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On the discrepancy in long wave scaling

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Abstract

Scaling in long wave theory is quite complex. In the present paper, the problems associated with scaling in long waves are described and characteristic length scales for the dynamical variables are developed. The vertical length scale over which the horizontal velocity varies is shown to be different from the vertical length scale over which the vertical velocity varies and are consistent with the properties of long wave theory.

Keywords: Nonlinear waves; Scaling; Long waves; Shallow water waves

1. Introduction

Scaling analysis plays a large part in the understanding of fluid flows. In a particular situation, they can give information on whether a physical process is dynamically important or not. This is usually done by introducing scales of motion that characterize the spatial and temporal variations of dynamical fields. Essentially, these are dimensional quantities expressing the overall magnitude of the variables under consideration. They are estimates rather than precisely defined quantities and are solely understood as *orders of magnitude* estimates of physical variables. In most situations, the key scales are those of time, length and velocity. Nondimensional variables are obtained by dividing the dimensional variables by their characteristic scales. Proper scaling of the dimensional equations results in nondimensional equations, which gives information on the dynamical importance of physical processes. The nondimensional variables obtained by scaling *should be O(1)* if meaningful information must emerge from the nondimensional equations.

In long wave theory, the characteristic length scales that represent the wave motion are wavelength, $2\pi/k$, wave height, a, and water depth h. These length scales give rise to three nondimensional numbers (only two are independent). They are:

$\epsilon = ka$—wave steepness
$\delta = a/h$—relative wave height and
$\mu = kh$—relative water depth

The proper scaling of long wave equations and appropriate boundary conditions ensures that at lowest order, the dominant terms are obvious. Scaling, however, has to preserve a few important properties of long waves. These are:

- All long waves correspond to $\mu \ll 1$. This means that the length of the wave is very long compared

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to the water depth. Thus, the long wave equations can be obtained by imposing the constraint $\mu \ll 1$ on the nondimensional set of equations.

- To the lowest order (i.e., $\mu \ll 1$), long wave theory implies the hydrostatic balance in the vertical momentum equation. Thus, proper scaling of the vertical momentum equation would result in the hydrostatic balance.
- All the terms in the vorticity equation should be of the same order, since the flow is assumed to be irrotational.
- The terms in the continuity equation must be of the same order.

Many researchers have performed scaling in long wave theory (e.g., Yoon and Liu, 1989; Nwogu, 1993; Wei et al., 1995) to obtain Boussinesq equations, which belong to the category of long wave theory with the additional constraint of $\delta/\mu^2 = O(1)$. The weakness of their scaling analysis is that they do not show the properties discussed above. Yoon and Liu (1989) obtain the hydrostatic balance through their scaling but are unable to properly scale the vorticity balance and the continuity equation. Nwogu (1993) is unable to obtain the terms in the vorticity balance to be of the same order. In this paper, the shortcomings of the scaling performed by the previous researchers are discussed and an alternate scaling which does not suffer from the above mentioned deficiencies is provided.

This paper is organized as follows. The next section introduces the governing equations for the wave motion. A linear solution is obtained, and in Section 3, these linear solutions are used to obtain characteristic scales for the long wave motion. In Section 4, these scales are used to develop the nondimensional equations to check the consistency of the scaling. Scaling of long waves by different researchers is discussed in Section 5. The final section is the Conclusion.

2. Equations

The governing equation for the inviscid, irrotational wave motion is the Laplace's equation. This can be stated as

$$\phi_{xx} + \phi_{zz} = 0 \tag{1}$$

where ϕ is the velocity potential, x is the horizontal coordinate, z is the vertical coordinate and subscripts denote differentiation with respect to a particular variable.(For simplicity, we consider the two-dimensional problem.). The imposed boundary conditions are

- The kinematic-free surface boundary condition

$$\eta_t + \phi_x \eta_x = \phi_z \qquad z = \eta \tag{2}$$

where η is the free surface elevation.

- The dynamic-free surface boundary condition

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = C(t) \qquad z = \eta \tag{3}$$

where g is the acceleration due to gravity, C is any function of time, and t is the time.

