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A comparison of two models for surface-wave propagation over rapidly varying topography

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Abstract: Comparisons are made between the predictions of two models for surface-wave propagation over rapidly-varying bottom topography, one based on the extended-mild-slope equation derived by Kirby (Kirby, J T *J Fluid Mechanics*, **162** (1986) 171–86),[1] and the other on the successive-application-matrix model described by O'Hare & Davies (O'Hare, T J & Davies A G. *Coastal Engineering*, **18** (1992) 251–66) [2] The models are applied to two types of undulating topography, namely sinusoidal and doubly-sinusoidal beds, and comparisons are made with existing laboratory data Both models provide similar, accurate, predictions for the first-order resonant reflection of surface waves having wavelength equal to approximately twice that of the sinusoidal bed components Agreement between the models and data is less good for higher-order resonances, due to the (different) formulations of the bottom boundary condition used in the models. In particular, some disagreement arises both when the surface wavelength is approximately equal to that of a bed component, and also when it corresponds to the sub-harmonic 'difference' wavelength. Generally, the successive-application-matrix model, which provides a more explicit formulation of the wave propagation problem than the extended-mild-slope equation model, gives better predictions of the data, but is computationally more demanding

1 INTRODUCTION

In recent years, considerable interest has been shown in the phenomenon of resonant reflection of surface waves by undulating bottom topography. Theoretical work by Davies,[3] and experiments by Davies & Heathershaw,[4] revealed that waves may be strongly reflected when their wavelength is approximately equal to twice that of the bottom undulations This process, which is analogous to the Bragg scattering of X-rays from crystal planes, may have important consequences for coastal protection. It has been suggested (e.g. Heathershaw & Davies,[5] Mei[6]) that wave reflection by offshore bars may provide a mechanism for the formation of new bars in the up-wave direction, and may protect a coastline from the full impact of incident waves. The evolution of bars beneath partially-standing waves, and the consequent enhancement of incident wave reflection has been demonstrated in the laboratory (O'Hare & Davies,[7] O'Hare[8]) In addition, the possible use of man-made bars to protect oil platforms and exposed coastlines has been discussed by Mei *et al*[9] and Bailard *et al*,[10] respectively.

Bars having spacings of approximately one-half of the surface wavelength and amplitudes sufficient to cause significant levels of reflection, represent topography which is *rapidly-varying*. It follows that established methods for predicting the interaction of waves and the bed, such as Berkhoff's[11] mild-slope equation, are not applicable in studies of such bars, and that reliable methods for determining the interaction between surface waves and such topography must be developed.

In the present paper, existing laboratory data for the reflection of waves by sinusoidal beds (Davies & Heathershaw[4]) and doubly-sinusoidal beds (Guazzelli *et al.*[12]) are utilized to provide a framework for the intercomparison of two models for wave propagation over rapidly-varying topography. The models considered are the extended-mild-slope model developed by Kirby[1] and the successive-application-matrix model of O'Hare & Davies [2]

In the following section, the two models are introduced and brief details of the methods of solution are given In Section 3, the models are applied to a series of test cases

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and comparisons made with laboratory data The results of these tests are discussed in Section 4

2 DESCRIPTION OF THE WAVE MODELS

2.1 Introduction

If the flow is irrotational, the velocity field $u(\mathbf{x}, z, t)$, $\mathbf{x} = \{x, y\}$, may be described in terms of the velocity potential $\phi(\mathbf{x}, z, t)$ as follows:

$$u = -\nabla\phi$$

where ∇ is the gradient operator

$$\nabla = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$$

The continuity equation, $\nabla u = 0$, may then be expressed in the form of Laplace's equation

$$\nabla^2\phi = 0$$

and the free-surface and bottom boundary conditions may be written without approximation as

$$\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y} + \frac{\partial\phi}{\partial z} = 0 \qquad z = \eta$$

$$g\eta - \frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi = 0 \qquad z = \eta$$

$$\frac{\partial\phi}{\partial n} = 0 \qquad z = -h(x)$$

where $\eta(\mathbf{x}, t)$ is the surface elevation and $\partial/\partial n$ represents differentiation in the direction normal to the bed.

Different mathematical representations of surface waves arise from the different approximations which are made for the free-surface and bottom boundary conditions. In the following two sections the assumptions underlying the models of Kirby[1] and O'Hare & Davies[2] are discussed, and the different approaches adopted to satisfy the bottom boundary condition for rapidly varying depth are explained Both models are then applied to an isolated one-dimensional region of topography, as shown in Fig 1

2.2 The extended-mild-slope equation model

Kirby[1] derived a general wave equation, applicable to linear water waves in intermediate or shallow water, which extended the 'mild-slope' approximation of Berkhoff[11]

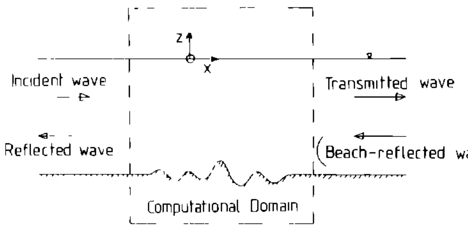


Fig. 1. Schematic diagram of the model configuration

to include rapidly varying, small-amplitude deviations from the slowly varying mean depth In his formulation, the total still-water depth $h'(\mathbf{x})$ is written as

$$h'(\mathbf{x}) = h(\mathbf{x}) - \delta(\mathbf{x})$$

where $h(\mathbf{x})$ is a slowly-varying depth satisfying the mild-slope assumption

$$\nabla_H h \ll kh \qquad \nabla_H = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$$

in which k is the wavenumber of the surface waves, and $\delta(x)$ represents *rapid* variations of the depth about the mean level

At the free surface, the boundary condition is linearized in respect of the wave amplitude, but at the bed it incorporates terms to first-order in the amplitude of the bed undulations, such that the boundary conditions are written as

$$\phi_{tt} + g\phi_z = 0 \qquad z = 0$$

$$\phi_z = -\nabla_H h\cdot\nabla_H\phi + \nabla_H(\delta\nabla_H\phi) \qquad z = -h$$

