# Forcing a three-dimensional, hydrostatic primitive-equation model for application in the surf zone, Part 1: Formulation

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**Abstract.** An Eulerian analysis for wave forcing of three-dimensional (3D) wave-averaged mean circulation in the surf zone is presented. The objective is to develop a dynamically consistent formulation for applications in a 3D primitive equation model. The analysis is carried out for the case of shallow-water linear waves interacting with wave-averaged depth-independent horizontal currents that vary on larger space and time scales. Variations in wave properties are governed by a wave action equation that includes wave-current interactions and dissipation representative of wave breaking. Wave forcing of the mean currents consists of a surface stress and a body force. The surface stress is proportional to the wave energy dissipation. The body force includes one term that is related to gradients of part of the radiation stress tensor and a second term that is related to the vortex force and is proportional to a product of the mean wave momentum and the vertical component of the mean vorticity vector. In addition there is a non-zero normal velocity at the mean surface that arises from the divergence of the mean Eulerian wave mass flux. This velocity results in an additional momentum flux forcing of the mean flow. Applications of this formulation to the DUCK94 field experiment are presented in Part 2.

## 1. Introduction

The interactions between waves and longer period flow and the forcing of mean currents by waves have been the subject of many previous studies [e.g., *Longuet-Higgins and Stewart*, 1962, 1964; *Bowen*, 1969; *Hasselmann*, 1970, 1971; *Longuet-Higgins*, 1973; *Mei*, 1989; *Garrett*, 1976; *Smith*, 2006 and the comprehensive discussion in *Phillips*, 1977]. Most of this work has been in the framework of one- or two-dimensional, depth-integrated currents. This approach has had considerable success in predicting wave-driven currents and wave-averaged surface elevation in the nearshore surf zone [e.g. recent studies by *Ruessink et al.*, 2001; *Reniers et al.*, 2004; *Long and Özkan-Haller*, 2005]. It seems clear, however, that development of a three-dimensional modeling capability for the wave-averaged circulation that resolves both vertical and horizontal spatial structure would be extremely useful, e.g., for application to sediment transport problems. A prerequisite for the development of that capability is a dynamically consistent derivation of the structure of the wave forcing. The primary objective of the present paper is to begin to develop such a rational formulation for wave forcing of a three-dimensional (3D) primitive equation model of the nearshore. An application of this formulation to conditions found during the DUCK94 field experiment,



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including detailed model-data comparisons, is presented in [*Newberger and Allen*, 2007] refered to here as NA2.

Initial efforts to look at depth-dependent effects of wave-forcing in the surf zone include one-dimensional vertical models with undertow calculated to balance the effects of the waves [e.g. *Svendsen*, 1984b; *Deigaard et al.*, 1991; *Garcez Faria et al.*, 2000]. Quasi-three-dimensional models have been developed which combine a horizontally two-dimensional shallow-water model with an approximate sub-model for the vertical current structure [e.g., *Svendsen and Putrevu*, 1994; *Putrevu and Svendsen*, 1999; *Van Dongeren, et al.*, 1999]. The increase in horizontal turbulent diffusion caused by three-dimensional dispersion is included in this type of model.

Other studies describing the effects of waves on mean currents have involved primarily non-breaking, deep-water waves. These include the work of *Craik and Leibovich* [1976] and of *Garrett* [1976] that describe possible generation mechanisms for Langmuir circulation. In these cases, it is the interactions between the waves and currents that are important. In another example, *Hasselmann* [1970] looks at mean currents generated by waves in a rotating environment and shows that rotation should not be neglected for times approaching an inertial period. In this study, we consider both the effects of wave breaking and of wave-current interactions with our results applicable to the nearshore surf zone.

The development of fully three-dimensional models of the wave-averaged circulation requires an understanding of the distribution of the wave forcing in the vertical. The wave forcing may include both a body force acting on the interior of the fluid and a surface stress. One relevant issue is that of the determination of the surface stress which acts to generate vorticity . *De Vriend and Kitou* [1991] discuss the forcing of a three-dimensional model and point out the inconsistency, when the waves are inviscid and irrotational, of non-zero surface stress seemingly indicated by Eulerian models. This issue has also been discussed, for example, by *Dingemans et al.* [1987].

A generalized Lagrangian mean (GLM) theory for non-linear waves has been developed by *Andrews and McIntyre* [1978]. *Groeneweg and Klopman* [1998] describe a three-dimensional GLM formulation for long-crested, non-breaking waves and apply a one-dimensional formulation of this model to explain laboratory observations of modification of the vertical profile of an imposed current by the presence of waves. A two-dimensional (vertical and across flume) GLM model with non-hydrostatic pressure correction is used by *Groeneweg and Battjes* [2003] to further study the same laboratory experiments. *Mellor* [2003, 2005] describes a three-dimensional model using phase averaging with wave following coordinates. In the context of Eulerian primitive-equation models, *McWilliams and Restrepo* [1999] and *McWilliams et al.* [2004] have derived wave forcing for non-dissipative waves and applied the results to oceanic simulations of wave influenced circulation.

The goal here is to derive approximate, dynamically consistent expressions for the wave forcing that can be used to extend an existing three-dimensional primitive equation model, the Princeton Ocean Model (POM) [*Blumberg and Mellor*, 1987], for application to the wave-averaged circulation in the surf zone. The unmodified POM includes forcing by surface stress from wind, surface fluxes of heat and fresh water and bottom stress calculated from a quadratic drag parameterization. A two-equation turbulent closure, the Mellor-Yamada level 2.5 scheme [*Mellor and Yamada*, 1982], is used to calculate turbulent eddy coefficients. In the surf zone, additional factors become important. Breaking waves exert a stress on the wave-averaged currents and create an increase of near surface turbulence. The onshore flux of mass in the waves must also be taken into account. Interactions of waves and currents near the bottom boundary increase the bottom stress felt by the mean currents above the boundary layer. These effects must be added to the model in a dynamically rational manner to simulate the surf-zone currents. In addition, a wave model is required to provide the wave-averaged wave energy density, dissipation rate and wavenumber needed to force the wave-averaged circulation.

In this paper we will develop an Eulerian formulation for wave forcing of a three-dimensional primitive equation model for wave-averaged currents. As an initial step in solving the general problem we will restrict our attention to shallow-water waves and depth-independent horizontal currents. Consistent with application to the surf zone, we will not assume that the currents are weak relative to the wave propagation speed so that the waves may be changed by the presence of the mean currents.

Although many aspects of the wave forcing of mean currents have been addressed previously, [e.g., *Hasselmann*, 1971; *Longuet-Higgins*, 1973; *Deigaard and Fredsøe*, 1989; *Deigaard*, 1993; *Rivero and Arcilla*, 1995], we have not been able to find a single, unified analysis that includes all the components necessary to formulate dynamically consistent forcing for a three-dimensional model of the wave-averaged currents in the nearshore. One important point is the inclusion of wave-current interactions in both the wave and current equations. Additionally, it is necessary to properly evaluate the correlation between the horizontal and vertical wave velocities. This correlation is zero for waves propagating without change of shape. It has been shown in particular cases that non-zero values of this term caused by dissipation [*Deigaard and Fredsøe*, 1989; *Deigaard*, 1993], shoaling [*Rivero and Arcilla*, 1995] or rotation [*Hasselmann*, 1970] cannot be ignored in calculating the three-dimensional forcing terms. We note that none of these results are directly applicable by themselves to the problem considered here with time variation in both the mean wave properties and the mean currents, sloping topography and wave-current interactions.

In section 2 we look at the general Eulerian framework for three-dimensional forcing extending the approach of *Hasselmann* [1971] to address calculation of the partition of the wave forcing into a surface stress and a body force. The special case of depth-independent horizontal currents and shallow-water waves is worked out in detail in sections 3 and 4. Discussion and conclusions are presented in section 5. Details of the derivations are presented in the Appendices. Numerical solutions obtained using the forcing derived in this paper are presented in NA2.

