsilon^2)^{1/2}}{1+(1-\epsilon^2)^{1/2}}. \quad (8)$$

Thus, the conditional probability of breaking waves given that Type I excursion has occurred can be evaluated from (5) and (8) taking into consideration of the breaking condition given in (4). That is,

Pr{breaking wave, Type I excursion}

$$= p_I \int_0^\infty \int_{\alpha(\bar{T}_m^2/m_0^{1/2})\lambda^2}^\infty f_I(\nu, \lambda) d\nu d\lambda. \quad (9)$$

Then, the probability of breaking waves associated with Type I excursion can be obtained by multiplying Eq. (9) by the probability of occurrence of Type I excursion. By taking account of the fact that the sum of the probabilities of Type I and Type II breaking waves is equal to unity when all waves have broken, we have,

Pr{breaking waves, Type I}

$$= \frac{p_I^2}{p_I^2 + p_{II}} \int_0^\infty \int_{\alpha(\bar{T}_m^2/m_0^{1/2})\lambda^2}^\infty f_I(\nu, \lambda) d\nu d\lambda. \quad (10)$$

For evaluating the breaking waves associated with Type II excursions, the joint probability density function of the excursion BA and the time interval is required. It is given in a dimensionless form as follows:

$$f_{II}(\nu, \lambda) = \frac{k(\epsilon)}{16(\pi)^{1/2}} \frac{[1+(1-\epsilon^2)^{1/2}]^5}{\epsilon(1-\epsilon^2)[\epsilon^2 + 2(1-\epsilon^2)^{1/2}]} \frac{\nu^2}{\lambda^5} \times \exp\left\langle -\frac{\nu^2}{2\epsilon^2}\left([1-\beta(\epsilon,\lambda)]^2 + \left\{\frac{\epsilon}{8}\frac{[1+(1-\epsilon^2)^{1/2}]^2}{(1-\epsilon^2)^{1/2}}\frac{1}{\lambda^2}\right\}^2\right)\right\rangle$$
$$\times \Bigg[\exp\left\{\frac{\lambda^2}{4\epsilon^2}[1-\beta(\epsilon,\lambda)]^2\right\} \cdot \left\langle 1 - \Phi\left\{\frac{\nu}{(2)^{1/2}\epsilon}[1+\beta(\epsilon,\lambda)]\right\}\right\rangle + \frac{[2(1-\epsilon^2)]^{1/2}}{1+\epsilon^2}\exp\left\{\frac{\nu^2}{2\epsilon^2}[(1-\epsilon^2) - 2\beta(\epsilon,\lambda)]\right\}$$
$$+ \frac{2(\pi)^{1/2}(1-\epsilon^2)^{1/2}}{\epsilon(1+\epsilon^2)^{3/2}} \nu[1-\beta(\epsilon,\lambda)] \cdot \exp\left\{\frac{\nu^2}{2(1+\epsilon^2)\epsilon^2}[1-\beta(\epsilon,\lambda)]\right\} \times \left\langle 1 - \Phi\left\{\frac{\nu}{\epsilon(1+\epsilon^2)^{1/2}}[\epsilon^2 + \beta(\epsilon,\lambda)]\right\}\right\rangle \Bigg],$$
$$0 \le \nu < \infty, \quad 0 \le \lambda < \infty, \quad (11)$$

where $\nu = \zeta/m_0^{1/2}$ with $\zeta = \xi_A - \xi_B$, ξ_B = magnitude of the positive minima and

$$\beta(\epsilon, \lambda) = \left(\frac{1+(1-\epsilon^2)^{1/2}}{2(2)^{1/2}}\frac{1}{\lambda}\right)^2.$$

The derivation of the joint probability density function is presented in Appendix A.

The probability of breaking waves can be evaluated from Eq. (11) by a procedure similar to that given in Eq. (10). That is,

Pr{Breaking waves, Type II}

$$= \frac{p_{II}}{p_I^2 + p_{II}} \int_0^\infty \int_{\alpha(\bar{T}_m^2/m_0^{1/2})\lambda^2}^\infty f_{II}(\nu, \lambda) d\nu d\lambda. \quad (12)$$

The total probability of breaking waves is the sum of the probabilities given in (10) and (12).

4. Computation of breaking waves in various sea severities

It is of interest to examine the effect of sea severity on the frequency of occurrence of breaking waves. For this, computations are carried out using a family of wave spectra for a specified sea severity. The family of wave spectra consists of eleven members including the one that is most likely to occur for a specified significant wave height, and it represents a variety of shapes of wave spectra observed in the North Atlantic Ocean (Ochi and Hubble, 1976).