To leading order, the velocity potential may then be expressed as

$$\phi(\mathbf{x}, z, t) = f(\mathbf{x}, z)\tilde{\phi}(\mathbf{x}, t) + \text{non-propagating modes}$$

where the potential $\tilde{\phi}$ refers to propagating wave modes, and

$$f = \frac{\cosh k(z+h)}{\cosh kh}$$

is a slowly-varying function of $\mathbf{x}$ on account of the slowly-varying water depth h, and wavenumber k. The angular frequency ω is related to h and k by the linear wave dispersion equation

$$\omega^2 = gk\tanh kh \qquad (1)$$

where g is the acceleration due to gravity

On this basis, Kirby[1] derived the following general, time-dependent, 'extended' mild-slope equation for two horizontal dimensions

$$\tilde{\phi}_{tt} - \nabla_H(CCg\nabla_H\tilde{\phi}) + (\omega^2 - k^2CCg)\tilde{\phi} + \frac{g}{\cosh^2 kh}\nabla_H(\delta\nabla_H\tilde{\phi}) = O(k\delta)^2$$

where C is the wave celerity (ω/k) and C_g is the group velocity $(\mathrm{d}\omega/\mathrm{d}k)$ of the waves

In the case of constant mean depth h, and with the velocity potential $\tilde{\phi}(\mathbf{x}, t)$ expressed in the form $\tilde{\phi}(\mathbf{x}, t) = \hat{\phi}(x)\exp(-\iota\omega t)$ for one horizontal direction (x) only, the following simpler equation is obtained

$$\hat{\phi}_{xx} + k^2\hat{\phi} - 4\left(\frac{\Omega'}{C_g}\right)(\delta\hat{\phi}_x)_x = 0 \qquad (2)$$

where Ω' is a slowly-varying parameter defined by

$$\Omega' = \frac{gk}{4\omega\cosh^2 kh}$$

Solution of eqn (2) may be accomplished numerically in finite difference form, subject to the following radiating boundary conditions at the ends of the region of undulating bed (shown in Fig 1):

$\hat{\phi}_x = -ik(\hat{\phi} - 2\hat{\phi}_I)$ up-wave of the computational grid

where $\hat{\phi}_I = \exp(ikx)$ is the incident wave of unit amplitude, and, assuming a purely progressive wave on the down-wave side

$\hat{\phi}_x = ik\hat{\phi}$ down-wave of the computational grid

The resulting tri-diagonal matrix may be inverted using a double-sweep algorithm

The reflection coefficient R, defined as the ratio of the reflected to incident wave amplitudes, may be evaluated by writing the reflected wave as

$$\hat{\phi}_R = R\exp(-ikx)$$

and using the expression

$$\hat{\phi} = \hat{\phi}_I + \hat{\phi}_R = \exp(ikx) + R\exp(-ikx)$$

up-wave of the computational grid.

2.3 The successive-application-matrix model

The successive-application-matrix model of O'Hare & Davies[2] is based on a model developed by Devillard *et al.*[13] for the propagation of monochromatic surface waves in one horizontal direction over a bottom profile comprising a succession of horizontal shelves separated by abrupt steps

Over the ith shelf, the water depth is denoted by h_i and the waves are characterized by their *local* wavenumber k_i obtained from eqn (1). Since the bed is horizontal between steps, the bottom boundary condition reduces to the simple form

$$\phi_z = 0 \qquad (z = -h)$$

but the solution must be matched (approximately) from step to step. This is the essential difference between Kirby's[1] extended-mild-slope equation model and the successive-application-matrix model In the former, the topography appears explicitly in the bottom boundary condition, whereas in the latter, the bottom boundary condition on any shelf is independent of changes in the bed topography.

At each step discontinuity ($x = x_i$), two 'wavefield parameters' (the velocity potential at the water surface and its horizontal gradient) are defined by the following equations

$$\Psi_i = [A_i\exp(ik_ix_i) + B_i\exp(-ik_ix_i)]\chi_i(0)$$

$$\Omega_i = -k_i^{-1}\left(\frac{\partial\Psi_i}{\partial x}\right)_{x_i}$$

where A_i and B_i are the (complex) amplitudes of the forward and backward waves respectively, and

$$\chi_i(z) = (2K/(Kh_i + \sinh^2 k_ih_i))^{1/2}\cosh[k_i(z + h_i)]$$

in which $K = \omega^2/g$ is the deep-water wavenumber of waves having angular frequency ω.

Devillard *et al*[13] related the values of Ψ and Ω on either side of a step by a series of matching conditions, the form of which depends upon the step geometry (i.e the bottom topography). They derived a matrix equation which relates the wavefield parameters at neighbouring steps:

$$\begin{bmatrix}\Psi_{i+1}\\ \Omega_{i+1}\end{bmatrix} = R_iM_i\begin{bmatrix}\Psi_i\\ \Omega_i\end{bmatrix} \tag{3}$$

in which R_i is a 'matrix of rotation' accounting for propagation of the forward and backward waves over the $(i+1)$th shelf

$$R_i = \begin{bmatrix}\cos\Theta & -\sin\Theta\\ \sin\Theta & \cos\Theta\end{bmatrix} \qquad \theta = k_{i+1}(x_{i+1} - x_i)$$

and M_i is a 'transfer matrix' which represents the interaction between the surface waves on either side of the vertical step discontinuity at $x = x_i$, and is based on Miles'[14] variational approximation The form of this transfer matrix depends not only upon the propagating plane-wave modes but also on the non-propagating modes generated at the discontinuity. Devillard *et al*[13] applied their matrix equation to the case of *large* steps separated by *long* shelves, for which the amplitudes of the non-propagating modes generated at one step were negligible at the neighbouring steps. For shorter shelf widths, these non-propagating modes have non-negligible amplitudes at neighbouring steps if the step discontinuity is large, and the variational approximation of Miles[14] breaks down.