## 2. Interactions of the mean current and waves

*Hasselmann* [1971] has examined the interactions between gravity waves and the larger-scale flow. We begin by using his equations (1-5) for the total flow modified to include the effects of rotation and vertical turbulent mixing. Thus, the governing equations in Cartesian $(x, y, z)$ coordinates are:

$$u_x + v_y + w_z = 0, \tag{1}$$

$$u_t + (u^2)_x + (uv)_y + (uw)_z + \frac{p_x}{\rho} - fv = \tau^x_z, \tag{2}$$

$$v_t + (uv)_x + (v^2)_y + (vw)_z + \frac{p_y}{\rho} + fu = \tau^y_z, \tag{3}$$

$$w_t + (uw)_x + (vw)_y + (w^2)_z + \frac{p_z}{\rho} = -g, \tag{4}$$

where $\mathbf{u} = (u, v, w)$ is the velocity vector, $x$ and $y$ the horizontal coordinates, $z$ the vertical coordinate, $p$ the pressure, $\rho$ the constant density, $f$ the Coriolis parameter and subscripts $x$, $y$, $z$ and $t$ denote partial differentiation. The terms $\tau^x_z$ and $\tau^y_z$ represent vertical gradients in the turbulent Reynolds stress associated with the vertical turbulent velocities.

We assume no imposed external stress at the surface (i.e. no wind stress) and that the bottom stress is zero. The surface and bottom boundary conditions are

$$\tau^x = \tau^y = 0, \quad p = p^a \quad \text{at } z = \eta, \tag{5a, b}$$

$$\eta_t + u\eta_x + v\eta_y - w = 0 \quad \text{at } z = \eta, \tag{6}$$

$$\tau^x = \tau^y = 0, \quad uH_{0x} + vH_{0y} + w = 0 \quad \text{at } z = -H_0, \tag{7a, b}$$

where $p^a$ is the atmospheric pressure assumed in the following to be uniform and therefore neglected, $\eta$ the free surface and $H_0$ the undisturbed water depth.

We define an averaging operator $\overline{(\ )}$ as an appropriate average over wave time scales which retains the slower time variations of the large scale currents and use it to separate the mean (wave-averaged) velocity and fluctuating components of the total flow such that, if $u = u_m + u'$,

$$\overline{u} = u_m, \quad \overline{u'} = 0. \tag{8}$$

This operator is applied to the equations for the total flow (1)-(4) to obtain the equations for the mean flow which, we assume, in addition, is in hydrostatic balance:

$$\overline{u}_x + \overline{v}_y + \overline{w}_z = 0. \tag{9}$$

$$p^m = \rho g(\overline{\eta} - z). \tag{10}$$

$$\overline{u}_t + (\overline{u}^2)_x + (\overline{u}\,\overline{v})_y + (\overline{u}\,\overline{w})_z + g\overline{\eta}_x - f\overline{v} = -(\overline{u'^2})_x - (\overline{u'v'})_y - (\overline{u'w'})_z - p^w_x + \overline{\tau^x}_z, \tag{11}$$

$$\overline{v}_t + (\overline{u}\,\overline{v})_x + (\overline{v}^2)_y + (\overline{v}\,\overline{w})_z + g\overline{\eta}_y + f\overline{u} = -(\overline{u'v'})_x - (\overline{v'^2})_y - (\overline{v'w'})_z - p^w_y + \overline{\tau^y}_z, \tag{12}$$

where $p^m$ is the mean pressure in the absence of waves and $p^w$ is the wave contribution to the mean pressure. Boundary conditions applied at the mean surface $\overline{\eta}$ are [*Hasselmann*, 1971],

$$\overline{\eta}_t + \overline{u}\,\overline{\eta}_x + \overline{v}\,\overline{\eta}_y - \overline{w} = -M^x_x - M^y_y \quad \text{at } z = \overline{\eta}, \tag{13}$$

where

$$M^x = \overline{\int_{\overline{\eta}}^{\eta} u dz}, \qquad M^y = \overline{\int_{\overline{\eta}}^{\eta} v dz}. \qquad (14a, b)$$

The boundary condition (13) at $z = \overline{\eta}$ is discussed further in Appendix A. At the bottom,

$$\overline{\tau^x} = \overline{\tau^y} = 0, \quad \overline{u} H_{0x} + \overline{v} H_{0y} + \overline{w} = 0$$

$$\text{at } z = -H_0. \qquad (15a, b)$$

Additional boundary conditions for $\overline{\tau^x}$ and $\overline{\tau^y}$ at $z = \overline{\eta}$ are required.

As pointed out by *Hasselmann* [1971] these equations are not closed. Equations for the fluctuating part of the flow are required. We will assume that the fluctuations are comprised of approximately linear waves with near-surface dissipation representing breaking waves. The inclusion of dissipation for the waves will allow us to apply the wave forcing to flow in the surf zone where breaking and dissipation at the surface are important processes. Breaking waves are clearly not linear, but this assumption allows analytical progress. It also retains physically important aspects of the lowest order wave dynamics that have made possible valuable results in previous studies of closely related problems [e.g., *Thornton and Guza*, 1983, 1986; *Deigaard and Fredsøe*, 1989]. We note, however, that applications to observed surf zone or laboratory flows frequently are significantly more accurate when the linear wave results are supplemented by the addition of a submodel for turbulent surface rollers [*Svendsen*, 1984a; *Fredsøe and Deigaard*, 1992; *Stive and De Vriend*, 1995; *Reniers and Battjes*, 1997; *Reniers et al.*, 2004]. Existing roller models are necessarily rather idealized approximations, but they have the advantage that they can be readily appended to other formulations. The addition of a roller submodel to the wave forcing formulation derived here is discussed, applied, and evaluated with model-data comparisons in NA2.

The waves act as forcing for the wave-averaged currents in three ways. First, as the body forces arising from the wave-averaged nonlinear wave terms that appear on the right hand side of (11) and (12). Additional forcing is in the form of a flux of mass at the wave-averaged surface (13). As discussed below, dissipating (i.e. breaking) waves will exert a stress at the wave-averaged surface. Wave dissipation in the wave bottom boundary layer is also important in some cases [*Deigaard and Fredsøe*, 1989; *Longuet-Higgins*, 2005] but will not be considered here.

We obtain equations for the wave- and depth-averaged wave momentum by integrating the equations for the total momentum (2) and (3) from the mean surface $\overline{\eta}$ to the free surface $\eta = \overline{\eta} + \eta'$ and applying the averaging operator. These integrals and averages are evaluated with the same assumptions and methods that are used in derivations of depth-integrated equations [e.g. *Phillips*, 1977; *Hasselmann*, 1971; *Smith*, 2006]. We assume that the velocity components can be analytically continued to the mean surface when the instantaneous surface is below $\overline{\eta}$ so that averages such as that in (14) can be defined. As in the case of the assumption of linear waves, analytic continuation is of questionable validity in the surf zone where the instantaneous surface is not clearly defined and the region between the trough and crest is an appreciable portion of the water column. It is consistent with the assumption of linear waves and will be employed here to allow estimation of the forcing terms.