- No flow through the bottom (flat-bottom for simplicity)

$$\phi_z = 0 \qquad z = -h \tag{4}$$

where h is the water depth.

The linear solution to the above boundary value problem is

$$\phi = \frac{ga}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx - \omega t) \tag{5}$$

where ω is the angular velocity.

$$\eta = a\cos(kx - \omega t) \tag{6}$$

and the dispersion relation is given as

$$\omega = \sqrt{gk\tanh(kh)} \tag{7}$$

These solutions can be used to obtain characteristic scales for the different variables.

3. Scaling

Characteristic scales for the dynamic variables can be obtained from the linear solution. The horizontal velocity, u, and the vertical velocity, w, can be obtained from the gradient of the potential. They can be expressed as

$$u = \phi_x = \frac{gak}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx-\omega t) \tag{8}$$

and

$$w = \phi_z = \frac{gak}{\omega}\frac{\sinh k(h+z)}{\cosh kh}\sin(kx-\omega t) \tag{9}$$

The horizontal and vertical velocities in long waves are obtained by evaluating the above expressions in the limit $\mu \ll 1$. This results in

$$u = \frac{gak}{\omega}\cos(kx-\omega t) \tag{10}$$

and

$$w = \frac{gak^2(h+z)}{\omega}\sin(kx-\omega t) \tag{11}$$

Thus, the characteristic velocity scales in the horizontal and vertical directions are gak/ω and gak^2h/ω, respectively. Also note that while w varies linearly with water depth, u is nearly constant over depth. Since u is nearly constant over the water depth, clearly h cannot be the characteristic length scale over which u varies. Hence, it is not a simple exercise to choose the vertical length scale for the dynamical variables. The linear solution is used to evaluate the vertical length scales for u and w. Evaluating $\partial u/\partial z$ and $\partial w/\partial z$, we obtain

$$\frac{\partial u}{\partial z} = \frac{gak^2}{\omega}\frac{\sinh k(h+z)}{\cosh kh}\cos(kx-\omega t) \tag{12}$$

and

$$\frac{\partial w}{\partial z} = \frac{gak^2}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx-\omega t) \tag{13}$$

Again, for the limit $\mu \ll 1$, the above expressions become

$$\frac{\partial u}{\partial z} = \frac{gak^3(h+z)}{\omega}\cos(kx-\omega t) \tag{14}$$

and

$$\frac{\partial w}{\partial z} = \frac{gak^2}{\omega}\sin(kx-\omega t) \tag{15}$$

Also, we can obtain expressions for $\partial u/\partial x$ and $\partial w/\partial x$. They are, in the shallow water limit,

$$\frac{\partial u}{\partial x} = \frac{gak^2}{\omega}\sin(kx-\omega t) \tag{16}$$

and

$$\frac{\partial w}{\partial x} = \frac{gak^2(h+z)}{\omega}\sin(kx-\omega t) \tag{17}$$

From Eqs. (12)–(17), we can obtain characteristic length scales for the variations of u and w.

$$\left|\frac{\partial u}{\partial z}\right| = \frac{gak^3h}{\omega} \tag{18}$$

$$\left|\frac{\partial w}{\partial z}\right| = \frac{gak^2}{\omega} \tag{19}$$

where $|\partial u/\partial z|$ and $|\partial w/\partial z|$ are the characteristic values of $\partial u/\partial z$ and $\partial w/\partial z$, respectively.

The characteristic vertical length scale over which u varies is $|u|/|\partial u/\partial z|$. Thus, the vertical length scale over which u varies is h/μ^2. Since μ is small in long waves, h/μ^2 is very large. This means that over the water depth, h, u is nearly constant. In contrast, the characteristic vertical length scale for w is $|w|/|\partial w/\partial z|$. In Eqs. (11) and (19), we can show that the vertical length scale for w is the water depth, h.This anisotropy in scaling makes the scaling of long waves quite complicated and causes inconsistencies if scaling is done improperly. All the previous researchers have ignored the difference in the vertical length scales for u and w in their scaling of the velocities.