By comparing the predictions of the successive-application-matrix model with the results of an exact potential solution for singly- and doubly-stepped beds, O'Hare & Davies[2] demonstrated that if smoothly-varying bottom topography is discretized into a series of *narrow* horizontal shelves separated by steps of *small* height, then the propagation of monochromatic surface waves over the topography may still be evaluated using the model of Devillard *et al*[13] Despite their close spacing, the shelves remain effectively uncoupled in respect of the non-propagating modes, if the step heights are sufficiently small.

The method of solution relies upon a knowledge of the wavefield at any *one* point in the computational domain It is usually possible to make some assumption about the wavefield at the down-wave end of the undulating bed region. For example, if there is perfect absorption of the forward wave at a beach, then the amplitude of the backward wave at the down-wave end of the computational domain will be zero Alternatively, there may be a known amount of reflection due to the presence of a sloping beach. In the absence of energy dissipation, the matrices R_i and M_i are independent of the absolute values of the forward and backward waves. Hence it is possible to make an arbitrary choice (a_T) for the amplitude of the forward (transmitted) wave at the down-wave end of the computational domain, and to assign to the backward wave an amplitude ($= R_B a_T$) and a relative phase angle (Θ_B), where R_B is the beach reflection coefficient. Thus the wavefield parameters Ψ and Ω may be specified at the right-hand end of the computational domain ($x = x_N$) using the relations

$$A_N = g a_T/\omega$$

$$B_N = g a_T R_B \exp(i\Theta_B)/\omega$$

Computation of the surface wavefield across the region of interest then involves the successive application of the matrix equation (3) at each position x_i between x_N and x_0. The amplitude of the forward (incident) wave at the up-wave end of the computational domain ($x = x_0$) is given finally by

$$a_0 = \frac{\omega|A_0|}{g}$$

This incident amplitude may be re-scaled to coincide with, for example, a measured value (a_I) by multiplying the calculated wave amplitudes by the factor a_I/a_0

O'Hare & Davies[2] showed that the successive-application-matrix method may be applied to smoothly-varying topography provided that the height of each vertical step (Δh) relative to the local water depth (h) does not exceed about $\Delta h/h = 0.02$. If this condition is satisfied, the matrix M_i reduces from a complicated form including a term involving the sum of all the non-propagating wave modes, to a simple plane-wave form

$$M_i = \frac{\chi_{i+1}}{\chi_i}\left(\frac{k_i}{k_{i+1}}\right)^{1/2}\begin{bmatrix}(N_i)^{J_i} & 0\\ 0 & (N_i)^{-J_i}\end{bmatrix}$$

where

$$N_i = 2K(k_1k_2)^{1/2}\sinh[k_2(h_2 - h_1)](k_1^2 - k_2^2)^{-1}$$
$$\times (Kh_1 + \sinh^2 k_1h_1)^{-1/2}(Kh_2 + \sinh^2 k_2h_2)^{-1/2}$$

$$J_i = \begin{cases} -1 & \text{if } h_{i+1} > h_i \\ +1 & \text{if } h_{i+1} < h_i \end{cases}$$

in which the smaller of h_i and h_{i+1} plays the role of h_1 and the larger the role of h_2

The reflection coefficient associated with a region of topography may be evaluated from $R = |B_0|/|A_0|$ with the velocity potential amplitudes $|A_0|$ and $|B_0|$ given by

$$|A_0| = \tfrac{1}{2}[\Psi_0^2 + \Omega_0^2 + 2\,\mathrm{Im}\,\Psi_0\,\mathrm{Re}\,\Omega_0 - 2\,\mathrm{Re}\,\Psi_0\,\mathrm{Im}\,\Omega_0]^{1/2}$$

$$|B_0| = \tfrac{1}{2}[\Psi_0^2 + \Omega_0^2 - 2\,\mathrm{Im}\,\Psi_0\,\mathrm{Re}\,\Omega_0 + 2\,\mathrm{Re}\,\Psi_0\,\mathrm{Im}\,\Omega_0]^{1/2}$$

3 MODEL PREDICTIONS

3.1 Sinusoidal beds

Both Kirby[1] and O'Hare & Davies[2] have separately compared their model results with the laboratory measurements of Davies & Heathershaw[4] for the reflection of monochromatic waves by a region of fixed sinusoidal bottom undulations on an otherwise flat bed. The measurements were made with conductivity-type wave gauges in a wave tank of dimensions 46 m × 0·9 m × 0·9 m. Initially, an inter-comparison of results from the two models is discussed for this case.

The variation of water depth along the tank (see Fig 2) may be expressed as

$$\begin{aligned} h(x) &= h_0 & x &< 0 \\ h(x) &= h_0 - b\sin(lx) & 0 \le 0 &\le 2\pi n/l \qquad (4)\\ h(x) &= h_0 & x &> 2\pi n/l \end{aligned}$$

where h_0 is the (constant) mean water depth, n is the number of sinusoidal bars in the undulating region, b is the bar amplitude and l is their wavenumber Down-wave of the bars, a wave-absorbing beach minimized back-reflection onto the bars. Measurements of the reflection coefficient up-wave of the bars were made for a range of wave periods In their experiments, Davies & Heathershaw[4] examined three bed configurations, namely:

(a) $n = 2$ $\quad b/h_0 = 0.32$
(b) $n = 4$ $\quad b/h_0 = 0.32$
(c) $n = 10$ $\quad b/h_0 = 0.16$

In each experiment, the surface wavenumber k was varied in the range $0.5 \le 2k/l \le 2.5$ chosen on the basis of the theoretical analysis of Davies [3] This had indicated the likelihood of constructive interference between waves back-reflected from individual bars when the surface wavelength is equal to twice the bar spacing (i.e $2k/l \approx 1$), giving rise to high reflection coefficients