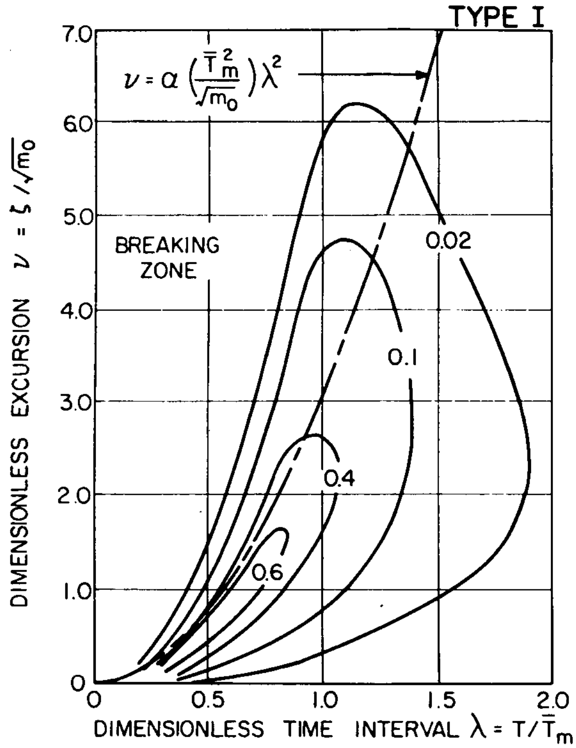

FIG. 4. Joint probability density function of the excursion and time interval for Type I excursion.

Prior to presenting the results of the computation, it may be well to show an example of the joint probability density function of the excursion and time interval used for evaluating the probability of breaking waves. Figs. 4 and 5 show the joint probability density functions for Type I and Type II excursions, respectively, of waves of significant height 12.2 m (40 ft) whose spectrum is shown in Fig. 6. Included also in each figure is the line indicating the breaking condition given in Eq. (4). If the wave excursion exceeds this line for a given time interval (this region is denoted by the breaking zone in the figure), then breaking takes place.

As shown in Eq. (10), the probability of breaking waves associated with Type I excursion is the product of the volume of the joint probability density function in the breaking zone indicated in Fig. 4 and the square of the frequency of occurrence of Type I excursion, which is equal to 40.66% for this example. On the other hand, the probability of breaking waves associated with Type II excursion is the product of the volume in the breaking zone indicated in Fig. 5 and the frequency of occurrence of Type II excursion, which is equal to 10.95% for this example. Thus, it appears that the probability of breaking waves of Type II excursion is approximately 26.9% of that associated with Type I excursion.

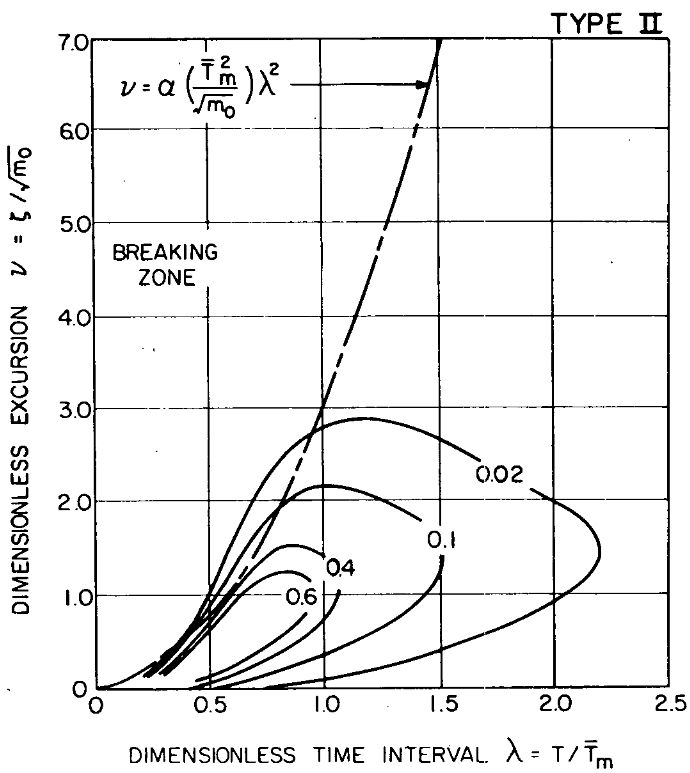

FIG. 5. Joint probability density function of the excursion and time interval for Type II excursion.

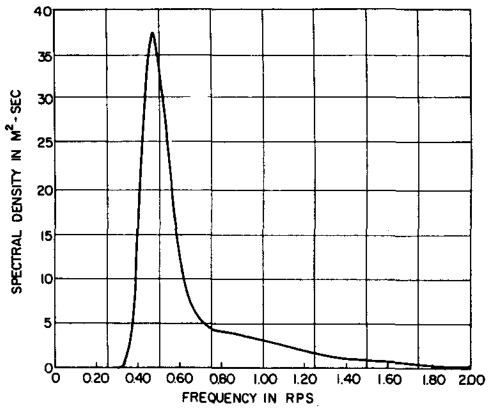

FIG. 6. Wave spectrum used for computation of the joint probability density functions shown in Figs. 4 and 5. Significant wave height 12.2 m.

Figure 7 shows the results of computation indicating the probabilities of occurrence of breaking in various sea severities. As can be seen in the figure, the probability of breaking increases significantly with increase in sea severity, in general, and no wave breaking is expected in seas of significant wave height less than approximately 4 m (13.1 ft) for this family of wave spectra.