Similarly, characteristic horizontal length scales for u and w can be derived. The horizontal length scale for u and w can be shown to be $1/k$. Table 1 compares the different scaling used by various researchers. It clearly shows that the vertical length scale of both the horizontal as well as the vertical velocities are scaled incorrectly by all the researchers. The scaling of the pressure term in the horizontal momentum equations is

Table 1
Comparison of various theories and their corresponding characteristic scales

	h_u	h_w	l_u	l_w	P_w	P_h
Correct	h/mu^2	h	$1/k$	$1/k$	ρgh	ρga
Nwogu (1993)	h	h	$1/k$	$1/k$	ρgh	ρgh
Yoon and Liu (1989)	h	h	$1/k$	$1/k$	ρgh	ρgh
Peregrine (1967)	h	h	$1/k$	$1/k$	ρgh	ρgh

h_u and h_w represent the vertical length scale over which u and w vary. Similarly, l_u and l_w represent the horizontal length scales over which u and w vary. P_w and P_u are the characteristic scales for pressure in the vertical momentum equations and horizontal momentum, respectively.

also incorrect. This is the main reason why their scaling is not able to satisfy all the criterion mentioned previously.

Next, we use observational data to show that while the vertical velocity (w) does vary within the range of the water depth, the horizontal velocity (u) does not. Velocity and surface elevation data were collected by Cox et al. (1995) which was later analyzed and reproduced in Veeramony and Svendsen (2000). The experiments were conducted in a wave flume with a plain beach of 1:35. The wave height at the wavemaker [see Fig. 3, Veeramony and Svendsen (2000)] was 11.5 cm; the water depth at beginning of the slope region was 0.4 m; and the wave period was 2.2 s. Fig. 1 shows the vertical profile of the velocity as a function of time (the maximum depth is 0.28 m). The surface elevation is also plotted to identify the phase over which the data is collected. Fig. 1 shows the vertical variation of the vertical velocity. It is clear that in the region where the surface elevation gradient is the greatest (third and fifth curves from left), the vertical velocity varies linearly. This shows that the characteristic vertical length scale of w is $O(h)$.Similarly, Fig. 2 shows the same for horizontal velocity. In contrast, horizontal velocity remains nearly constant