Both models were run for each of the above bed configurations, using a grid spacing corresponding to one-hundredth of the bar wavelength, and assuming that only an outgoing progressive wave was present down-wave of the bars The results of these tests are shown in Figs 3a, 3b and 3c together with Davies & Heathershaw's[4] data. Both models give good agreement with the data, with the widths of the resonant peaks at $2k/l \approx 1$ being well predicted Most of the discrepancy

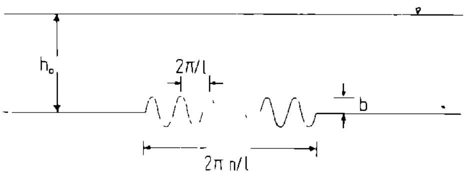


Fig. 2. Schematic diagram of sinusoidal bottom topography used in the laboratory experiments of Davies & Heathershaw [4]

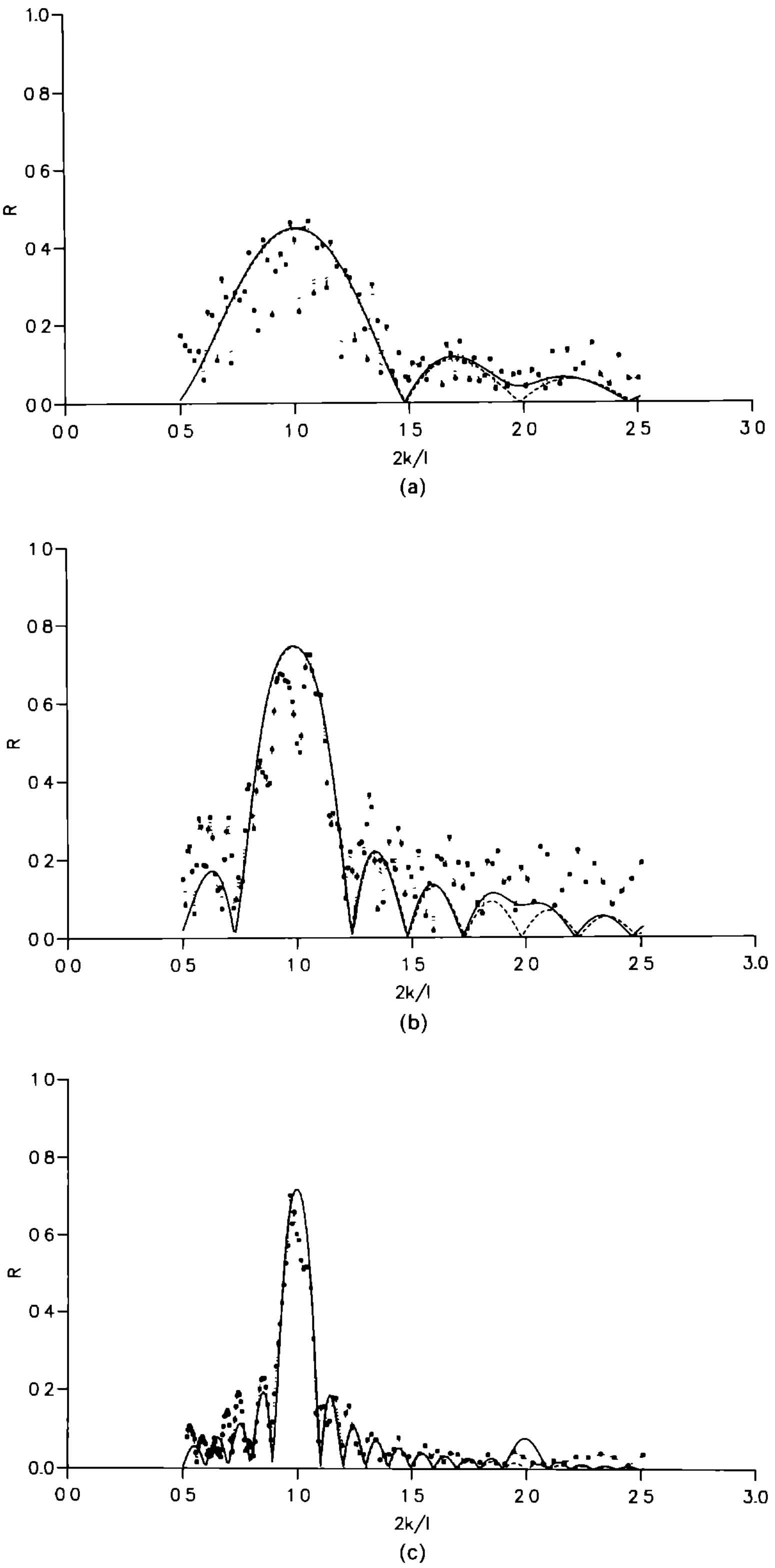


Fig. 3. (a) Wave reflection from sinusoidal bottom topography with $n = 2$: $b/h_0 = 0{\cdot}32$, —— successive-application-matrix model, – – – extended-mild-slope equation model, (●) experimental data (b) As Fig 3a but with $n = 4$ $b/h_0 = 0{\cdot}32$ (c) As Fig 3a but with $n = 10$ $b/h_0 = 0{\cdot}16$.

between the model predictions and laboratory data may be attributed to the assumption in the models of a purely outgoing wave on the down-wave side of the bars In the experiments, a small amount of reflection from the absorbing beach was measured (the reflection coefficient R_B being typically less than 0·1), leading to a possible error in the range $\pm R_B$ in the reflection coefficients measured up-wave of the bars. In addition, the models take no account of frictional dissipation at the bed and side-walls of the channel, the effect of which is to reduce slightly the measured reflection coefficients from their theoretical values

Despite the generally close agreement between the predictions of the models, two main differences are evident in the results. Firstly, the matrix model predicts a greater reflection coefficient when $2k/l \approx 2$ (i e surface wavelength = bar spacing), compared with the extended mild-slope model which predicts a zero in the reflection coefficient in each case when $2k/l \approx 2$. This is most clearly evident in Fig. 3c Secondly, the predictions of the matrix model are shifted slightly towards the left (i e to lower wavenumber) This is most pronounced in Fig 3b

3.2 Doubly-sinusoidal beds

A more complicated situation arises when the bed consists of the superposition of two sinusoidal components (chosen here to be of equal amplitude) By extension of eqn (4), the variation of the water depth may be written in this case as follows

$$\begin{aligned} h(x) &= h_0 & x &< 0 \\ h(x) &= h_0 - b[\sin(lx) + \sin(mlx)] & 0 &\leq x \leq 2\pi n/l \\ h(x) &= h_0 & x &> 2\pi n/l \end{aligned} \tag{5}$$

Here, n is the number of sinusoidal bars having the larger bed wavelength $(2\pi/l)$ and m is the ratio of the larger and smaller bed wavelengths.