To evaluate the terms in the mean wave momentum equations we will further assume that the total near surface current can be expressed as a Taylor series about the value at the mean surface. From (2) and (3), we obtain, respectively,

$$
\begin{aligned}
& M_t^x + 2\left(\overline{u} M^x\right)_x + \left(\overline{u} M^y + \overline{v} M^x\right)_y - \overline{u}\left(M_x^x + M_y^y\right) \\
& \quad - \overline{u'w'} - f M^y + \frac{1}{2} g \left(\overline{\eta'^2}\right)_x + \left(\overline{u'^2}\right)\overline{\eta}_x + \left(\overline{u'v'}\right)\overline{\eta}_x \\
& \quad = -\overline{\tau^x} \quad \text{at } z = \overline{\eta}, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
& M_t^y + 2\left(\overline{v} M^y\right)_y + \left(\overline{u} M^y + \overline{v} M^x\right)_x - \overline{v}\left(M_x^x + M_y^y\right) \\
& \quad - \overline{v'w'} + f M^x + \frac{1}{2} g \left(\overline{\eta'^2}\right)_y + \left(\overline{u'v'}\right)\overline{\eta}_x + \left(\overline{v'^2}\right)\overline{\eta}_y \\
& \quad = -\overline{\tau^y} \quad \text{at } z = \overline{\eta}. \qquad (17)
\end{aligned}
$$

These equations relate the wave-averaged surface stress components $\overline{\tau^x}$ and $\overline{\tau^y}$ at the mean surface $\overline{\eta}$ to $\overline{u'w'}(\overline{\eta})$ and $\overline{v'w'}(\overline{\eta})$ as well as to time and space variations in the wave momentum and the near-surface mean velocity component $\overline{u}$ and $\overline{v}$ . Note that the wave-current interaction terms [*Garrett*, 1976] that were omitted in *Hasselmann*'s [1971] analysis are included. With an appropriate wave model, (16) and (17) provide formulae for evaluating the mean surface stress.

*Kirby and Chen* [1989] discuss the effects of weak vertically-sheared flows on surface waves. Their results point out the difficulty of describing the waves in the presence of vertically sheared mean flow. The determination of an appropriate wave model that includes at least, shoaling, dissipation and wave-current interactions in the presence of mean currents with O(1) vertical shear is not addressed here and remains a topic for future research. In the sections 3 and 4 we will develop expressions for the wave forcing in the special case of shallow-water waves and currents using the expressions (16) and (17) to calculate the surface stress.

## 3. Shallow water currents with linear waves

We specialize the results of Section 2 to shallow-water waves in the presence of depth-independent horizontal mean currents. Approximate wave solutions are obtained for the case of slowly varying mean currents and topography. Consistent with application to the surf zone, the mean currents may be comparable in magnitude to the wave propagation velocity. In Section 4 these wave solutions are utilized to calculate the time-averaged wave forcing terms for the mean flow.

We define scales appropriate for linear waves in shallow water and slowly varying depth-independent horizontal currents and topography. Length scales are $L_w = \hat{K}^{-1} \ll L_B$ for the waves and mean currents, denoted by subscripts $w$ and $B$ respectively, where $\hat{K}$ is a typical wavenumber. Other length scales are $\hat{A}$ for the wave amplitude and $\hat{H}$ for the water depth, both assumed to be much smaller than $L_w$. The mean surface elevation $\overline{\eta}$ is assumed to scale with $\hat{H}$. The time scale for the waves $T_w$, and currents $T_B$, scale as $T_w = \hat{\omega}^{-1} \ll T_B$ where $\hat{\omega}$ is a typical wave frequency. The wave scales are related by the shallow-water dispersion relationship so that a typical wave speed

$\hat{c} \equiv \left(g\hat{H}\right)^{1/2} = \hat{\omega}/\hat{K} = L_w/T_w$. The velocity scales are $\hat{U}_w = \hat{c}\hat{A}/\hat{H}$ for the waves and $\hat{U}_B = \hat{c}$ for the currents. From the continuity equation, the vertical velocities scale so that $\hat{W}_w = \hat{H}\hat{U}_w/L_w$ and $\hat{W}_B = \hat{H}\hat{U}_B/L_B$. We assume that

$$\epsilon = L_w/L_B = T_w/T_B \ll 1. \tag{18}$$

The waves are assumed to be linear and shallow water; that is, the parameters

$$\mu = \hat{H}\hat{K} \ll 1, \qquad \alpha = \hat{A}\hat{K} \ll 1 \tag{19a, b}$$

and in addition,

$$\beta \equiv \hat{A}/\hat{H} = \alpha/\mu \ll 1 \tag{19c}$$

is required for shallow-water waves to be approximately linear. We further assume that

$$\beta \ll \epsilon. \tag{19d}$$

The mean wave properties, such as amplitude and wave number, are assumed to vary on the same large space and time scales as the mean velocities and topography. In the shallow-water approximation, the vertical gradient of the stress $\tau_z^x$ and $\tau_z^y$ must be independent of depth except in frictional boundary layers. We assume that these stress terms are of at most order $\epsilon$.

We assume that the velocity components $u$, $v$ and $w$, and surface elevation $\eta$ can be expressed as a sum of slowly varying quantities $U_B$, $V_B$, $W_B$, and $\eta_B$ corresponding to the mean currents, with time and space scales $T_B$ and $L_B$, and wave quantities $u_w$, $v_w$, $w_w$, and $\eta_w$, varying on shorter wave scales. The Eulerian average (8) can be considered as an average at a fixed spatial point over multiple wave periods, $T_w$, where the averaging time is small compared to the time scale of the currents $T_B$. The terms $\tau_z^x$ and $\tau_z^y$ are assumed to be the sum of mean and fluctuating parts.

We define the non-dimensional horizontal velocity and surface elevation

$$\mathbf{u}^* = \mathbf{u}/\hat{U}_B = \beta\mathbf{u}_w/\hat{U}_w + \mathbf{U}_B/\hat{U}_B = \beta\mathbf{u}_w^* + \mathbf{U}_B^*, \tag{20a}$$

$$\eta^* = \eta/\hat{H} = \beta\eta_w/\hat{A} + \eta_B/\hat{H} = \beta\eta_w^* + \eta_B^*, \tag{20b}$$

slow time,

$$T^* = t/T_B = \epsilon\hat{\omega}t = \epsilon t^*, \tag{21}$$

and coordinates

$$\mathbf{X}^* = \mathbf{x}/L_B = \epsilon\hat{K}\mathbf{x} = \epsilon\mathbf{x}^*. \tag{22}$$

Note that $\mathbf{u}_w^*$ and $\eta_w^*$ are functions of the variables $(x^*, y^*, t^*)$ while $\mathbf{U}_B^*$ and $\eta_B^*$ are functions of the slow variables $(X^*, Y^*, T^*)$. With this scaling, the depth-integrated and wave-averaged wave momentum vector $\mathbf{M} = (M^x, M^y)$ (14) in non-dimensional form is

$$\begin{aligned} \mathbf{M}^* &= \mathbf{M}/\hat{H}\hat{U}_B = \beta^2\mathbf{M}/\hat{A}\hat{U}_w \\ &= \beta^2\mathbf{M}_w^* = \beta\overline{\int_{\eta_B^*}^{\eta^*} \mathbf{u}_w^* dz^*} \approx \beta^2\overline{\mathbf{u}_w^*\eta_w^*}. \end{aligned} \tag{23}$$

Using this scaling, the non-dimensional equations for the shallow-water approximation from (1)-(4) are (omitting the superscript asterisks):

$$\beta\left(u_{wx}+v_{wy}+w_{wz}\right)+\epsilon\left(U_{BX}+V_{BY}+W_{Bz}\right)=0, \qquad (24)$$

$$\begin{aligned}
&\beta u_{wt}+\epsilon U_{BT}+\beta^2\Big((u_w u_w)_x+(v_w u_w)_y+(w_w u_w)_z\Big)\\
&\quad+\beta\epsilon\Big((u_w U_B)_X+(v_w U_B)_Y+(w_w U_B)_z\Big)\\
&\quad+\beta\Big((U_B u_w)_x+(V_B u_w)_y+(W_B u_w)_z\Big)+\epsilon\Big((U_B U_B)_X\\
&\quad+(V_B U_B)_Y+(W_B U_B)_z\Big)-\epsilon f_0\left(V_B+\beta v_w\right)\\
&\quad=-\left(\epsilon\eta_{BX}+\beta\eta_{wx}\right)+\epsilon\tau_z^x, \qquad (25)
\end{aligned}$$

