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Coherence and Band Structure of Inertial Motion in the Sea

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NOTATION

λ, east longitude.

φ, latitude.

$\mu = \int_0^\varphi \sec \varphi \, d\varphi$, Mercator latitude.

z, height (0 at surface, $-h$ at bottom).

u, v, w, eastward, northward, upward velocity components.

q, velocity amplitude.

$\rho\ (xyzt)$, density.

$\bar{\rho}(z)$, ρ_0, density averages.

p, perturbation pressure $\div\ \rho_0$.

$b = g(\rho_0 - \rho) \div \rho_0$, buoyancy.

$N(z) = [-(g/\rho_0)\ (d\rho/dz)]^{1/2}$, buoyancy (Väisälä) frequency.

$\bar{N}$, *mean* buoyancy frequency, 25 cycles per day (cpd); $n = N(z) \div \bar{N}$.

a, Ω, radius and angular velocity of earth.

ω, s, temporal, zonal frequencies [exp $i(s\lambda - \omega t)$].

$\sigma = \omega/\Omega$ dimensionless frequency, in cycles per day (cpd).

$\alpha = s \sec \varphi$, zonal frequency, in cycles per earth circumference.

β, latitudinal wave number (quite generally).

$\gamma = 2\ \pi r \dfrac{a/h}{\bar{N}/\Omega} \approx \sqrt{10^5 r}$, $r = 1, 2, \cdots$ dimensionless vertical wave number.

$\kappa(z)$, dimensional (local) vertical wave number (equation 12).

k_i, $i = 1, 2, 3$, local Cartesian wave number components eastward, northward, upward.

$\varphi_S(\sigma, s, \gamma)$, spheroidal turning latitude (equation 28).

φ_0, $\sigma_0 = 2 \sin \varphi_0$, reference latitude and frequency.

φ_T, $\sigma_T = 2 \sin \varphi_T$, turning latitude and frequency.

$L = (\gamma^2 \sin 2\ \varphi)^{-1/3}$, Airy scale.

η, Airy argument (equation 43).

$F(\sigma)$, $C(\sigma)$, $Q(\sigma)$, power, co- and quadrature spectra.

$R(\sigma)$, $\phi(\sigma)$, coherence and phase (equation 53).

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Abstract. It is now well established by observation that a peak in the spectrum of horizontal motion should be anticipated everywhere in the ocean near the local inertia frequency, 2Ω sine (latitude). The theory of wave motion in a weakly stratified, rotating ocean of constant depth explains this observation either by the existence of a frequency condensation point in wave-number space or, alternatively, by the vanishing of the meridional group velocity. This explanation is independent of a specific generating mechanism, such as tidal forcing. The details of the wave structure and dispersion relation are readily obtained when, as seems both likely and desirable, it is permissible to ignore the discrete normal-mode-producing effects of distant lateral boundaries.

This theory predicts a spectral peak slightly above the inertia frequency, and this displacement depends on the zonal and vertical wave numbers. The peak frequency in the North Atlantic measurements by Fofonoff and Webster implies vertical modes of $O(10)$ and a zonal wave number of O (several hundred cycles per earth circumference). When these numbers are applied to a simple coherence model, assuming phase independence between different wave numbers, one can account for the observed lack of coherence between stations separated in depth or longitude. This theory also defines a latitudinal scale; for vertical wave number 10 this is, typically, of O(25 km), which is in qualitative agreement with Hendershott's observations in the eastern North Pacific.

The present theoretical model is appropriate for random distributed sources. The observations, however, indicate a higher degree of intermittency than is implied by this model. We conclude that both random distributed sources and intermittent discrete sources must be taken into account for a satisfactory description of the phenomena.

INTRODUCTION

The existence of horizontal 'inertial' currents rotating (clockwise in northern hemisphere) at a rate of 2 sine (latitude) revolutions per day is one of the few features about which there is any kind of agreement among oceanographers. Table 1 gives a summary of some of these observations; most of these observations are close to 30° latitude where there is an overlap between (diurnal) tidal and inertial frequency. *Webster* [1968*b*] gives a more complete summary.