With this form of bed, the possibility exists not only for first-order resonances to occur at $2k/l \approx 1$ and $2k/l \approx m$, but also for higher-order resonances at $2k/l \approx (m-1)$ and $2k/l \approx (m+1)$ (e g Mattioli,[15] Guazzelli *et al*[12]) These additional resonances are referred to as the 'difference' and 'sum' interactions respectively (There is also the possibility of second-order resonances, due to the individual sinusoidal components, which may occur when $2k/l \approx 2$ and $2k/l \approx 2m$ as discussed by Davies *et al*[16])

Kirby[1] applied his model to two doubly sinusoidal beds:

(a) $n = 4 \quad m = 2 \quad b/h_0 = 0\ 32$
(b) $n = 4 \quad m = 15/8 \quad b/h_0 = 0\ 32$

In these tests, the first sinusoidal component corresponded to case (b) of the earlier sinusoidal bed tests (Section 3.1) The significance of Kirby's[1] choice for m is that in case (a) the zeros of the (oscillatory) reflection coefficient for each bed component occur at the same values of $2k/l$ (according to Davies & Heathershaw's[4] analytical treatment), whereas in case (b) they occur at different values.

To examine the correspondence between the model results for this more complicated topography, the two models were run for each of the bed configurations listed above Again, the grid spacing corresponded to one-hundredth of the larger bed wavelength. Comparisons between the predictions of the two models are shown in Figs 4a and 4b As in the cases involving purely sinusoidal beds, the models give very similar predictions, with the results from the matrix model again being shifted slightly to lower wavenumbers compared with the results from the extended-mild-slope model In Fig 4a, zeros in R are predicted by both models whereas in Fig 4b, R is always positive, for the reason given above

Similar doubly sinusoidal bed configurations have been examined experimentally by Guazzelli *et al*[12] Their experiments were of the same type as those of Davies & Heathershaw,[4] but were performed in a smaller flume using a laser system for measuring the surface wave envelope and, hence, determining the reflection coefficient up-wave of the bars

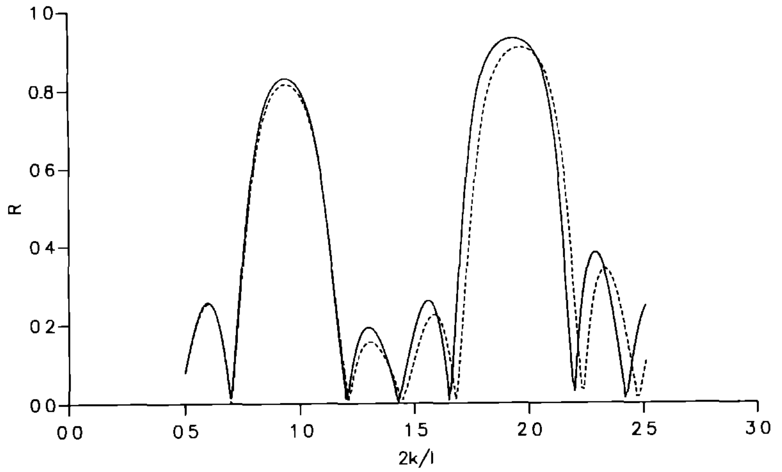
Both models were run for three of the bed configurations examined by Guazzelli *et al*[12] in which $m = 2$, namely

(a) $n = 4 \quad m = 2 \quad b/h_0 = 0\ 25$
(b) $n = 4 \quad m = 2 \quad b/h_0 = 0\ 33$
(c) $n = 4 \quad m = 2 \quad b/h_0 = 0\ 40$

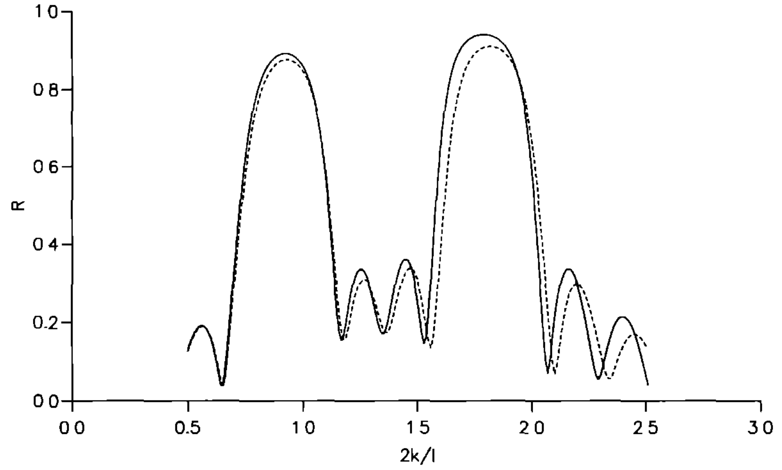
In these runs, the grid spacing corresponded to one-hundredth of the *smaller* bed wavelength. The model predictions are shown in Figs 5a, 5b and 5c together with the laboratory data. Again, the model predictions are generally similar to each other, and also show reasonable agreement with the data in the vicinity of the first-order peaks