$$\begin{aligned}
&\beta v_{wt}+\epsilon V_{BT}+\beta^2\Big((u_w v_w)_x+(v_w v_w)_y+(w_w v_w)_z\Big)\\
&\quad+\beta\epsilon\Big((u_w V_B)_X+(v_w V_B)_Y+(w_w V_B)_z\Big)\\
&\quad+\beta\Big((U_B v_w)_x+(V_B v_w)_y+(W_B v_w)_z\Big)+\epsilon\Big((U_B V_B)_X\\
&\quad+(V_B V_B)_Y+(W_B V_B)_z\Big)+\epsilon f_0\left(U_B+\beta u_w\right)\\
&\quad=-\left(\epsilon\eta_{BY}+\beta\eta_{wy}\right)+\epsilon\tau_z^y, \qquad (26)
\end{aligned}$$

where the horizontal velocity $\mathbf{u}$ is depth-independent, $f_0=\hat{f}/\epsilon\hat{\omega}$ and $\hat{f}$ is a typical value of the Coriolis parameter assumed to be order $\epsilon$ as are the stress terms. The parameter $f_0$ is included in the equations as an O(1) dimensionless place holder for the Coriolis parameter $f$. The boundary conditions are given by (5)-(7).

### 3.1. Mean flow equations

Applying the averaging operator, $\overline{(\ )}$, to (24)-(26) we obtain equations for the mean flows:

$$U_{BX}+V_{BY}+W_{Bz}=0, \qquad (27)$$

$$\begin{aligned}
&U_{BT}+U_B U_{BX}+V_B U_{BY}-\frac{\beta^2}{\epsilon}F_b^x-f_0 V_B\\
&\quad=-\eta_{BX}+\epsilon\overline{\tau_z^x}, \qquad (28)
\end{aligned}$$

$$\begin{aligned}
&V_{BT}+U_B V_{BX}+V_B V_{BY}-\frac{\beta^2}{\epsilon}F_b^y+f_0 U_B\\
&\quad=-\eta_{BY}+\epsilon\overline{\tau_z^y}, \qquad (29)
\end{aligned}$$

where the continuity equation (27) has been used to rewrite the nonlinear terms in (28) and (29). The wave forcing terms are given by

$$\begin{aligned}
F_b^x&=-\Big(\overline{(u_w u_w)_x}+\overline{(v_w u_w)_y}+\overline{(w_w u_w)_z}\Big)\\
&=-\left(\overline{u_w u_{wx}}+\overline{v_w u_{wy}}\right), \qquad (30)
\end{aligned}$$

and

$$\begin{aligned}
F_b^y&=-\Big(\overline{(u_w v_w)_x}+\overline{(v_w v_w)_y}+\overline{(w_w v_w)_z}\Big)\\
&=-\left(\overline{u_w v_{wx}}+\overline{v_w v_{wy}}\right), \qquad (31)
\end{aligned}$$

where the continuity equation for the wave variables, (24)−(27), has been used to eliminate the $w_{wz}$ terms in (30) and (31). The velocity components $(U_B, V_B, W_B)$ are defined in the region $-H_0<z<\eta_B$. Note that wave forcing terms such as

$$\overline{u_w u_{wx}} = \epsilon \frac{1}{2} \left(\overline{u_w u_w}\right)_X \tag{32}$$

are derivatives of wave-averaged quantities and therefore of order ($\epsilon$). These terms will be evaluated below.

The non-dimensional kinematic boundary condition at the mean surface $\eta_B$, equivalent to the dimensional boundary condition (13) [*Hasselmann*, 1971], is

$$\begin{aligned} &\eta_{BT} + U_B \eta_{BX} + V_B \eta_{BY} - W_B(\eta_B) \\ &\quad = -\beta^2 (M^x_{wX} + M^y_{wY}) \quad \text{at } z = \eta_B. \end{aligned} \tag{33}$$

The bottom boundary condition is

$$U_B H_{0X} + V_B H_{0Y} + W_B(-H_0) = 0 \quad \text{at } z = -H_0. \tag{34}$$

Equations (27), (33) and (34) imply that the mean depth-integrated continuity equation is

$$\eta_{BT} + (HU_B)_X + (HV_B)_Y = -\beta^2 (M^x_{wX} + M^y_{wY}), \tag{35}$$

where

$$H = H_0 + \eta_B \tag{36}$$

is the mean water depth. The boundary conditions for the stress terms are determined below.

### 3.2. Wave equations

The equations for the waves, obtained by subtracting (27), (28) and (29) from (24), (25) and (26) are

$$u_{wx} + v_{wy} + w_{wz} = 0, \tag{37}$$

$$\begin{aligned} &u_{wt} + \epsilon\,(u_w U_{BX} + v_w U_{BY}) + U_B u_{wx} + V_B u_{wy} - \epsilon f_0 v_w \\ &\quad = -\eta_{wx} + \frac{\epsilon \tau'^x|_{z=\eta_B}}{\beta H}, \end{aligned} \tag{38}$$

$$\begin{aligned} &v_{wt} + \epsilon\,(u_w V_{BX} + v_w V_{BY}) + U_B v_{wx} + V_B v_{wy} + \epsilon f_0 u_w \\ &\quad = -\eta_{wy} + \frac{\epsilon \tau'^y|_{z=\eta_B}}{\beta H}, \end{aligned} \tag{39}$$

where the continuity equations (27) and (37) have been used to rewrite the nonlinear wave-mean flow terms and where the relatively small O($\beta$) nonlinear wave-wave terms have been omitted. We have also assumed that the wave breaking process leads to a time-dependent stress ($\tau'^x$, $\tau'^y$) at the mean surface $\eta_B$. Effects of wave-resolved near surface stresses associated with wave breaking are discussed and modeled, for example, by *Veeramony and Svendsen* [2000], utilizing the analysis of laboratory measurements of *Svendsen et al.* [2000].

The boundary conditions are

$$\begin{aligned} &\eta_{wt} + \epsilon \left\{ u_w \eta_{BX} + v_w \eta_{BY} + \eta_w \left( U_{BX} + V_{BY} \right) \right\} \\ &\quad + U_B \eta_{wx} + V_B \eta_{wy} - w_w = 0 \qquad \text{at } z = \eta_B, \end{aligned} \tag{40}$$

and

$$\epsilon \left\{ u_w H_{0X} + v_w H_{0Y} \right\} + w_w = 0 \qquad \text{at } z = -H_0, \tag{41}$$

The depth integral of (37), together with the kinematic boundary conditions for the waves (40) and (41), implies that the depth-integrated continuity equation for the waves is

$$\eta_{wt} + H\left\{u_{wx} + v_{wy}\right\} + U_B\eta_{wx} + V_B\eta_{wy} \\ + \epsilon\left\{u_w H_X + v_w H_Y + \eta_w\left(U_{BX} + V_{BY}\right)\right\} = 0. \tag{42}$$

From (38), (39) and (42), the corresponding equation for the wave energy density

$$E_w = \frac{\eta_w^2}{2} + H\left(\frac{u_w^2 + v_w^2}{2}\right), \tag{43}$$

is

$$E_{wt} + \Big\{\eta_w H u_w + E_w U_B\Big\}_x + \Big\{\eta_w H v_w + E_w V_B\Big\}_y \\ = -\epsilon\bigg\{\left(H u_w^2 + \frac{\eta_w^2}{2}\right)U_{BX} + \left(H v_w^2 + \frac{\eta_w^2}{2}\right)V_{BY} \\ + H u_w v_w\Big(U_{BY} + V_{BX}\Big) - \frac{1}{\beta}\left(\tau'^x u_w + \tau'^y v_w\right)\bigg\}, \tag{44}$$

where the relatively small $O(\beta^2)$ wave terms on the right hand side of (35) are neglected.