It is noted in Fig. 7 that the probability of breaking waves varies considerably in a given sea severity depending on the shape of the wave spectrum. For example, in a sea of significant wave height 10.7 m (35 ft), the probability ranges from 3.97 to 22.92% with a most probable of 10.38% which is computed by using the most probable wave spectrum of the family. Fig. 8 shows three spectra of the family for the sea of significant wave height 10.7 m which yield the largest, most probable, and smallest probability of breaking identified as Spectra I, II, and III, respectively.

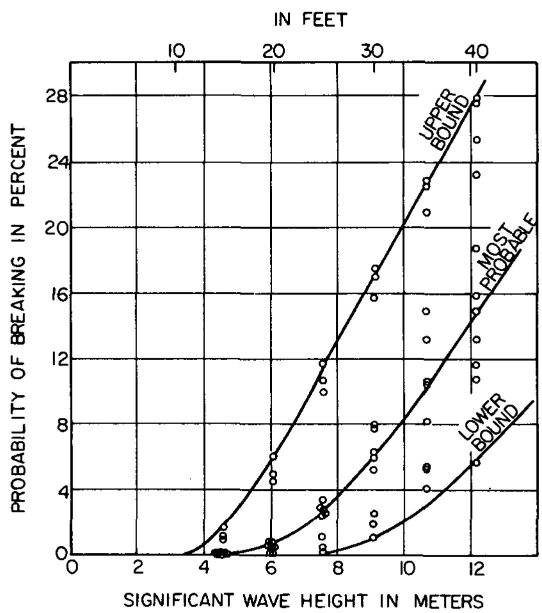

FIG. 7. Probability of occurrence of breaking waves using the six-parameter wave spectrum family.

The results of computations in various sea severities show that the frequency of occurrence of breaking waves is a function of the fourth moment m_4 of the wave spectrum. In order to elaborate on this statement, Fig. 9 shows the probabilities of occurrence of breaking in six different sea states of the six-parameter wave spectra family plotted against the dimensionless fourth moment of each spectrum. As can be seen in the figure, the frequency of breaking increases significantly with increase in the fourth moment of the wave spectrum.

The conclusion given in the above paragraph can be verified from the breaking condition given in Eq. (4). That is, from the definition of the average time interval $\bar{T}_m$, Eq. (4) can be expressed in the following form:

$$\nu \geq (4\pi)^2\alpha\left[\frac{(1-\epsilon^2)^{1/2}}{1+(1-\epsilon^2)^{1/2}}\right]^2\frac{m_0^{1/2}}{m_2}\lambda^2$$

$$= (4\pi)^2\alpha\frac{(1-\epsilon^2)^{1/2}}{[1+(1-\epsilon^2)^{1/2}]^2}\frac{\lambda^2}{m_4^{1/2}}. \qquad (13)$$

It is noted that $(1-\epsilon^2)^{1/2}/[1+(1-\epsilon^2)^{1/2}]^2$ in the above equation is almost constant for the practical range of ϵ-value of ocean waves, say 0.3–0.8. Therefore, the ν-value reduces in proportion to the square of the increase in the fourth moment m_4 of the wave spectrum. This implies that a significant increase of the probability of breaking is expected with increase in the fourth moment.

In order to substantiate the conclusion given above, computations of the probability of occurrence of breaking as well as the joint probability function of wave excursion and time interval are carried out for

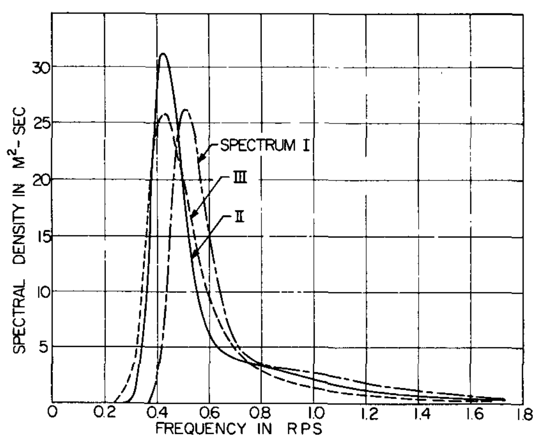

FIG. 8. Spectra which yield the largest (I), most probable (II) and the smallest (III) probability of breaking in seas of significant wave height 10.7 m of the six-parameter wave spectra family.

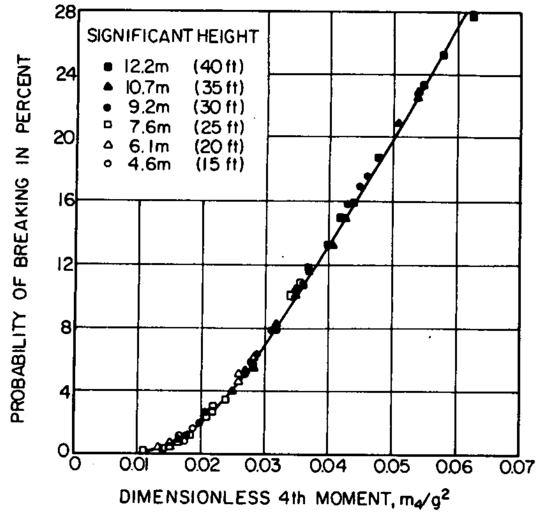

FIG. 9. Probabilities of occurrence of breaking waves plotted against the dimensionless fourth moment of each spectrum.