The data reveal that as b/h_0 increases, the first-order peaks are shifted to lower wavenumbers. This shift is better predicted by the matrix model than by the extended-mild-slope model When $b/h_0 = 0\ 4$ (Fig 5c), the extended-mild-slope model predictions show a considerable discrepancy in the positioning of the peak in reflection coefficient at $2k/l \approx 2$ (i e the first-order resonance associated with the smaller bed wavelength and also, possibly, the second-order resonance associated with the larger bed wavelength). In addition, the experimental data suggest some enhancement of the reflection coefficient when $2k/l \approx 3$ due to the sum interaction, and when $2k/l \approx 4$ due to the second-order resonance associated with the smaller bed wavelength The matrix model predicts peaks in the reflection coefficient at these values of $2k/l$, whereas the extended-mild-slope model does not, though these peaks are too large

For all the doubly sinusoidal beds considered above, the value of m is such that the difference interaction occurs at the same value of $2k/l$ as the first-order resonance associated with the larger wavelength (i.e $2k/l = 1$) Thus it is not clear how well the models perform in the vicinity of the difference interaction



(a)



(b)

Fig. 4. (a) Wave reflection from doubly sinusoidal bottom topography with $n = 4$ $m = 2$ $b/h_0 = 0{\cdot}32$, —— successive-application-matrix model; – – – extended-mild-slope equation model. (b) As Fig 4a but with $n = 4$ $m = 15/8$ $b/h_0 = 0{\cdot}32$.

Guazzelli *et al.*[12] performed a further set of experiments with $m = 1{\cdot}5$, in which the first-order resonances occurred at $2k/l \approx 1$ and $2k/l \approx 1{\cdot}5$, the sum interaction occurred at $2k/l \approx 2{\cdot}5$ and the difference interaction at the clearly distinct value of $2k/l \approx 0{\cdot}5$

The two models were run for three of the bed configurations of this kind examined experimentally by Guazzelli *et al.*,[12] namely

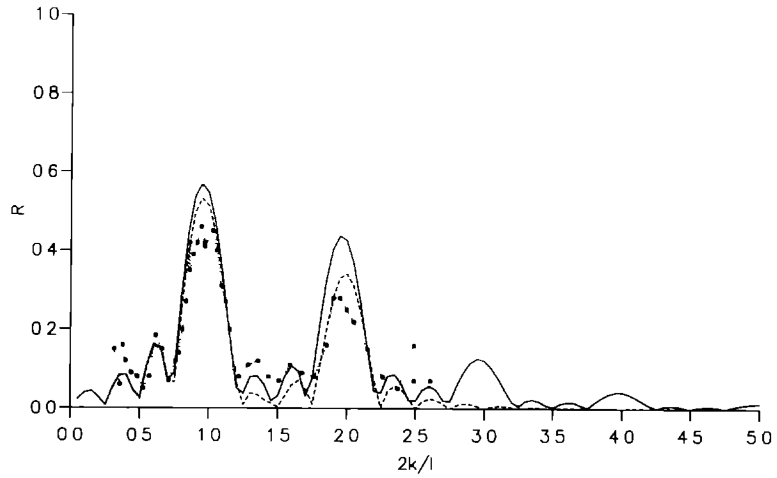
(a) $n = 8$ $m = 1{\cdot}5$ $b/h_0 = 0{\cdot}25$
(b) $n = 8$ $m = 1{\cdot}5$ $b/h_0 = 0{\cdot}33$
(c) $n = 8$ $m = 1{\cdot}5$ $b/h_0 = 0{\cdot}40$

with a grid spacing corresponding to one-hundredth of the larger bed wavelength The results of these comparisons are shown in Figs 6a, 6b and 6c. Again, both models show reasonable agreement with each other and with the data in the vicinity of the first-order resonances The matrix model better predicts the observed shift of the peaks in reflection coefficient to lower wavenumbers as b/h_0 increases As before, a sum interaction (at $2k/l \approx 2{\cdot}5$) is predicted by the matrix model, but not by the extended-mild-slope model, though in this region there are no experimental data to validate either set of model predictions *Both* models predict enhanced reflection associated with the difference interaction at $2k/l \approx 0{\cdot}5$, but tend to underestimate the experimental data (except for case (c)). The matrix model performs slightly better than the extended-mild-slope model, but the agreement between both models and the data gets worse as b/h_0 decreases (i.e. in deeper water).

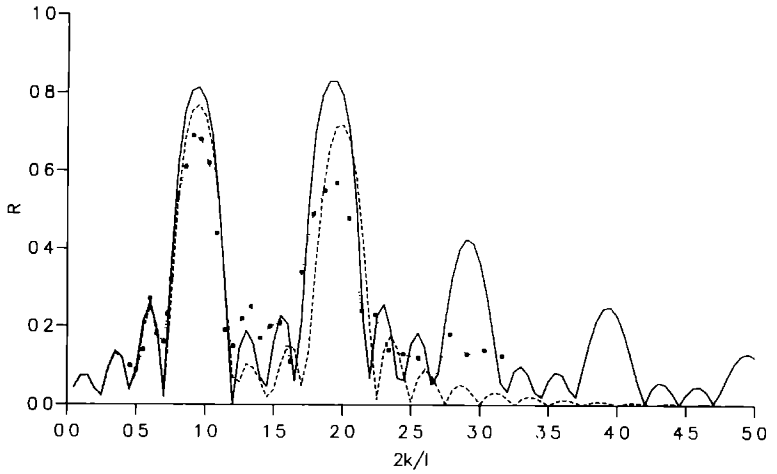
4 DISCUSSION

The results of the model comparisons described in Section 3 may be summarized as follows:

(1) Both models perform well in the vicinity of the first-order resonances (i.e. when the surface wavelength is approximately equal to *twice* the wavelength of a sinusoidal bed component) with larger peak values predicted by the matrix model than by the extended-mild-slope model.