### 3.3. Solution of the wave equations

The evaluation of the wave forcing terms (30) and (31) in the mean flow equations requires a solution of (38), (39) and (42) for the waves. We obtain the relevant approximate solution by using slow variables

$$X = \epsilon x, \quad Y = \epsilon y, \quad T = \epsilon t, \tag{45}$$

and assuming that the horizontal wave velocities can be expressed to order $\epsilon$ as

$$u_w = \left[U_0(X,Y,T) + i\epsilon U_1(X,Y,T)\right] \\ \exp\Big(i\Theta(X,Y,T)/\epsilon\Big), \tag{46}$$

$$v_w = \left[V_0(X,Y,T) + i\epsilon V_1(X,Y,T)\right] \\ \exp\Big(i\Theta(X,Y,T)/\epsilon\Big), \tag{47}$$

and that the surface elevation is given by

$$\eta_w = A(X,Y,T)\exp(i\Theta(X,Y,T)/\epsilon), \tag{48}$$

where $A$ is the complex amplitude and $\Theta$ is the phase function, such that

$$\Theta_T = -\omega, \quad \Theta_X = k, \quad \Theta_Y = l, \tag{49}$$

with $\omega$ the absolute frequency and $(k,l) = \mathbf{k}$ the wave number vector. We define $K$ to be the magnitude of $\mathbf{k}$,

$$K = \left(k^2 + l^2\right)^{1/2}. \tag{50}$$

These definitions imply

$$k_Y = l_X, \quad k_T = -\omega_X, \quad l_T = -\omega_Y. \tag{51}$$

We assume that $(\tau'^x, \tau'^y)$ is a real multiple of the wave velocity vector $(u_w, v_w)$ and that it scales so that

$$\tau'^x = -\beta R u_w, \quad \tau'^y = -\beta R v_w, \tag{52a, b}$$

where $R > 0$ is of order one and is a function of the slow variables $X, Y, T$. As a result,

$$\tau'^x u_w + \tau'^y v_w = -\beta R(u_w^2 + v_w^2), \tag{53}$$

where $R$ may depend on the wave amplitude and frequency, water depth and mean velocity. This assumption is clearly an idealization of the dissipative processes due to breaking waves at the surface, but it provides a useful method to incorporate general effects of wave-breaking into the present formulation.

Substituting $u_w$ (46), $v_w$ (47), and $\eta_w$ (48) into (38), (39) and (42) gives at first order:

$$iU_0 \left(-\omega + U_B k + V_B l\right) = -ikA, \tag{54}$$

$$iV_0 \left(-\omega + U_B k + V_B l\right) = -ilA, \tag{55}$$

$$iA \left(-\omega + U_B k + V_B l\right) + iHU_0 k + iHV_0 l = 0. \tag{56}$$

With the relative frequency defined as

$$\omega_r = \omega - U_B k - V_B l, \tag{57}$$

(54) and (55) may be written

$$U_0 = \frac{Ak}{\omega_r}, \quad V_0 = \frac{Al}{\omega_r}, \tag{58a, b}$$

Substituting (58a,b) in (56) implies

$$\omega_r^2 = HK^2, \tag{59}$$

so that the relative frequency $\omega_r$ satisfies the nondimensional shallow-water dispersion relation.

The wave-averaged energy density $E = \overline{E_w}$ is given by

$$E = \overline{E_w} = \frac{\eta_w \eta_w^*}{4} + \frac{H\left(U_0 U_0^* + V_0 V_0^*\right)}{4} = \frac{AA^*}{2}, \tag{60}$$

where $^*$ indicates the complex conjugate. The contribution from the potential energy is equal to that of the kinetic energy. The phase speed $c$ and group speed $c_g$ are equal

$$c \equiv \frac{\omega_r}{K} = (H)^{1/2}, \quad c_g \equiv \frac{\partial \omega_r}{\partial K} = (H)^{1/2}, \tag{61a, b}$$

and the components of the wave velocity vector are

$$\mathbf{c} \equiv (c^x, c^y) \equiv (ck/K, cl/K). \tag{61c}$$

At order $\epsilon$, from (42) the complex amplitude $A$ satisfies

$$\begin{aligned} &A_T + (HU_0)_X + (HV_0)_Y - HU_1 k - HV_1 l \\ &\quad + (U_B A)_X + (V_B A)_Y = 0, \end{aligned} \tag{62}$$

where from (38) and (39)

$$\begin{aligned} U_1 \omega_r = &-\left[U_{0T} + U_B U_{0X} + V_B U_{0Y} + U_0 U_{BX}\right. \\ &\left. + V_0 U_{BY} - f_0 V_0 + A_X + RU_0\right] \end{aligned} \tag{63}$$

$$V_1 \omega_r = -\left[V_{0T} + U_B V_{0X} + V_B V_{0Y} + U_0 V_{BX}\right.$$

$$+V_0 V_{BY} + f_0 U_0 + A_Y + RV_0 \big] . \qquad (64)$$

We substitute the velocity components from (58a,b), (63) and (64) into (62) to determine the equation for the amplitude $A$. The resulting amplitude equation implies, after considerable but straightforward manipulation, the wave action equation with dissipation:

$$\left(\frac{E}{\omega_r}\right)_T + \left\{ \left(c^x + U_B\right) \frac{E}{\omega_r} \right\}_X + \left\{ \left(c^y + V_B\right) \frac{E}{\omega_r} \right\}_Y = -\frac{\varepsilon_d}{\omega_r}, \qquad (65)$$

where $\varepsilon_d = RE$ is the energy dissipation.

## 4. Wave forcing of the mean circulation

The three components of the wave forcing of the mean currents can be calculated from the wave velocities given by (46), (47), (58a,b), (63) and (64). The first component, calculated in section 4.1 is the flux of mass through the mean surface (33) [*Hasselmann*, 1971]. The second component, determined in section 4.2 and Appendix B is the body force (30), (31) arising from the average of the wave nonlinear terms. The remaining term is the surface stress caused by the breaking waves and is discussed in 4.3. The relationship of this formulation to the radiation stress gradient forcing for depth-averaged mean currents is discussed in Appendix D.

### 4.1. Surface boundary condition

The surface kinetic boundary condition (33) is equivalent to a non-zero velocity perpendicular to the mean surface $\eta_B$.

$$\begin{aligned} W_\perp &= W_B - \eta_{BT} - U_B \eta_{BX} - V_B \eta_{BY} \\ &= \beta^2 (M^x_{wX} + M^y_{wY}) = \beta^2 \left\{ \left(\frac{Ek}{\omega_r}\right)_X + \left(\frac{El}{\omega_r}\right)_Y \right\} . \qquad (66) \end{aligned}$$

Note that this boundary condition results in a mean momentum flux forcing at the surface. This forcing appears explicitly, for example, in the depth-integrated momentum balance for the mean flow (D7) in Appendix D.

### 4.2. Body forces

We calculate the body forces as

$$F^x_b = -\left\{ \overline{\Re u_w \, \Re u_{w\,x}} + \overline{\Re v_w \, \Re u_{w\,y}} \right\} \qquad (67)$$

$$F^y_b = -\left\{ \overline{\Re u_w \, \Re v_{w\,x}} + \overline{\Re v_w \, \Re v_{w\,y}} \right\} , \qquad (68)$$

where $\Re$ is the real part of a complex number and the complex velocities are given by (46) and (47). Evaluation of the body force $F^x_b$ is discussed below with most of the calculations outlined in Appendix B. A similar procedure gives $F^y_b$.