Typical velocities are a few centimeters per second. The oscillations are intermittent, but, when they do occur, the inertial frequency is quite prominent. (Stating it another way, the inertial motion is associated with a prominent spec-

TABLE 1. Serial Measurements of Inertial Currents

	Location	Depth, m	Duration, days	Instrument
Ekman and Helland-Hansen [1931]	Atlantic, 30°13′N	0–1000	6	Ekman meters
Gustafson and Kullenberg [1933]	Baltic, 57°49′N	14	7	Ekman meters
Knauss [1962]	Pacific, 28°48′N 28°12′N	700–2300		Swallow floats
Reid [1962]	Pacific, 30°06′N	Surface	15	GEK
Hendershott [1964]	Pacific, 29°15′N 29°36′N 30°06′N	Surface	19	GEK
Day and Webster [1965]	Atlantic, 28°07′N	50, 100	87	Richardson meters
Webster [1968*a*]	Atlantic, 39°20′N	120	43	Richardson meters

tral peak, typically of $\pm 1\%$ bandwidth with peak intensity 10 db above the surrounding level.) Another point of agreement is that, whenever simultaneous measurements are made at nearby locations, the records appear remarkably dissimilar, apart from having the same prominent frequency, whether the separation is east–west, north–south, or up–down. Thus, *Webster* [1968*a*, *b*] found small coherence between 7- and 87-meter depths at one station and only moderate coherence for 3-km west–east separation. Hendershott remarks about the lack of any obvious correlation for a 50-km north–south separation. The observations of Fofonoff and Webster reveal an unexpected degree of intermittency in both space and time.

What determines the observed features of these oscillatory currents (their frequency, width, spatial coherence, and intermittency)? We shall attempt to discuss these questions, using the superposition of many modes of the appropriate planetary-gravity (pg) waves as a model. The very active development of pg wave theory in the last few years has emphasized the *normal* modes of various ocean basins, although normal modes have not been observed. Our emphasis, however, is on pg wave solutions that are not sensitive to lateral boundary conditions. These *interior* solutions provide insight into some observed features and establish design criteria for a coherent pg wave array.

THE SPHEROIDAL WAVE EQUATION

Fundamental equations. The small density variations in the ocean and the smallness of the frequencies of interest justify the assumptions of incompressibility, i.e. the Boussinesq system. The resulting perturbation equations for a stratified ocean are

$$\partial u/\partial t - 2\Omega v \sin\varphi = -(1/a\cos\varphi)(\partial p/\partial\lambda) \tag{1}$$

$$\partial v/\partial t + 2\Omega u \sin\varphi = -(1/a)(\partial p/\partial\varphi) \tag{2}$$

$$\partial w/\partial t = -(\partial p/\partial z) + b \tag{3}$$

$$\partial b/\partial t + N^2(z)w = 0 \tag{4}$$

$$(\partial w/\partial z) + (1/a\cos\varphi)[(\partial u/\partial\lambda) + (\partial v\cos\varphi/\partial\varphi)] = 0 \tag{5}$$

For frequencies $\omega \ll N$, $\partial w/\partial t$ in (3) is negligible. The Coriolis terms $-2\Omega w \cos\varphi$ and $+2\Omega u\cos\varphi$ have been omitted in (1) and (3), the 'traditional approximation' [*Eckart,* 1960]. Some brief remarks on their effect are made at the end of the section on Airy solutions.

A convenient separation of variables is given by

$$\begin{bmatrix} u\cos\varphi \\ v\cos\varphi \\ p \end{bmatrix} = \Omega a \operatorname{Re} Z(z)\exp i(s\lambda - \omega t)\begin{bmatrix} U(\varphi) \\ iV(\varphi) \\ \Omega a P(\varphi) \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} w \\ b \end{bmatrix} = \Omega a \operatorname{Re} W(z)\exp i(s\lambda - \omega t)\, P(\varphi)\begin{bmatrix} i \\ \dfrac{N^2(z)}{\omega} \end{bmatrix} \tag{7}$$