four spectra measured in the North Atlantic. Two of the spectra, NW 1 and NW 154 shown in Fig. 10, are those in a sea of significant wave height 4.6 m, while the other two spectra, NW 314 and JHC 75 shown in Fig. 11, are those in a sea of 10.8 m. The results of the computation show that there is almost no wave breaking for the two spectra in the sea of significant wave height 4.6 m (1.1 percent for NW 1 and none for NW 154). On the other hand, the probability of breaking is 11.90 and 13.05% for the spectrum NW 134 and JHC 75, respectively, in the sea of significant wave height 10.8 m. A comparison of the joint probability function of Type I excursion and time interval $f(\nu, \lambda)$ for the spectrum NW 154 (significant wave height 4.6 m) and JHC 75 (significant wave height 10.8 m) is shown in Fig. 12. Included also in the figure is the line indicating the breaking condition. As can be seen in the example shown in Fig. 12, the shapes of the dimensionless joint probability density function of excursion and time interval are nearly equal for

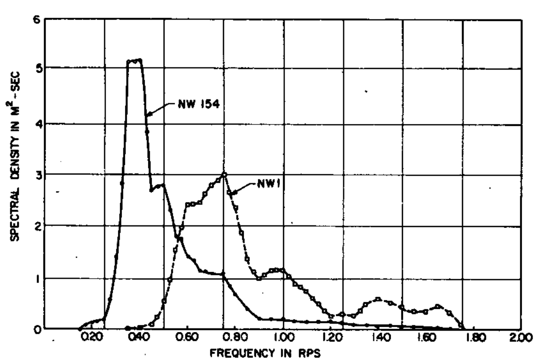

FIG. 10. Spectra of significant wave height 4.6 m measured in the North Atlantic.

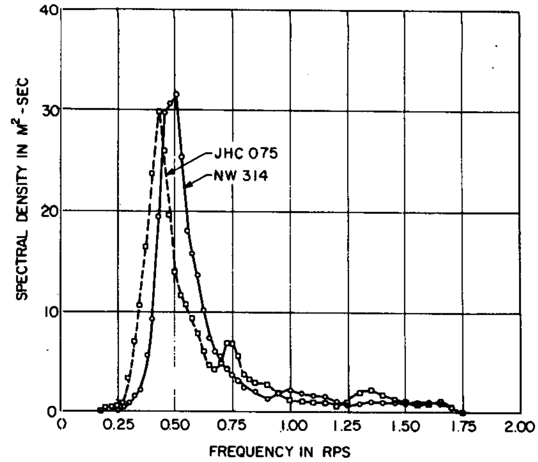

FIG. 11. Spectra of significant wave height 10.8 m measured in the North Atlantic.

these sea states. However, the location of the line indicating the breaking condition differs substantially as shown in the figure depending on the magnitude of the fourth moment of the spectrum. This in turn yields the difference in the probability of breaking according to the formulation given in Eq. (13).

It is of interest to note that the relationship between the probability of breaking and the fourth moment of the spectrum evaluated for various spectra measured in the North Atlantic agrees very well with the functional relationship given in Fig. 9 which is obtained using the six-parameter family of wave spectra.

Since breaking increases significantly with increase in the fourth moment of the wave spectrum, in general, breaking is very sensitive to the existence of high-frequency energy. In other words, breaking is more likely to occur in high-frequency waves, and this agrees with the concept of the equilibrium range of a spectrum (Phillips, 1958).

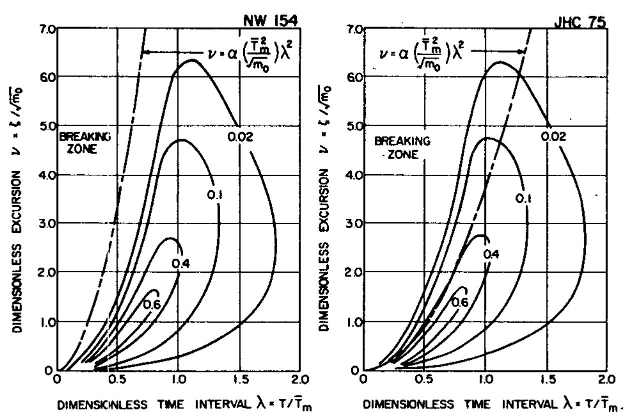

FIG. 12. Comparison of the joint probability density function of Type I excursion and time interval for spectrum NW 154 (significant height 4.6 m) and JHC 75 (significant height 10.8 m).

5. Experiments on the frequency of breaking waves

An experimental study was carried out in which irregular waves were generated in the tank to examine whether or not the observed frequencies of occurrence of breaking waves agree with theoretical results. Unfortunately, because of the limitation in the capability of the wave-maker, it was difficult to generate high-frequency component waves that were sufficiently steep to produce breaking associated with Type II excursions. Although records taken during the tests showed a number of Type II excursions in the generated waves, their steepness was insufficient to result in breaking. This is in agreement with the results of computations carried out on the generated waves which indicate the probability of breaking of Type II excursions is extremely small.