The first term in the body force $F^x_b$ is

$$\begin{aligned} \left(\overline{\Re u_w \, \Re u_{wx}}\right) &= \frac{1}{4} \left\{ \overline{\left(u_w + u^*_w\right)\left(u_{wx} + u^*_{wx}\right)} \right\} \\ &= \frac{\epsilon}{4} \left(U_0 U^*_{0X} + U_{0X} U^*_0\right) \\ &= \frac{\epsilon}{4} \left(\frac{k^2 A A^*}{\omega_r^2}\right)_X = \frac{\epsilon}{2} \left(\frac{k^2 E}{\omega_r^2}\right)_X . \qquad (69) \end{aligned}$$

The other term $\overline{\Re v_w \, \Re u_{wy}}$ is calculated in Appendix B and is given by (B7). From (69) and (B7) we find that

$$F_b^x = -\epsilon \left[ \frac{K^2}{2\omega_r^2} (E)_X + \frac{kE}{\omega_r} \left( \frac{k}{\omega_r} \right)_X + \frac{lE}{\omega_r} \left( \frac{k}{\omega_r} \right)_Y - \frac{lEK^2}{\omega_r^3} (V_{BX} - U_{BY} + f_0) + \frac{lE}{\omega_r^3} (k\omega_{rY} - l\omega_{rX}) \right]. \quad (70)$$

With (59) and wave momentum

$$\mathbf{M}_w = (M_w^x, M_w^y) = (Ek/\omega_r, El/\omega_r), \quad (71)$$

(70) reduces to

$$F_b^x = \epsilon \left\{ -\frac{1}{2} \left( \frac{E}{H} \right)_X + \frac{M_w^y}{H} (V_{BX} - U_{BY} + f_0) \right\}. \quad (72)$$

From a similar calculation we find

$$F_b^y = \epsilon \left\{ -\frac{1}{2} \left( \frac{E}{H} \right)_Y - \frac{M_w^x}{H} (V_{BX} - U_{BY} + f_0) \right\}. \quad (73)$$

The terms involving $(E/H)_X$ and $(E/H)_Y$ can be seen to come directly from part of the radiation stress in the two-dimensional depth-averaged formulation [*Longuet-Higgins*, 1973; *Smith*, 2006 and Appendix D]. Without restriction to shallow-water waves, *Longuet-Higgins* [1973] shows that, in the depth-integrated case with weak currents and no time variation of the wave field, the radiation stress forcing can be expressed as the sum of two terms, one a gradient term which reduces to the first terms of (72) and (73) in the limit of shallow-water waves and the other proportional to the wave dissipation which we will show to be the surface stress in this formulation.

The other terms in (72) and (73) are related to the vortex force [*Leibovich*, 1980]. These terms result from the wave-current interactions [*Garrett*, 1976; *McWilliams et al.*, 2004; *Smith*, 2006] and involve a product of the wave momentum and the vertical component of the vorticity of the mean velocities $V_{Bx} - U_{By}$ and the Coriolis parameter $f_0$. We note that an identical form of this depth-independent vortex force was found by *McWilliams et al.* [2004] in the shallow water limit (Section 12) of their depth-dependent results for non-dissipative waves and weak currents.

### 4.3. Surface stress

Equations (16) and (17) for the surface stress in terms of the wave-averaged wave momentum are derived by integrating the momentum equations (25)-(26) for the total flow from the mean surface $\eta_B$ to the free surface $\eta = \eta_B + \beta\eta_w$ and wave-averaging. With the assumptions of Section 3, the non-dimensional wave-averaged surface stress (16) in the $X$ direction reduces to

$$\frac{\overline{\tau^x}}{\beta^2} = -M_{wT}^x - (2U_B M_w^x)_X - (U_B M_w^y + V_B M_w^x)_Y + U_B (M_{wX}^x + M_{wY}^y) - \frac{E_X}{2} + f_0 M_w^y + \frac{1}{\epsilon} \left( \overline{\Re u_w \, \Re w_w} \right) |_{z=\eta_B} - E\frac{k^2}{\omega_r^2}\eta_{BX} - E\frac{kl}{\omega_r^2}\eta_{BY}. \quad (74)$$

The term $\left( \overline{\Re u_w \, \Re w_w} \right) |_{z=\eta_B}$ is evaluated in Appendix C and is given by:

$$\left( \overline{\Re u_w \, \Re w_w} \right) |_{z=\eta_B} = \epsilon \left\{ \frac{E_X}{2} - \frac{El^2}{2\omega_r^2} H_X \right.$$

$$-M_w^y (V_{BX} - U_{BY} + f_0) + \frac{Ek^2}{2\omega_r^2} H_X - (M_w^x c^x)_X$$
$$\left. - (M_w^x c^y)_Y + \frac{Elk}{\omega_r^2} H_Y - \frac{Ek^2}{\omega_r^2} H_{0X} - \frac{Elk}{\omega_r^2} H_{0Y} \right\}. \quad (75)$$

An equation for the time-rate of change of the $x$ and $y$ components of the wave-momentum vector can be derived from the wave action equation (65) by multiplying by $k$ and $l$, respectively. The resulting equation for the $X$ component is

$$M_{wT}^x + [M_w^x (c^x + U_B)]_X + [M_w^x (c^y + V_B)]_Y + \frac{E}{2H} H_X$$
$$+ M_w^x U_{BX} + M_w^y V_{BX} = -\frac{\varepsilon_d k}{\omega_r}. \quad (76)$$

Substituting the value of $\left(\overline{\Re u_w \, \Re w_w}\right)|_{z=\eta_B}$ from (75) and $M_{wT}^x$ from (76) into (74) we obtain

$$\frac{\overline{\tau^x}}{\beta^2} = \frac{\varepsilon_d k}{\omega_r}, \quad \text{at } z = \eta_B. \quad (77)$$

It follows similarly that the surface stress in the $Y$ direction is

$$\frac{\overline{\tau^y}}{\beta^2} = \frac{\varepsilon_d l}{\omega_r}, \quad \text{at } z = \eta_B. \quad (78)$$

Consequently, the surface stress vector has a magnitude that is directly related to the near-surface wave dissipation and a direction that is aligned with that of the wave propagation, in agreement with the findings of *Deigaard and Fredsøe* [1989] and *Deigaard* [1993] (see also *Fredsøe and Deigaard* [1992] chapter 6).

It may be noted that the calculation of $\left(\overline{\Re u_w \, \Re w_w}\right)|_{z=\eta_B}$ (75) shows explicitly that the gradient of the pressure contribution to the radiation stress $E_X/2$, which arises in (74) from integration over the surface layer from $\eta_B$ to $\eta$, does not contribute directly to a surface stress, consistent with physical reasoning [*De Vriend and Kitou*, 1991]. Likewise, the combined substitution of (75) and (76) into (74) shows the same result for the wave current interaction terms involving products of $U_B$, $V_B$ and $M_w^x$, $M_w^y$.

The results found in subsections 4.1-4.3 provide the required expressions for the wave forcing terms in the mean flow equations (27)-(33). We note that, as discussed in Appendix D, the depth-integral of the resulting equations, with the body force specified as in 4.2 and the boundary conditions found in 4.1 and 4.3, agrees exactly with previously derived depth-integrated equations for the mean flow [*Smith*, 2006].

## 5. Discussion

With the objective of establishing a rational approximation in an Eulerian frame of reference for forcing of wave-averaged circulation in the surf zone represented by three-dimensional primitive-equation models, we derive the forcing for the case of depth-independent horizontal currents and shallow-water waves. We include wave-current interaction terms in both the wave and mean equations. We find that the surface stress is non-zero only when there is dissipation of wave energy near the surface. The body force consists of two parts, one related to the gradients in part of the radiation stress tensor with modifications for variation is water depth. The other is a wave-current interaction term that involves a product of the wave momentum and the vertical vorticity of the mean current plus the Coriolis parameter. It is related to the vortex force of *Craik and Leibovich* [1976] [see *Leibovich*, 1983; *Garrett*, 1976; *Smith*, 2006]. We find that the evaluation of the terms involving $(\overline{u_w w_w})$ and $(\overline{v_w w_w})$ is critically important for a consistent estimation of the forcing. The proper specification of the wave related surface stress cannot be determined without the correct evaluation of these terms.