We will consider ω as positive and s as either positive or negative. U, V, P, Z, and W are dimensionless, and (4) is automatically satisfied. In terms of the non-dimensional frequency[2] $\sigma = \omega/\Omega$, equations 1–3 produce

$$\sigma U(\varphi) + 2\sin\varphi V(\varphi) = sP(\varphi) \tag{8}$$

$$2\sin\varphi U(\varphi) + \sigma V(\varphi) = -\cos\varphi\, dP/d\varphi \tag{9}$$

$$N^2(z)W(z) = a\sigma\Omega^2\, dZ/dz \tag{10}$$

and the continuity equation (5) yields a separability condition, with the separation constant γ^2:

$$-\frac{4a}{\sigma Z}\frac{dW}{dz} = \frac{4}{\sigma P\cos^2\varphi}\left(sU + \cos\varphi\frac{dV}{d\varphi}\right) = \gamma^2 \tag{11}$$

Vertical equation. The z part of (11) together with (10) provides an equation for W

$$d^2W/dz^2 + \kappa^2 W = 0 \qquad \kappa(z) = (\gamma/2a)(N(z)/\Omega) \tag{12}$$

For the bottom boundary condition we set $W = 0$ at $z = -h$, thus ignoring the important dynamic effects of bottom topography. At the (perturbed) surface the total pressure $=\rho_0\,(p - gz)$ vanishes; hence $\partial p/\partial t - gw = 0$. The solutions (6) and (7) for (p, w) and the expression for Z from the first equation (11) lead to the surface boundary condition

$$dW/dz - (g\gamma^2/4\Omega^2a^2)W = 0 \qquad \text{at} \quad z = 0$$

Both boundary conditions imply a zero flux of energy across the surface in question, thereby ignoring wave generation by atmospheric effects and wave dissipation in bottom boundary layers.

For the simple case of uniform and weak stratification, i.e. $N^2(z) = \text{constant} = \bar{N}^2 \ll g/h$, the solutions are trigonometric:

$$W = \sin\bar{\kappa}(z + h) \qquad \bar{\kappa} = (\gamma/2a)(\bar{N}/\Omega) \tag{13}$$

This satisfies the boundary condition at the sea bottom; at the surface we find

$$\tan\frac{\bar{N}h}{C} = \frac{C\bar{N}}{g} \qquad C = \frac{2\Omega a}{\gamma} \tag{14}$$

where C, as defined, will be the phase speed of a (hydrostatic) gravity wave in a nonrotating system. The allowable values (to be designated γ_r, C_r, $r = 0, 1, 2, \cdots$) are discrete and can be found to a first approximation by setting first $\bar{N}h/C \ll 1$, then $C\bar{N}/g \ll 1$ in (14).

External mode ($r = 0$):

$$C_0 = \sqrt{gh} \sim 200 \quad \text{m/sec}^{-1} \qquad \gamma_0^{\,2} \sim 22 \tag{15}$$

Internal modes ($r \geq 1$):

$$C_r = h\bar{N}/\pi r \sim 2.5/r \quad \text{m/sec}^{-1} \qquad \gamma_r^{\,2} \sim 10^5 r^2 \tag{16}$$

[2] Hence, σ is in the convenient unit of cycles per day (cpd). Most investigators prefer the definition $\sigma = \omega/2\Omega$, which eliminates many factors of 2 in the equations.

The numbers are based on $h = 4$ km and $\bar{N} = 2 \times 10^{-3}$ sec^{-1}. The largeness of γ_r for the internal modes will play an important role in the Airy solutions of the next section. For $r = 0$, W is essentially a linear function of z; for $r \geq 1$, $W = \sin r\pi(1 + z/h)$. r is therefore the number of nodal levels in $W(z)$.