(a)



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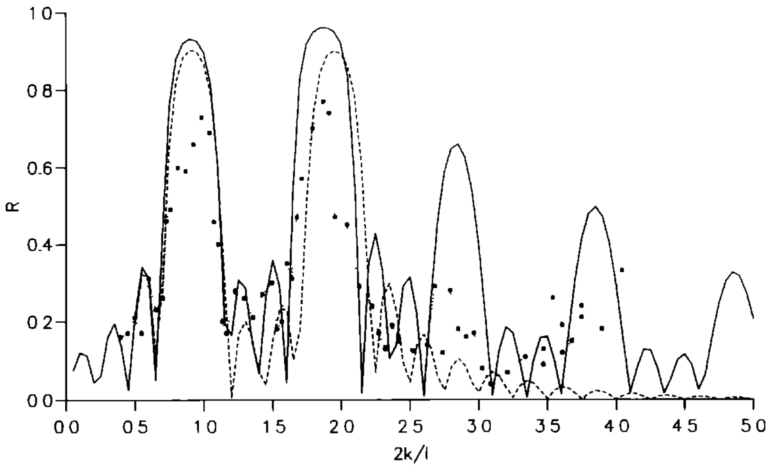

(c)

Fig. 5. (a) Wave reflection from doubly sinusoidal bottom topography with difference interaction coincident with the first-order resonance Parameter settings $n = 4$ $m = 2 \cdot b/h_0 = 0{\cdot}25$, —— successive-application-matrix model; – – – extended-mild-slope equation model, (●) experimental data (b) As Fig 5a but with $n = 4 : m = 2$ $b/h_0 = 0\ 33$ (c) As Fig 5a but with $n = 4$ $m = 2$ $b/h_0 = 0\ 4$.

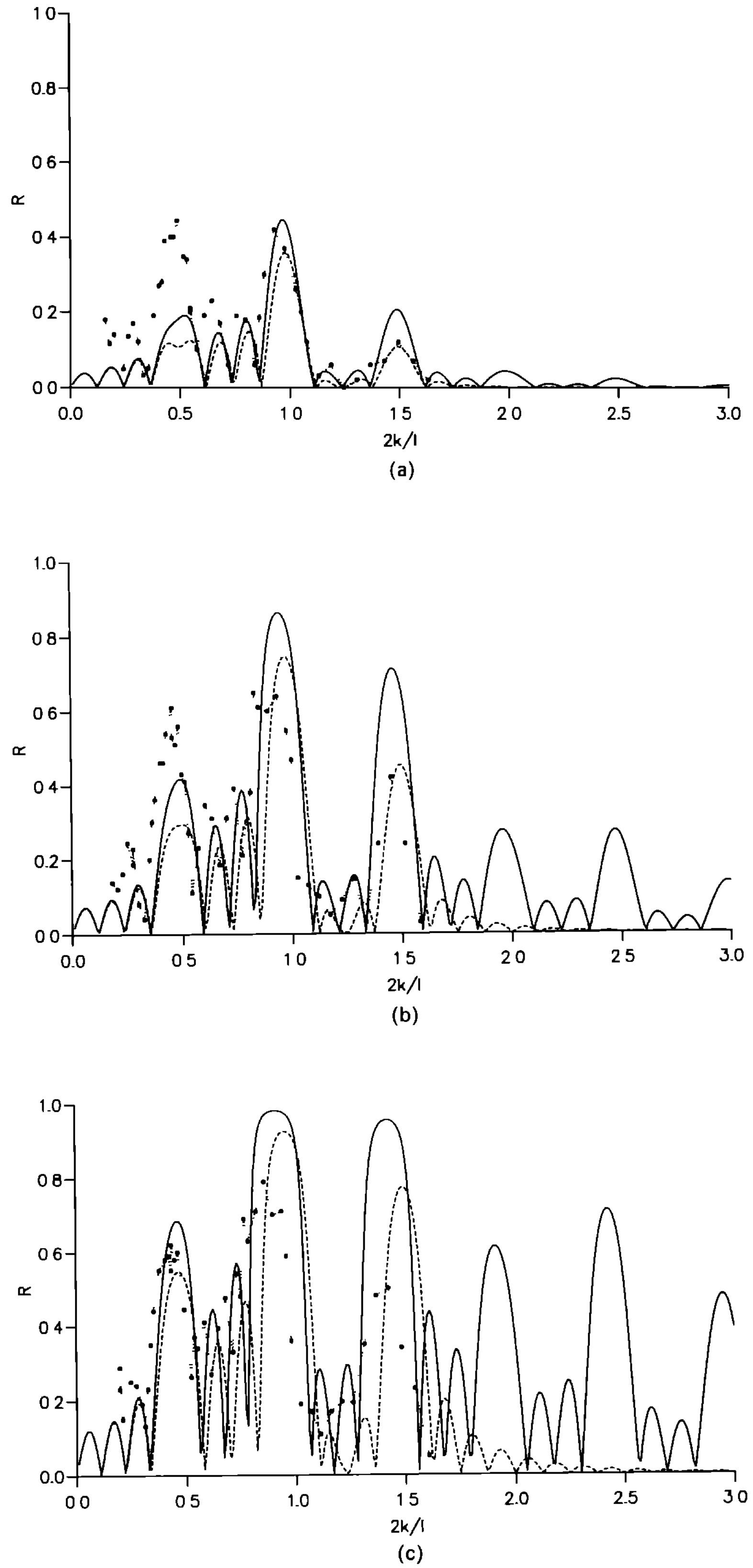

Fig. 6. (a) Wave reflection from doubly sinusoidal bottom topography with difference interaction distinct from the first-order resonances. Parameter settings: $n = 8$ $m = 1{\cdot}5$ $b/h_0 = 0{\cdot}25$, —— successive-application-matrix model; – – – extended-mild-slope equation model; (●) experimental data (b) As Fig 6a but with $n = 4$ $m = 1{\cdot}5$ $b/h_0 = 0{\cdot}33$ (c) As Fig. 6a but with $n = 4$ $m = 1{\cdot}5$ $b/h_0 = 0{\cdot}4$