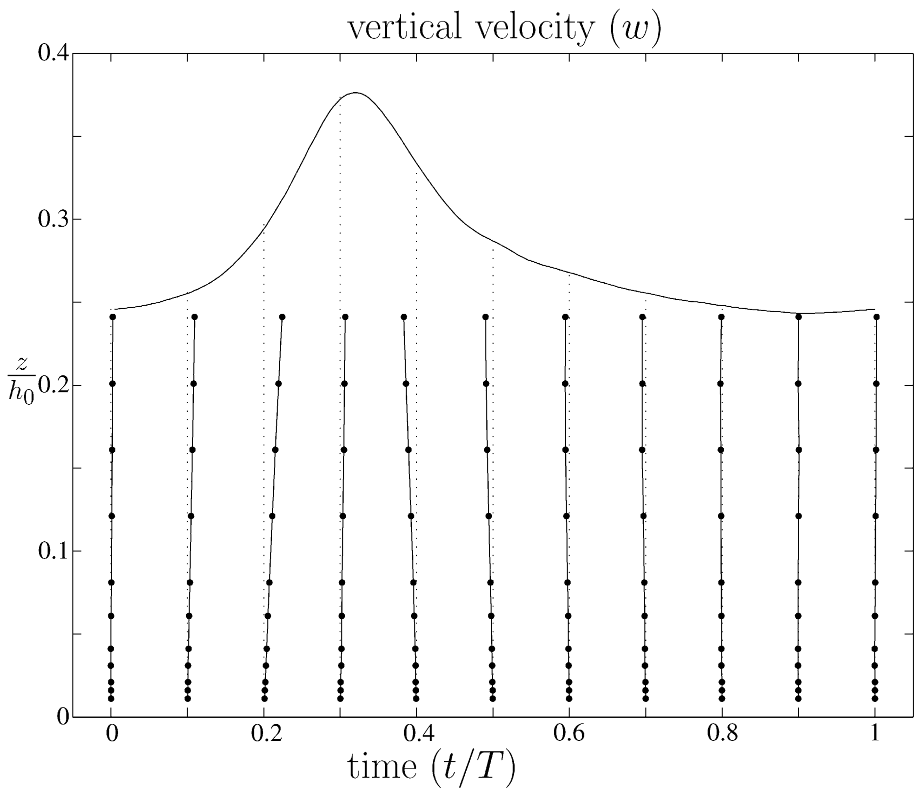

Fig. 1. The vertical profile of the vertical velocity (w) is plotted over one wave period. z/h_0 is the dimensionless form of the depth and t/T is the dimensionless time. In this case, $h_0=0.28$ m and $T=2.2$ s.

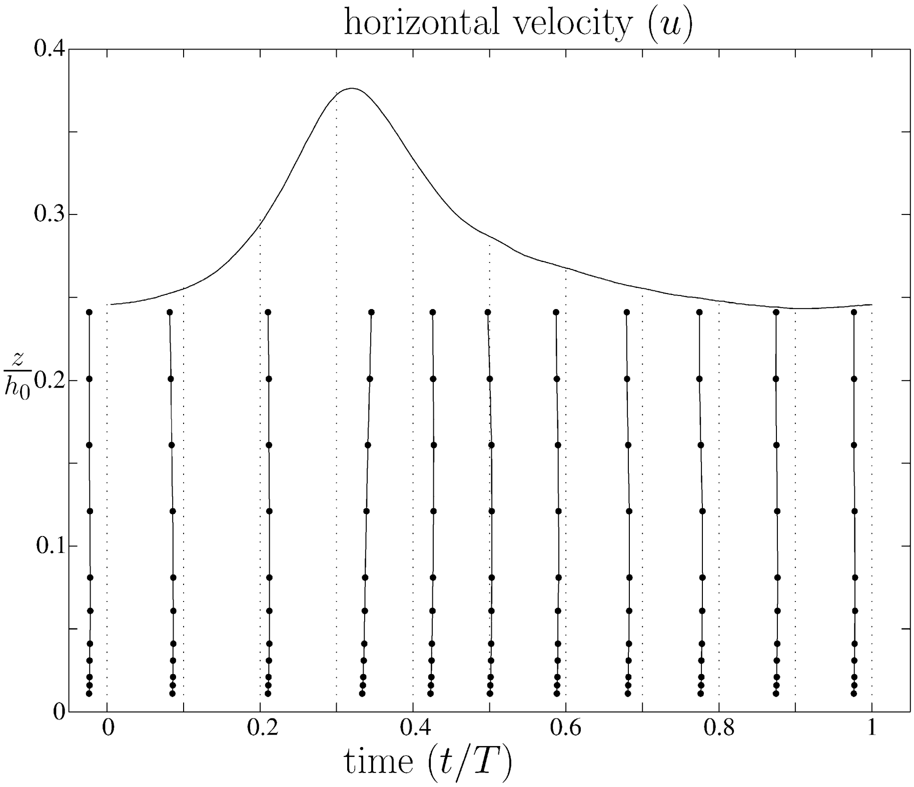

Fig. 2. The vertical profile of the horizontal velocity (u) is plotted over one wave period. z/h_0 is the dimensionless form of the depth and t/T is the dimensionless time. In this case, $h_0=0.28$ m and $T=2.2$ s.

for all the phases. This implies that the vertical length scale over which u varies is much larger than $O(h)$. These results are consistent with the length scales developed in this paper.

In the next section, the scaling of Euler's equations, the continuity equation, and the irrotationality condition is introduced to show the consistency in the scaling system adopted here.