For nonuniform $N(z)$, the distortion of the high-order modes can be allowed for with WKBJ approximation

$$W = \left[\frac{\bar{N}}{N(z)}\right]^{1/2} \sin \psi(z) \qquad \frac{dW}{dz} = \frac{[\bar{N}N(z)]^{1/2}}{C_r} \cos \psi(z)$$
$$\psi(z) = \frac{1}{C_r}\int_{-h}^{z} N(z)\, dz \qquad C_r = \frac{h\bar{N}}{r\pi} = \frac{1}{r\pi}\int_{-h}^{0} N\, dz \tag{17}$$

The 'local' vertical wave number is $\kappa(z) = d\psi/dz$, in agreement with (12).

Large γ approximation. The latitude equations come from (8), (9), and the latitude part of (11). It is convenient to introduce the Mercator coordinate μ, which removes the polar singularities to infinity:

$$\mu = \ln\,[(1 + \sin\varphi)/\cos\varphi] \qquad \sin\varphi = \tanh\mu \tag{18}$$
$$d\varphi/d\mu = \cos\varphi = \operatorname{sech}\mu$$

[cf. *Eckart,* 1960]. Equation 8 stays as it is:

$$\sigma U = sP - 2\sin\varphi V \tag{19}$$

and can be used to eliminate U from (9) and (11):

$$\frac{dP}{d\mu} + \frac{2s}{\sigma}\sin\varphi\, P = \left(\frac{4\sin^2\varphi}{\sigma} - \sigma\right)V \tag{20}$$

$$\frac{dV}{d\mu} - \frac{2s}{\sigma}\sin\varphi\, V = \left(\frac{\gamma^2\sigma\cos^2\varphi}{4} - \frac{s^2}{\sigma}\right)P \tag{21}$$

The latitude eigenfunctions traditionally have been analyzed in terms of the pressure function P. *Longuet-Higgins* [1965] and *Dikii* [1966] have shown, however, that the equation for V has some advantages when γ is large. Elimination of P gives the *primitive wave equation*

$$\frac{d^2V}{d\mu^2} + \beta^2 V = \left[\frac{\gamma^2\sigma\cos^2\varphi\sin\varphi}{s^2 - \frac{1}{4}\gamma^2\sigma^2\cos^2\varphi}\right]\left[\tfrac{1}{2}\sigma\frac{dV}{d\mu} - s\sin\varphi\, V\right] \tag{22}$$

where

$$\beta^2 = \gamma^2\cos^2\varphi(\tfrac{1}{4}\sigma^2 - \sin^2\varphi) - (2s/\sigma)\cos^2\varphi - s^2 \tag{23}$$

Longuet-Higgins and Dikii present arguments that the right side of (22) is negligible for large γ (and small s); in that event one obtains the *spheroidal wave equation*

$$(d^2V/d\mu^2) + \beta^2 V = 0 \tag{24}$$

Two important special cases of the dispersion relation (23), written as $\sigma = \sigma\,(\alpha, \beta, \gamma; \varphi)$ with $\alpha = s\sec\varphi$, are

$$\sigma = \sigma(\alpha, \beta, \gamma; |\varphi| \ll 1) \qquad \sigma = \sigma(\alpha, |\beta| \ll 1, \gamma; \varphi)$$

corresponding to *equatorial* and *turning latitude* approximations, respectively. The former viewpoint is equivalent to results obtained from the 'equatorial beta plane' approximation $\sin \varphi = \varphi$, $\cos \varphi = 1$ [*Rattray*, 1964]:[3]

$$\beta^2 \approx \gamma^2(\tfrac{1}{4}\sigma^2 - \varphi^2) - 2s/\sigma - s^2 = \gamma(2n + 1) - \gamma^2\varphi^2$$

Discrete modes naturally arise in this case because integer values of n are required for finiteness of V at large φ.

We shall be concerned instead with the second viewpoint, which leads to Airy solutions valid in the vicinity of the (spheroidal) turning latitude, φ_s (defined by $\beta^2(\varphi_s) = 0$). These solutions are essentially confined to a strip of order $\gamma^{-2/3}$ in $\varphi - \varphi_s$ and are therefore insensitive to conditions farther away. This freedom allows us to ignore specific boundary conditions in latitude and to interpret the approximate solutions as a *continuous* spectrum. This procedure is meaningful for the internal modes ($\gamma_r^{2/3} \sim 50\, r^{2/3}$) but is dubious for the external mode ($\gamma_0^{2/3} \sim 3$). *Dikii* [1966] and *Longuet-Higgins* [1965, 1968] have analyzed the discrete normal modes determined by polar boundary conditions.