In order to compare the computed and observed frequencies of wave breaking, two wave probes were installed in the tank separated by a distance of 14.6 m (48 ft). An example of wave spectra measured at these two locations is shown in Fig. 13. As can be seen in the figure, wave energy was substantially reduced because of wave breaking that took place while the waves were traveling between the two locations, and this results in a reduction in the magnitude of significant wave height from 8.96 cm to 7.32 cm. The probability of breaking waves at each location was computed by using the spectrum measured at that location. The probability of breaking waves thus computed was compared with that obtained experimentally. The latter was evaluated by dividing the number of waves visually observed to break between these two locations during the tests by the total number of waves. The results of the comparisons made for four (4) different sea conditions generated in the tank are tabulated as follows:

Sea condition	I	II	III	IV
Experimentally obtained probability of breaking	0.060	0.015	0.051	0.040
Computed probability of breaking	0.048	0.021	0.043	0.045

As can be seen in the above table, reasonably good agreement is obtained between the theoretical and experimentally obtained results.

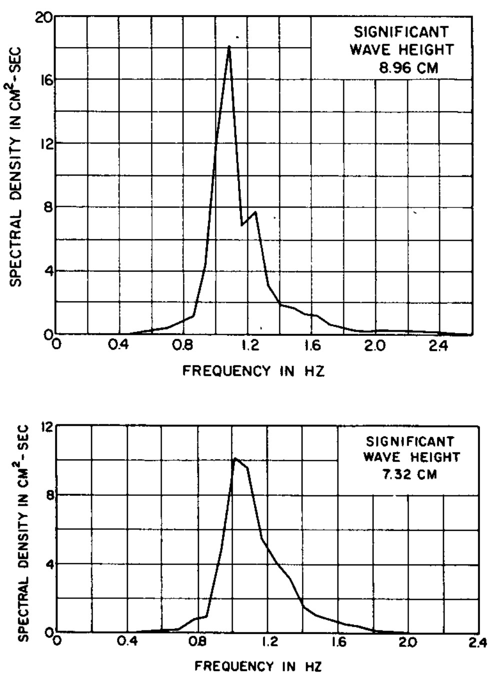

FIG. 13. Examples of spectra measured at two locations in the tank.

6. Conclusions

This paper discusses a method to evaluate the frequency of occurrence of breaking waves in deep water. Breaking waves defined in the present study are those associated with steepness, and the breaking criterion was obtained from measurements of irregular waves generated in the tank. It was found that the functional relationship between wave height and period at the time of breaking of the irregular waves is by and large different from that known for regular waves. From the results of the experimental study, the following relationship is obtained as the criterion for breaking of irregular waves:

$$H \geqslant 0.020gT^2.$$

In the derivation of the prediction formula, the joint probability distribution of wave excursion and associated time interval for a non-narrow-band random process is used. Wave breaking that takes place along an excursion crossing the zero-line (Type I excursion) as well as that which occurs along an excursion above the zero-line (Type II excursion) is considered.

Comparisons between computed and observed frequencies of occurrence of wave breaking made for four different sea conditions generated in the tank show reasonably good agreement.

Computations of the probability of occurrence of breaking are made in various sea severities using a family of wave spectra developed from analysis of wave data measured in the North Atlantic Ocean. From results of computations, the following conclusions may be drawn:

1) In evaluating the probability of breaking waves, the frequencies of occurrence of Type I and Type II excursions have to be taken into consideration. The probability of breaking waves associated with Type II excursion cannot be neglected; for some wave spectra,

the probability is approximately 27% of that associated with the Type I excursion.

2) The results of computations indicate that no wave breaking is expected in seas of significant wave height less than 4 m for this family of spectra.

3) The probability of occurrence of breaking waves depends to a great extent on the shape of the wave spectrum. The fourth moment of the spectrum is a dominant parameter which influences the occurrence of breaking. The probability increases significantly with increase in the fourth moment of the spectrum irrespective of sea severity.

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APPENDIX A

Joint Probability Density Function of Type II Excursion and Associated Time Interval

In order to derive the joint probability density function of Type II excursion (BA in Fig. 1) and the associated time interval T we may first consider the joint probability density function of the positive minima B and the associated acceleration. The joint probability density function will then be transformed to the joint probability density function of the positive minima and the time interval, T.

Let $\bar{N}_B$ be the expected number of positive minima per unit time, and let $\bar{N}_{\xi_B,\ddot{x}}$ be the expected number of positive minima per unit time above a level ξ_B with its associated acceleration $\ddot{x}$. These are given as follows:

$$\bar{N}_B = \int_0^\infty \int_0^\infty \ddot{x} f(x, 0, \ddot{x}) d\ddot{x} dx, \qquad \text{(A1)}$$

$$\bar{N}_{\xi_B \ddot{x}} = \int_{\xi_B}^\infty \ddot{x} f(x, 0, \ddot{x}) dx,$$

where $f(x, \dot{x}, \ddot{x})$ is the joint probability density function of the displacement, velocity and acceleration of a Gaussian random process with zero mean and the covariance matrix,

$$\mathbf{\Sigma} = \begin{pmatrix} m_0 & 0 & -m_2 \\ 0 & m_2 & 0 \\ -m_2 & 0 & m_4 \end{pmatrix}.$$