The wave-current interactions that result in the vortex force term are frequently omitted from surf zone models based on scaling arguments that assume relatively weak currents. We have made a deliberate choice to scale the currents here so that these interaction terms are retained. One advantage is that it helps demonstrate the relation of surf zone forcing models to the substantial, but generally separate, set of wave-current interaction results that originated in studies of Langmuir circulation [e.g. *Leibovich*, 1983]. A second advantage is that the present formulation remains valid in the weak current limit. Thus, if the currents are weak in the applications, the wave-current interaction forces will be correspondingly small, but the remaining wave-forcing will be properly specified. In the companion paper NA2, we use this formulation to force a three-dimensional primitive equation model and compare the results with observations from the DUCK94 field experiment. It is shown there that the wave-current interaction forcing terms can play an appreciable role. The extension of the present results to include depth-dependent currents is clearly needed and is a topic for future research.

## Appendix A: Wave-averaged surface boundary conditions

The wave-averaged boundary condition (13) at the mean surface $\overline{\eta}$ was derived originally by *Hasselmann* [1971]. Physically, it states that the divergence of the horizontal time-averaged wave mass flux, which in an Eulerian formulation occurs between the wave crests and troughs [e.g., *Phillips*, 1977], is balanced by a mean normal mass flux at the mean free surface. This same boundary condition naturally arises in the asymptotic wave-averaged Eulerian analyses of *McWilliams and Restrepo* [1999] and of *McWilliams et al.* [2004]. Results that might appear to be different are obtained, however, in the recent analyses of *Mellor* [2003, 2005] where time-averages in an Eulerian frame of reference are not utilized, but rather time-averages in a wave following coordinate system are used.

Questions naturally arise concerning the differences, especially with respect to implementation in three-dimensional models in Eulerian coordinates. To obtain some insight into these issues, it is useful to reexamine the original *Hasselmann* [1971] derivation. Accordingly, following *Hasselmann* [1971], taking $(x, y)$ derivatives of (14a,b) and adding, we find

$$M^x_x + M^y_y = \overline{(u\,\eta_x + v\,\eta_y)}|_{z=\eta} - \left(\overline{u}\,\overline{\eta}_x + \overline{v}\,\overline{\eta}_y\right)|_{z=\overline{\eta}} + \overline{\int_{\overline{\eta}}^{\eta} (u_x + v_y)\, dz}. \qquad \text{(A1)}$$

Integrating the last term in (A1) after substitution of $w_z$

from (1), we obtain

$$\overline{(\eta_t + u\eta_x + v\eta_y - w)}|_{z=\eta}$$
$$= (\overline{\eta_t} + \overline{u}\,\overline{\eta_x} + \overline{v}\,\overline{\eta_y} - \overline{w})\,|_{z=\overline{\eta}} + M^x_x + M^y_y, \qquad \text{(A2)}$$

where the equation $\overline{(\eta_t)}|_{z=\eta} = (\overline{\eta_t})\,|_{z=\overline{\eta}}$ has been added. From (6), the left hand side of (A2) is zero which implies (13). The point to be emphasized here is that (A2) is an equation that converts a time-average at the wave-following free surface $\eta$ into an equivalent relationship between time-averaged variables at $\overline{\eta}$ and the divergence of the time-averaged Eulerian wave mass flux. It seems clear that if equations are being formulated for implementation in an Eulerian coordinate model, time-averages at wave-following locations need to be properly converted to time-averages at fixed spatial locations as in (A2). We also call attention to Appendix D where it is shown that boundary condition (13), which implies (33), is necessary to provide consistency with separately formulated depth-integrated equations for the mean flow [*Smith*, 2006].

## Appendix B: Evaluation of $(\overline{\Re v_w\,\Re u_{wy}})$

Evaluation of $(\overline{\Re v_w\,\Re u_{wy}})$ is required to find the body force term in Section 4.2. We divide the terms of $(\overline{\Re v_w\,\Re u_{wy}})$ into parts that can be more easily evaluated:

$$\left(\overline{\Re v_w\,\Re u_{wy}}\right) = \frac{\epsilon}{4}\left[E_1 + E_2 + E_3 + E_4\right], \qquad \text{(B1)}$$

where

$$E_1 = \frac{l^2}{\omega_r^2}\left[A\left(\frac{A^*k}{\omega_r}\right)_T + A^*\left(\frac{Ak}{\omega_r}\right)_T\right]$$
$$-\frac{lk}{\omega_r^2}\left[A\left(\frac{A^*l}{\omega_r}\right)_T + A^*\left(\frac{Al}{\omega_r}\right)_T\right]$$
$$= \frac{2AA^*l}{\omega_r^3}\left[lk_T - kl_T\right] = \frac{2AA^*l}{\omega_r^3}\left[k\omega_Y - l\omega_X\right], \qquad \text{(B2)}$$

$$E_2 = \frac{l^2}{\omega_r^2}\left[AU_B\left(\frac{A^*k}{\omega_r}\right)_X + A^*U_B\left(\frac{Ak}{\omega_r}\right)_X\right.$$
$$+AV_B\left(\frac{A^*k}{\omega_r}\right)_Y + A^*V_B\left(\frac{Ak}{\omega_r}\right)_Y + 2U_{BX}\frac{AA^*k}{\omega_r}$$
$$\left.+2U_{BY}\frac{AA^*l}{\omega_r}\right]$$
$$= \frac{l^2}{\omega_r^2}\left[U_B\frac{k}{\omega_r}(AA^*)_X + V_B\frac{k}{\omega_r}(AA^*)_Y\right.$$
$$+2U_B AA^*\left(\frac{k}{\omega_r}\right)_X + 2V_B AA^*\left(\frac{k}{\omega_r}\right)_Y$$
$$\left.+2AA^*\frac{k}{\omega_r}U_{BX} + 2AA^*\frac{l}{\omega_r}U_{BY}\right], \qquad \text{(B3)}$$

$$E_3 = -\frac{lk}{\omega_r^2}\left[AU_B\left(\frac{A^*l}{\omega_r}\right) + A^*U_B\left(\frac{Al}{\omega_r}\right)_X\right.$$
$$+AV_B\left(\frac{A^*l}{\omega_r}\right)_Y + A^*V_B\left(\frac{Al}{\omega_r}\right)_Y + 2V_{BX}\frac{AA^*k}{\omega_r}$$
$$\left.+2V_{BY}\frac{AA^*l}{\omega_r}\right]$$

$$= -\frac{lk}{\omega_r^2}\Big[U_B \frac{l}{\omega_r}(AA^*)_X + V_B \frac{l}{\omega_r}(AA^*)_Y$$
$$+2U_B AA^* \left(\frac{l}{\omega_r}\right)_X + 2V_B AA^* \left(\frac{l}{\omega_r}\right)_Y$$
$$+2AA^* \frac{k}{\omega_r} V_{BX} + 2AA^* \frac{l}{\omega_r} V_{BY}\Big], \qquad \text{(B4)}$$

$$E_4 = \frac{l^2}{\omega_r^2}\left[(AA^*)_X - 2\frac{f_0 AA^* l}{\omega_r}\right] - \frac{lk}{\omega_r^2}\Big[(AA^*)_Y$$
$$+ 2\frac{f_0 AA^* k}{\omega_r}\Big] + \frac{l}{\omega_r}\left[\left(\frac{AA^* k}{\omega_r}\right)_Y + AA^*\left(\frac{k}{\omega_r}\right)_Y\right]$$
$$= \frac{l^2}{\omega_r^2}(AA^*)_X + 2\frac{AA^* l}{\omega_r}\left(\frac{k}{\omega_r}\right)_Y - 2\frac{lK^2}{\omega_r^3} f_0 AA^*. \qquad \text{(B5)}$$

Using $\omega_r = \omega - kU_B - lV_B$ we find

$$E_1 + E_2 + E_3 = \frac{2AA^*}{\omega_r^3}\Big[lk\omega_{rY} - l^2\omega_{rX}$$
$$+l\left(k^2 + l^2\right)(U_{BY} - V_{BX})\Big]. \qquad \text{(B6)}$$