(2) Only the matrix model predicts enhanced reflection associated with second-order resonances (i e when the surface wavelength is approximately *equal* to the wavelength of a sinusoidal bed component)
(3) The matrix model predicts the shift of the resonant peaks towards lower wavenumbers, as the ratio b/h_0 increases, better than the extended-mild-slope model
(4) The matrix model predicts the presence of the sum interaction, but overestimates the experimental values. The extended-mild-slope model predicts no enhancement of reflection due to the sum interaction
(5) Both models predict enhanced reflection due to the difference interaction but tend to underestimate the experimental values, especially for lower values of b/h_0 The matrix model is generally closer to the experimental data than the extended-mild-slope model

The reflection of waves from doubly sinusoidal beds has also been modelled by Mattioli[15,17] (who also considered rectangular submerged bar systems), and by Guazzelli *et al*[12] (see Rey[18] for a full description of the model used) These models are essentially identical, and are based on the approach of Takano[19] in which the bed is discretized into a series of horizontal shelves However, the models differ from that of O'Hare & Davies,[2] which also divides the bed into horizontal shelves, by the inclusion of the effects of the *interactions* of non-propagating (or evanescent) wave modes generated at each discontinuity with neighbouring steps

If the region is divided into N steps, and P non-propagating modes are included in the calculation, the problem ultimately reduces to the solution of a system of $2N(P+1)$ simultaneous equations, with terms corresponding to the interactions of both propagating (i.e forward and backward) and all the non-propagating wave modes, with each of the N steps Solution of the system of equations relies on the inversion of a $2N(P+1) \times 2N(P+1)$ complex matrix, a task which is impractical for all but the simplest topographies Both Mattioli[15,17] and Guazzelli *et al*[12] simplify the algebra by considering usually just one, and no more than three, non-propagating modes, but have different approaches to limiting the number of steps used in their computations Mattioli[15,17] discretizes the entire undulating bed profile into relatively few shelves (14 per bar wavelength), whereas Guazzelli *et al*[12] sub-divide the full profile into four smaller 'patches', each discretized into 61 shelves (with at least 20 per bar wavelength), and then combine the solutions for the individual patches. In this approach, it is assumed that the patches are *not* coupled by the non-propagating modes

Both these models provide good approximations to the full solution for surface wave propagation over rapidly-varying topography, but require large amounts of computational effort and are thus not suitable for general use However, the results presented by Mattioli[15,17] and Guazzelli *et al*[12] provide good insight into certain failings of both the successive-application-matrix model and the extended-mild-slope model

In all cases, the numerical results of Guazzelli *et al*[12] agree well with their experimental data Not only are the first-order peaks predicted correctly, but also the sum and difference resonances are accurately described Moreover, the model works well for all of the b/h_0 values examined Detailed examination of the role of the non-propagating modes by Mattioli[15] indicates that their inclusion has the effect of

(1) reducing the size of the second-order resonance predicted for sinusoidal beds,
(2) increasing the size of the first-order resonances predicted for both sinusoidal and doubly sinusoidal beds,
(3) shifting first-order resonances further towards lower wavenumbers,
(4) reducing the size of the sum resonance,
(5) increasing the size of the difference resonance

Thus the inclusion of the non-propagating wave modes has the general effect of shifting the resonances towards lower wavenumbers and reducing the size of all the higher-order resonances, except for the difference interaction which is enhanced This is exactly the change required to bring the matrix model predictions, described in Section 3, into agreement with the experimental data of Guazzelli *et al.*[12] This result is not surprising, since the successive-application-matrix model is simply an implementation of the 'exact' model of Guazzelli *et al*[12] but without the inclusion of any non-propagating modes Belzons *et al*[20] discuss this shortcoming of the successive-application-matrix model and describe how the amplitudes and phases of the propagating waves over the patch must be continually renormalized by the non-propagating modes if accurate predictions are to be obtained

The extended-mild-slope model derived by Kirby[1] would not be expected to predict the higher-order resonances, since Kirby's equation only includes terms to *first-order* in the amplitude of the bed undulations However, since these higher-order resonances are *inhibited* in the full solution by the inclusion of the non-propagating modes, this deficiency may not be too serious in many practical applications

In addition to the accuracy of the model predictions, there are a number of other important factors which need to be considered in choosing the most appropriate model for a particular application Perhaps the most important of these is the computational efficiency of the method Both of the models compared earlier are simple to implement, and Kirby's[1] model provides a small speed advantage over the matrix model for a given grid spacing However, the successive-application-matrix model provides a more explicit formulation of the wave propagation problem, through its division of the surface wavefield into forward and backward components.

Both models have been extended to allow for the effects of bottom friction (Tsay *et al.*,[21] O'Hare[8]), but only the extended-mild-slope model has been applied to cases where variations in topography occur in two horizontal directions However, for studies involving erodible beds, the discretization of the bottom profile into a series of horizontal shelves in the matrix model has the advantage of allowing it to be linked easily with bottom boundary layer and sediment transport models to provide predictions of both surface wave *and* bed evolution (O'Hare[8]).

5 CONCLUSIONS

Both Kirby's[1] extended-mild-slope equation model and O'Hare & Davies[2] successive-application-matrix model provide reasonable predictions of the reflection characteristics for sinusoidal and doubly sinusoidal bottom topographies The models perform less well in the vicinity of the higher-order resonance which occurs at the difference wavenumber of the two bed components in the latter case In general, the predictions of the matrix model show slightly better agreement with the experimental data, but at a somewhat higher cost in terms of computational effort The matrix model is more generally applicable than the extended-mild-slope model in one-dimensional problems, as it may be applied to rectangularly shaped beds as well as to smoothly varying topography

The use of either of these models for wave propagation over rapidly varying topography would appear to be justified on the basis of the comparisons presented here However, there is room for improvement in the predictions of both models, and further work is required if an accurate and reliable, simple model for wave propagation over rapidly varying bottom topography is to be produced

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