4. The nondimensional equations

Euler's equations, in the non-rotating framework, can be expressed as (in the dimensional form)

$$u_t + uu_x + wu_z = -\frac{p_x}{\rho_0} \tag{20}$$

$$w_t + uw_x + ww_z = -\frac{p_z}{\rho_0} - g \tag{21}$$

The continuity equation is

$$u_x + w_z = 0 \tag{22}$$

and the irrotationality constraint is

$$u_z - w_x = 0 \tag{23}$$

The characteristic scales for the different variables are

$$u \sim \frac{gak}{\omega}; \quad w \sim \frac{gak^2h}{\omega}; \quad x \sim \frac{1}{k}; \quad t \sim \frac{1}{\omega}; \quad \eta \sim a$$

The pressure is scaled as follows: Characteristic scale for the pressure, p, is $\rho_0 gh$ when it varies over the vertical coordinate, z (which is scaled by h) and the scale for the dynamic pressure is $\rho_0 ga$ when scaled over the horizontal coordinate, x (which is scaled by $1/k$).

The vertical length scale depends on the variable chosen. For the variable, w, the vertical length scale is h and for the variable, u, the vertical length scale is h/μ^2.

Introducing the scaling results in the nondimensional equations. Note that the nondimensional variables (which are denoted by primes) are of $O(1)$. The nondimensional equations are

$$u'_{t'} + \delta(u'u'_{x'} + \mu^2 w'u'_{z}) = -p'_{x'} \tag{24}$$

The horizontal momentum equations shows that in the shallow water limit (or, long wave limit) the dominant balance is between the pressure gradient, the local acceleration term, and the nonlinear term. If further, the weakly nonlinear assumptions is made ($\delta \ll 1$), then the nonlinear terms become small t and the local acceleration term balances the pressure gradient.

$$\delta\mu^2 w'_{t'} + \delta^2\mu^2(u'w'_{x'} + w'w'_{z'}) = -p'_{z'} - 1 \tag{25}$$

For the vertical momentum equation in the long wave limit ($\mu^2 \ll 1$),the two terms on the left-hand side drop out leaving behind the hydrostatic equation. This is an important property of long wave theory.

$$u'_{x'} + w'_{z'} = 0 \tag{26}$$

Scaling of the continuity equation shows both terms to be of the same order. Supposing one term were to dominate over the other, for e.g., if at the lowest order, we of fund $u'_{x'} = 0$, then at the lowest order, the only solution this would permit is for u to be constant throughout the water column. This is clearly a trivial solution. Hence, both terms in the continuity equation need to be of the same order and is obtained from proper scaling analysis.

$$u'_{z'} + w'_{x'} = 0 \tag{27}$$

The above nondimensional equation implies that the terms in the vorticity equation are all of the same order. Instead, if one of the terms, say, $u'_{z'} = 0$ is obtained at lowest order, then vorticity would be generated inside the domain from no apparent source. This would be a violation of physical principles. Hence, the correct nondimensional form of the equation must include both the terms at lowest order. Hence, the equations are scaled consistently with the properties of long wave theory. Similarly, proper scaling of the Boussinesq equations can be done using the scaling variables used here.

5. Discrepancy in scaling

Researchers have either used the Euler's or the Laplace's equation as governing equations for deriving the Boussinesq equation. In this section, we describe the scaling used by the different researchers and show that the properties of long waves are not preserved.

5.1. Euler's equation

Nwogu (1993), Peregrine (1967) and Yoon and Liu (1989)all used the Euler equations for their investigation of Boussinesq equations. Nwogu (1993) used h as the characteristic length scale for the vertical coordinate. This produced a discrepancy in the scaling of the continuity equation.