Combining this result with the definition of $E_4$ completes the computation.

$$\left(\overline{\Re v_w\, \Re u_{wy}}\right) = \frac{\epsilon}{2}\Big[\frac{l^2}{2\omega_r^2}(AA^*)_X + \frac{lAA^*}{\omega_r}\left(\frac{k}{\omega_r}\right)_Y$$
$$-lAA^* \frac{K^2}{\omega_r^3}(V_{BX} - U_{BY} + f_0)$$
$$+\frac{lAA^*}{\omega_r^3}(k\omega_{rY} - l\omega_{rX})\Big]$$
$$= \epsilon\Big[\frac{l^2}{2\omega_r^2}E_X + \frac{lE}{\omega_r}\left(\frac{k}{\omega_r}\right)_Y - \frac{lEK^2}{\omega_r^3}(V_{BX} - U_{BY}$$
$$+f_0) + \frac{lE}{\omega_r^3}(k\omega_{rY} - l\omega_{rX})\Big]. \qquad \text{(B7)}$$

The expression (B7) may be rewritten in a form useful in Appendix C.

$$-\frac{H}{\epsilon}\left(\overline{\Re v_w\, \Re u_{wy}}\right) = \frac{1}{2}(M_w^x c^x)_X + M_w^y (V_{BX} - U_{BY}$$
$$+f_0) + \frac{El^2}{2HK^2}H_X - \frac{E_X}{2}, \qquad \text{(B8)}$$

where we use

$$(M_w^x c^x)_X = \left(\frac{Ek^2}{K^2}\right)_X = \frac{k^2 E_X}{K^2} + \frac{Ek_X^2}{K^2}$$
$$-\frac{2Ek^2 K_X}{K^3}. \qquad \text{(B9)}$$

## Appendix C: Evaluation of $\left(\overline{\Re u_w\, \Re w_w}\right)|_{z=\eta_B}$

Evaluation of $\left(\overline{\Re u_w\, \Re w_w}\right)|_{z=\eta_B}$ is required to find the surface stress in Section 4.3. We evaluate $\left(\overline{\Re u_w\, \Re w_w}\right)|_{z=\eta_B}$ as

$$\left(\overline{\Re u_w\, \Re w_w}\right)|_{z=\eta_B} = \int_{-H_0}^{\eta_B}\left(\overline{\Re u_w\, \Re w_w}\right)_z dz$$
$$+\left(\overline{\Re u_w\, \Re w_w}\right)|_{z=-H_0}, \qquad \text{(C1)}$$

where

$$\left(\overline{\Re u_w \, \Re w_w}\right)|_{z=-H_0} = -\epsilon \left\{ \overline{\Re (u_w)^2} H_{0X} + \overline{\Re u_w \, \Re v_w} H_{0Y} \right\}. \tag{C2}$$

The continuity equation for the waves (37) implies that

$$\left(\overline{\Re u_w \, \Re w_w}\right)_z = -\left\{ \overline{\Re u_w \, \Re u_{wx}} + \epsilon \left(\overline{\Re u_w \, \Re v_w}\right)_Y - \overline{\Re v_w \, \Re u_{wy}} \right\}, \tag{C3}$$

where $\left\{\left(\overline{\Re u_w \, \Re u_{wx}}\right)\right\}$ and $\left\{\left(\overline{\Re v_w \, \Re u_{wy}}\right)\right\}$ are calculated in (69) and (B8) respectively, and

$$\left(\overline{\Re v_w \, \Re u_w}\right)_Y = \frac{1}{2}\left(\frac{AA^* kl}{\omega_r^2}\right)_Y = \left(\frac{Ekl}{\omega_r^2}\right)_Y. \tag{C4}$$

Substituting (C2), (C3) and (C4) into (C1) we find (75).

## Appendix D: Comparison with the depth-integrated equations

Integrating the total horizontal momentum equations (25) and (26) from the bottom $H_0$ to the free surface $\eta$ and wave-averaging gives the equations for the total wave-averaged momentum $\hat{\mathbf{M}} = (\hat{M}^x, \hat{M}^y)$ in terms of the radiation stresses [*Phillips*, 1977; *Garrett*, 1976; *Smith*, 2006]. The total wave-averaged momentum in the $x$ direction is given by

$$\hat{M}^x = \overline{\int_{-H_0}^{\eta} u \, dz} = M_B^x + \beta^2 M_w^x, \tag{D1}$$

where

$$M_B^x = \int_{-H_0}^{\eta_B} U_B dz, \tag{D2}$$

is the contribution from the mean flow and $M_w^x$ (23) is the contribution from the waves. For depth-independent horizontal currents and shallow-water waves this becomes

$$\begin{aligned} M_{BT}^x &+ \left\{H {U_B}^2\right\}_X + \left\{H(U_B \, V_B)\right\}_Y + H\eta_{BX} - f_0 M_B^y \\ &= \beta^2 \left\{ -M_{wT}^x + f_0 M_w^y - \left\{S_{xx} + 2U_B M_w^x\right\}_X \right. \\ &\left. - \left\{S_{xy} + U_B M_w^y + V_B M_w^x\right\}_Y \right\}, \end{aligned} \tag{D3}$$

where we again assume that $\tau$ is zero on the free surface $\eta$ and where $S_{xx}$ and $S_{xy}$ are components of the radiation stress tensor

$$\begin{aligned} S_{xx} &= \overline{\int_{-H_0}^{\eta} u_w^2 + p^w \, dz} \\ &= E\frac{kc^x}{\omega_r} + \frac{E}{2} = M_w^x c^x + \frac{E}{2}, \end{aligned} \tag{D4}$$

$$S_{xy} = \overline{\int_{-H_0}^{\eta} u_w \, v_w \, dz} = E\frac{kc^y}{\omega_r} = M_w^x c^y = M_w^y c^x. \tag{D5}$$

In addition,

$$M_B^x = HU_B. \tag{D6}$$

Subtracting the wave momentum equation (76) from (D3), we obtain the equation for the depth-integrated and wave-averaged momentum [*Garrett*, 1976; *Smith*, 2006],

$$M^x_{BT} + \left\{HU_B^2\right\}_X + \left\{H(U_B\,V_B)\right\}_Y + H\eta_{BX} - f_0 M^y_B$$
$$= \beta^2 \left\{ M^y_w \left(V_{Bx} - U_{By} + f_0\right) + \frac{k\varepsilon_d}{\omega_r} \right.$$
$$\left. -U_B \left(M^x_{wX} + M^y_{wY}\right) - \frac{H}{2}\left(\frac{E}{H}\right)_X \right\}. \qquad \text{(D7)}$$

The same equation is obtained by depth integrating (28) for $U_{BT}$ and by using (27), (66), (72) and (77). The wave forcing terms on the right hand side of (D7) are from left to right: the depth integral of the vortex contribution to the three-dimensional body force plus the wave Coriolis term, the surface stress from wave breaking, the advection of mean velocity by the non-zero surface velocity perpendicular to the mean surface and a term from the part of the radiation stress gradient that includes the effects of sloping bottom and changes in wave energy density. Note that the surface boundary condition (13), which translates in Section 3 to (33), results in the momentum flux term $-U_B(M^x_{wX} + M^y_{wY})$ which is necessary to provide the agreement that should be found between the depth-integral of (28) and the separately derived depth-integrated equation (D7) [*Smith*, 2006]. Consistent with Smith [2006] the only part of the radiation stress gradients that remains explicitly when the wave momentum is subtracted from the depth-integrated equation for the total flow is the finite depth term, which for shallow-water is $-H(E/2H)_X$. The remainder of the forcing consists of wave-current interaction terms and the surface stress arising from dissipation in the surface layer. The latter, of course, is related to radiation stress gradients resulting from wave energy variations produced by dissipative processes near the surface [*Longuet-Higgins*, 1973].

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