Then, the joint probability density function of the positive minima ξ_B and the associated acceleration $\ddot{x}$ is given by

$$f(\xi_B, \ddot{x}) = \frac{d}{d\xi}\left[1 - \frac{\bar{N}_{\xi_B}, \ddot{x}}{\bar{N}_B}\right] = \frac{\ddot{x} f(\xi_B, 0, \ddot{x})}{\int_0^\infty \int_0^\infty \ddot{x} f(x, 0, \ddot{x}) d\ddot{x} dx}. \qquad \text{(A2)}$$

By carrying out the integration involved in the denominator of (A2), we have,

$$f(\xi_B, \ddot{x}) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\ddot{x}}{(\Delta m_4)^{1/2}[1 - (1 - \epsilon^2)^{1/2}]} \times e^{-(1/2\Delta)(m_4\xi_B^2 + 2m_2\xi\ddot{x} + m_0\ddot{x}^2)}$$

$$0 \leq \xi_B < \infty, \quad 0 \leq \ddot{x} < \infty \qquad \text{(A3)}$$

where $\Delta = m_0 m_4 - m_2^2$.

Next, consider the joint probability density function of the positive maxima ξ_A the positive minima ξ_B and associated acceleration $\ddot{x}$. Since it may be safely assumed that the positive maxima and the positive minima are statistically independent, the joint probability density function is given simply by the product of two probability density functions, $f(\xi_A)$ and $f(\xi_B, \ddot{x})$. Here, the probability density function of the positive maxima is given by [Ochi, 1973],

$$f(\xi_A) = \frac{\dfrac{2}{m_0^{1/2}}}{1 + (1 - \epsilon^2)^{1/2}} \left\langle \frac{\epsilon}{(2\pi)^{1/2}} \exp\left[-\frac{1}{2\epsilon^2}\left(\frac{\xi_A}{m_0^{1/2}}\right)^2\right] + (1 - \epsilon^2)^{1/2}\left(\frac{\xi_A}{m_0^{1/2}}\right) \exp\left[-\frac{1}{2}\left(\frac{\xi_A}{m_0^{1/2}}\right)^2\right] \times \left\{1 - \Phi\left[-\frac{(1 - \epsilon^2)^{1/2}}{\epsilon}\left(\frac{\xi_A}{m_0^{1/2}}\right)\right]\right\}\right\rangle,$$

$$0 \leq \xi_A < \infty, \qquad \text{(A4)}$$

where

$$\Phi(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^u e^{-t^2/2} dt.$$

Hence, from (A3) and (A4), we have,

$$f(\xi_A, \xi_B, \ddot{x}) = \frac{2(2)^{1/2}}{(\pi)^{1/2}} \frac{(m_0 m_4)^{1/2}}{\Delta^{3/2}} \ddot{x} \left\langle \frac{\epsilon}{(2\pi)^{1/2}} \exp\left[-\frac{1}{2\epsilon^2}\left(\frac{\xi_A}{m_0^{1/2}}\right)^2\right] + (1 - \epsilon^2)^{1/2}\left(\frac{\xi_A}{m_0^{1/2}}\right) \exp\left[-\frac{1}{2}\left(\frac{\xi_A}{m_0^{1/2}}\right)^2\right] \times \left\{1 - \Phi\left[-\frac{(1 - \epsilon^2)^{1/2}}{\epsilon}\left(\frac{\xi_A}{m_0^{1/2}}\right)\right]\right\}\right\rangle \times \exp\left[-\frac{1}{2\Delta}(m_4\xi_B^2 + 2m_2\xi_B\ddot{x} + m_0\ddot{x}^2)\right]. \qquad \text{(A5)}$$

In order to facilitate the integration involved in further analysis, the following approximation is made in Eq. (A5);

$$1 - \Phi\left(-\frac{(1-\epsilon^2)^{1/2}}{\epsilon}\frac{\xi_A}{m_0^{1/2}}\right) \sim 1. \quad \text{(A6)}$$

This assumption results in Eq. (A5) no longer being the probability density function. In order to maintain the condition required for (A5) to be a probability density function, it should be multiplied by a factor which is derived from the condition given by

$$\int_0^\infty \int_0^\infty \int_0^\infty f(\xi_A, \xi_B, \ddot{x}) d\xi_A d\xi_B d\ddot{x} = 1. \quad \text{(A7)}$$