Nwogu derived his nondimensional continuity equation as:

$$w'_{z'} + \mu^2 u'_{x'} = 0 \tag{28}$$

By definition, $w'_{z'}$ and $u'_{x'}$ are both $O(1)$. An $O(1)$ term cannot balance an order μ^2 term. Hence, Nwogu's scaling implies that at long wave limit, $w'_{z'}$ has to be zero. Also, the rate of variation of u with respect to x is an order of magnitude smaller than variation of w with respect to z. This is physically incorrect. Hence, the nondimensional variables $w'_{z'}$ and $u'_{x'}$ in Nwogu's incorrect scaling are *not* $O(1)$.The correct form of this nondimensional equation is:

$$w'_{z'} + u'_{x'} = 0 \tag{29}$$

Nwogu also scaled his pressure by ρga in the vertical momentum equation. He thus obtained:

$$\mu^2 w'_{t'} + \delta^2(u'w'_{x'} + 1/\mu^2 w'w'_{z'}) = -\delta p'_{z'} - 1 \tag{30}$$

For fully nonlinear ($\delta = O(1)$)shallow water theory ($\mu^2 \ll 1$),Nwogu's scaling implies $w' w'_{z'} = 0$. The lowest order balance obtained from Nwogu's scaling is not the hydrostatic assumption. The equation obtained from Nwogu's scaling is physically meaningless.

Yoon and Liu (1989)derived a set of Boussinesq equations using Euler's equations as their starting point. They scale the vertical variation of both u and w with h. Also, they do not distinguish between the pressure scale being different in the vertical and horizontal momentum equations. They scale pressure by ρgh. From their scaling, they obtain the nondimensional continuity equation to be:

$$\mu w'_{z'} + u'_{x'} = 0 \tag{31}$$

At lowest order, this suggest, $u'_{x'} = 0$. Hence, at the lowest order this would not permit solutions with horizontal variation in u. This is clearly incorrect. The correct form is shown in Eq. (29). The scaling of horizontal momentum equation gives:

$$\mu u'_{t'} + \mu^2 (u' u'_{x'} + \delta' w u'_z) = -p'_{x'} \tag{32}$$

At lowest order ($\mu \ll 1$),this gives $p'_{x'} = 0$. That is, at the lowest order, there is no horizontal pressure gradient. This again is not correct. For fully nonlinear and weakly dispersive long waves, the correct form of the lowest order equations is $u'_{t'} + u' u'_{x'} = -p'_{x'}$. However, Yoon and Liu (1989) are able to obtain the hydrostatic equation from their scaling of the vertical momentum equation.

Peregrine (1967) used h as the length scale for both horizontal and vertical motions. Again, in his case, the scaling of the vertical momentum equation does not lead to the hydrostatic equation.

5.2. Laplace's equation

Gobbi et al. (2000) and Wei et al. (1995) use the Laplace's equation as the governing equation from which they derive the Boussinesq equations. They choose h as the characteristic vertical length scale over which the potential, ϕ, varies. This is equivalent to having the same depth scale for both the depth scales, u and w. Thus, the anisotropy in scaling is not taken into account. Upon scaling, they obtain:

$$\phi'_{z'\,z'} + \mu^2 \phi'_{x'\,x'} = 0 \tag{33}$$

Hence, for the long wave limit, this seems to suggest that the lowest order is $\phi_{z'\,z'}{}' = 0$. This is would imply that the vertical velocity varies much faster over depth that the horizontal velocity varies over the horizontal. This is incorrect and similar to what Nwogu obtains. The correct form of this non-dimensional equation would be:

$$\phi'_{z'\,z'} + \phi'_{x'\,x'} = 0 \tag{34}$$

6. Conclusion

The scaling of long waves is quite complex. It is important to understand how the different dynamical variables vary spatially. Many previous researchers have obtained an inconsistent set of nondimensional governing equations due to improper scaling. This is because the nondimensional variables obtained from scaling are not $O(1)$. In this paper, the deficiencies in the scaling arguments of previous studies are pointed out and a different set of scaling variables that are consistent with the assumptions of long wave theory is developed.

The horizontal velocity in inviscid long wave theory is associated with a vertical length scale of h/μ^2. Often, the vertical length scale over which u varies has been chosen to be the water depth, h, which is incorrect. This is due to the inviscid nature of the boundary value problem. In an inviscid domain, the horizontal velocity is usually non-zero at the bottom and the u is nearly constant over the water depth (in the case of long wave motion). The vertical length scale of the vertical velocity, however, is the water depth. These scaling arguments can also be used in correcting the improper scaling in long wave equations, such as the Boussinesq equations.

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