The above approximation does not cause any serious effect on the joint probability density function of waves whose dimensionless height is less than unity. Thus, the approximate joint probability density function of ξ_A, ξ_B, and $\ddot{x}$ becomes,

$$f(\xi_A, \xi_B, \ddot{x}) = \left(\frac{1+(1-\epsilon^2)^{1/2}}{\epsilon^2 + 2(1-\epsilon^2)^{1/2}}\right)\frac{2(2)^{1/2}}{(\pi)^{1/2}}\frac{(m_0 m_4)^{1/2}}{\Delta^{3/2}} \times \ddot{x}\left\{\frac{\epsilon}{(2\pi)^{1/2}}\exp\left[-\frac{1}{2\epsilon^2}\left(\frac{\xi_A}{m_0^{1/2}}\right)^2\right] + (1-\epsilon^2)^{1/2}\left(\frac{\xi_A}{m_0^{1/2}}\right)\exp\left[-\frac{1}{2}\left(\frac{\xi_A}{m_0^{1/2}}\right)^2\right]\right\} \times \exp\left[-\frac{1}{2\Delta}(m_4\xi_B^2 + 2m_2\xi_B\ddot{x} + m_0\ddot{x}^2)\right]. \quad \text{(A8)}$$

Next, let us consider the excursion BA, denoted by $\zeta = \xi_A - \xi_B$, and derive the joint probability density function of ξ_A, ζ, and $\ddot{x}$. This can be simply obtained by substituting $(\xi_A - \zeta)$ for ξ_B in (A8). Then, the joint probability density function of the excursion ζ and the acceleration $\ddot{x}$ can be obtained as the marginal probability density function of $f(\xi_A, \zeta, \ddot{x})$. That is,

$$f(\zeta, \ddot{x}) = \int_0^\infty f(\xi_A)\cdot f(\xi_A - \zeta, \ddot{x})d\xi_A + \int_\zeta^\infty f(\xi_A)\cdot f(\xi_A - \zeta, \ddot{x})d\xi_A,$$

$$-\infty < \zeta = \xi_A - \xi_B < \infty, \quad 0 \leqslant \ddot{x} < \infty. \quad \text{(A9)}$$

The first term of Eq. (A9) is for $\zeta < 0$, and the second term is for $\zeta > 0$. Since wave breaking is associated with $\xi_A > \xi_B$, namely $\zeta > 0$, we consider only the second term for the present problem. The integration of the second term of (A9) yields,

$$\int_\zeta^\infty f(\xi_A)\cdot f(\xi_A - \zeta, \ddot{x})d\xi_A = \left(\frac{1+(1-\epsilon^2)^{1/2}}{\epsilon^2 + 2(1-\epsilon^2)^{1/2}}\right)\frac{2}{(\pi)^{1/2}}\frac{\ddot{x}}{(\Delta m_4)^{1/2}}\exp\left[-\frac{1}{2\Delta}(m_4\zeta^2 - 2m_2\zeta\ddot{x} + m_0\ddot{x}^2)\right]$$
$$\times\left[\exp\left[\frac{(m_4\zeta - m_2\ddot{x})^2}{4\Delta m_4}\right]\left\{1 - \Phi\left[\left(\frac{2m_4}{\Delta}\right)^{1/2}\left(\zeta - \frac{m_4\zeta - m_2\ddot{x}}{2m_4}\right)\right]\right\} + \frac{[2(1-\epsilon^2)]^{1/2}}{1+\epsilon^2}\exp\left[-\frac{1+\epsilon^2}{2}\left(\frac{m_4}{\Delta}\right)\zeta^2 + \frac{1}{\Delta}(m_4\zeta - m_2\ddot{x})\zeta\right] + \frac{2(\pi)^{1/2}(1-\epsilon^2)^{1/2}}{(1+\epsilon^2)^{3/2}}\frac{m_4\zeta - m_2\ddot{x}}{(\Delta m_4)^{1/2}}\exp\left[\frac{1}{1+\epsilon^2}\frac{(m_4\zeta - m_2\ddot{x})^2}{2\Delta m_4}\right] \times\left\{1 - \Phi\left[\frac{(1+\epsilon^2)^{1/2}}{\epsilon}\frac{1}{m_0^{1/2}}\left(\zeta - \frac{1}{1+\epsilon^2}\frac{m_4\zeta - m_2\ddot{x}}{m_4}\right)\right]\right\}\right],$$

$$0 \leqslant \zeta < \infty, \quad 0 \leqslant \ddot{x} < \infty, \quad \text{(A10)}$$

It is noted that Eq. (A10) is derived by truncating the probability density function given in (A9) at $\zeta = 0$. Hence, (A10) has to be multiplied by a constant such that it satisfies the condition required to being a probability density function. Taking this condition into consideration, (A10) can be expressed in the following dimensionless form:

$$f(\nu, \tau) = k(\epsilon)\left[\frac{1+(1-\epsilon^2)^{1/2}}{\epsilon^2 + 2(1-\epsilon^2)^{1/2}}\right]\frac{2}{(\pi)^{1/2}}\frac{\tau}{\epsilon}\exp\left\{-\frac{1}{2\epsilon^2}[\nu^2 - 2(1-\epsilon^2)^{1/2}\nu\tau + \tau^2]\right\}$$
$$\times\left[\exp\left\{\frac{1}{4\epsilon^2}[\nu - (1-\epsilon^2)^{1/2}\tau]^2\right\}\left\langle 1 - \Phi\left\{\frac{1}{(2)^{1/2}\epsilon}[\nu + (1-\epsilon^2)^{1/2}\tau]\right\}\right\rangle + \frac{[2(1-\epsilon^2)]^{1/2}}{1+\epsilon^2} \times\exp\left\{\frac{(1-\epsilon^2)^{1/2}}{2\epsilon^2}[(1-\epsilon^2)^{1/2}\nu^2 - 2\nu\tau]\right\} + \frac{2(\pi)^{1/2}(1-\epsilon^2)^{1/2}}{\epsilon(1+\epsilon^2)^{3/2}}[\nu - (1-\epsilon^2)^{1/2}\tau] \times\exp\left\{\frac{1-\epsilon^2}{2(1+\epsilon^2)\epsilon^2}[\nu - (1-\epsilon^2)^{1/2}\tau]^2\right\} \times\left\langle 1 - \Phi\left\{\frac{1}{\epsilon(1+\epsilon^2)^{1/2}}[\epsilon^2\nu + (1-\epsilon^2)^{1/2}\tau]\right\}\right\rangle\right],$$

$$0 \leqq \nu \leqq \infty, \quad 0 \leqq \tau < \infty, \quad \text{(A11)}$$

where ν = dimensionless excursion = $\zeta/m_0^{1/2}$ with $\zeta = \xi_A - \xi_B$, τ = dimensionless acceleration = $\ddot{x}/m_4^{1/2}$.

The function $k(\epsilon)$ in the above equation is associated with the truncation of the probability density function at $\zeta = 0$, and it is given by,

$$k(\epsilon) = \frac{1}{\int_0^\infty \int_0^\infty f(\nu, \tau) d\nu d\tau}. \quad \text{(A12)}$$

The results of computations using Eq. (A12) for various ϵ-values have shown that $k(\epsilon)$ can be approximately expressed by the following function of ϵ with sufficient accuracy in the practical range of ϵ-values for wind-generated seas, say, $0.3 \leq \epsilon < 0.9$:

$$k(\epsilon) = 1.13 + 0.011 e^{6.68(\epsilon - 0.4)}. \quad \text{(A13)}$$

Next, the joint probability density function given in (A11) can be converted from an acceleration to a time interval by the same procedure as was used for the derivation of the joint probability density function of Type I excursion and the associated time interval (Arhan *et al.*, 1976; Cavanié *et al.*, 1976; Ezraty *et al.*, 1977).

$$\ddot{x} = \frac{\xi_A - \xi_B}{2} \left(\frac{2\pi}{T}\right)^2. \quad \text{(A14)}$$

In the dimensionless form, Eq. (A14) yields,

$$\tau = \frac{1}{8} \frac{[1 + (1 - \epsilon^2)^{1/2}]^2}{(1 - \epsilon^2)^{1/2}} \frac{\nu}{\lambda^2}, \quad \text{(A15)}$$

where λ is defined in (4).

From Eq. (A11) and (A15) the joint probability density function of ν and τ can be transformed to that of ν and λ which is the desired joint probability density function of the Type II excursion given in (11).

APPENDIX B

Nomenclature

- f — wave frequency in Hz,
- $f(x, \dot{x}, \ddot{x})$ — joint probability function of the displacement, velocity and acceleration of a Gaussian random process,
- $f(\zeta, \ddot{x})$ — joint probability function of the excursion and the associated acceleration,
- $f(\xi_A, \xi_B, \ddot{x})$ — joint probability density function of the positive maxima, the positive minima and the associated acceleration,
- $f(\xi_B, \ddot{x})$ — joint probability function of the positive minima and the associated acceleration,
- $f(\xi_A)$ — probability density function of the positive maxima,
- $f_I(\nu, \lambda), f_{II}(\nu, \lambda)$ — joint probability density function for the dimensionless Type I or Type II excursion and the associated time interval,
- g — gravitational acceleration,
- H — wave height,
- L_* — Stokes limiting wave length,
- m_j — the jth moment of the wave spectral density function,
- $\bar{N}_A, \bar{N}_B$ — expected number of the positive maxima and minima per unit time, respectively,
- $\bar{N}_{CA}, \bar{N}_{BA}$ — expected number of Type I and Type II excursion per unit time, respectively,
- $\bar{N}_{\xi_B, \ddot{x}}$ — the expected number of positive minima per unit time above a specified level ξ_B with its associated acceleration $\ddot{x}$,
- p_I, p_{II} — probability of occurrence of Type I or Type II excursion, respectively,
- T — wave period or time interval between successive positive maxima,
- $\bar{T}_m$ — average time interval between successive positive maxima,
- $x, \dot{x}, \ddot{x}$ — displacement, velocity and acceleration of the water surface,
- α — constant associated with dimensionless wave breaking condition,
- Δ — $m_0 m_4 - m_2^2$,
- Σ — covariance matrix of the Gaussian random process,
- ϵ — band-width parameter of the spectrum,
- ζ — excursion,
- λ — dimensionless time interval between positive maxima, $T/\bar{T}_m$,
- ν — dimensionless excursion $\zeta/m_0^{1/2}$,
- ξ_A, ξ_B — positive maxima and positive minima, respectively,
- τ — dimensionless acceleration, $\ddot{x}/m_4^{1/2}$,
- $\Phi(u)$ — $\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{u} e^{-t^2/2} dt.$

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