Stationarity. Accepting the spheroidal equation (24) as being valid for large γ, we can interpret β as the (local) south–north wave number. Equation 23, written as $\sigma = \sigma(\alpha, \beta, \gamma; \varphi)$, then expresses frequency as a (latitude dependent) function of the west–east ($\alpha = s \sec \varphi$), south–north (β), and down–up (γ) wave numbers. Following *Blandford* [1966], we expect frequency spectra to have maxima where σ is stationary in α, β, γ space:

$$\partial\sigma/\partial\alpha = 0 \qquad \partial\sigma/\partial\beta = 0 \qquad \partial\sigma/\partial\gamma = 0 \tag{25}$$

Upon partial differentiation of (23), these expressions yield

$$\sigma = -\cos\varphi/\alpha \qquad \beta = 0 \qquad \sigma = 2\sin\varphi \tag{26}$$

respectively. The first of these stationarity conditions is probably not applicable for inertial frequencies because for any longitudinal wavelength of interest $\alpha \gg 1$ and, thus, $\sigma \ll 1$. The third condition yields the familiar inertial frequency. We shall come back to $\beta = 0$.

There are two ways of looking at equations 25: (1) For a *global* generation of waves, there will be at any one latitude certain narrow ranges of σ corresponding to wide ranges in α, β, γ, and we expect these 'condensation points' to be prominent. (2) For a *local* generation, the disturbance will leak off, except for the components associated with very small group velocity. Equations 25 are precisely the condition of zero group velocity.

Dispersion. Under the stationarity condition $\beta = 0$, we find from (23)

$$0 = \cos^2\varphi_S[\gamma^2(\tfrac{1}{4}\sigma^2 - \sin^2\varphi_S) - 2s/\sigma] - s^2 \tag{27}$$

or

$$2\sin^2\varphi_S = 1 + \tfrac{1}{4}\sigma^2 - \frac{2s}{\sigma\gamma^2} - \left[\left(1 - \frac{\sigma^2}{4} + \frac{2s}{\sigma\gamma^2}\right)^2 + \frac{4s^2}{\gamma^2}\right]^{1/2} \tag{28}$$

[3] The notation 'beta' $= a^{-1}\, d(2\Omega \sin\varphi)/d\varphi$, introduced by Rossby, has no relation to 'β', the north–south wave number.

This determines the spheroidal 'turning latitude' φ_s (σ; s, γ), for at this latitude $V(\mu)$ changes from an oscillatory to an exponential function, going poleward. We can write (27) in the form

$$\beta^2 = \gamma^2(\sin^2 \varphi_S - \sin^2 \varphi)(\cos^2 \varphi + s^2\gamma^{-2} \sec^2 \varphi_S)$$

to make explicit the vanishing of β at $\varphi = \varphi_s$. φ_s exists if $\gamma^2 > 4\,\sigma^{-2}\,(s^2 + 2s/\sigma)$. Plots of $\sigma = \sigma(s)$ for fixed φ_s and various γ are shown in Figure 1. A critical point

$$s_m = -\frac{\cos^2 \varphi_S}{\sqrt{2} \sin \varphi_S} \qquad \sigma_m = \sqrt{2} \sin \varphi_S \qquad \gamma_m = \frac{\cos \varphi_S}{\sin^2 \varphi_S}$$

separates gravity waves from Rossby waves; $\gamma \geq \gamma_m$ in order for s to be real.

Gravity waves of inertia frequency $\sigma = 2 \sin \varphi_S$ occur at $s = 0$ and $s = -\sqrt{2}\, s_m$, but, as $\gamma \to \infty$, $\sigma \to 2 \sin \varphi_S$ for all s. Because $\gamma = 10^{5/2}r$ is large for all internal modes, we can expect the observations at any latitude φ to be prominently associated with the local inertial frequency

$$\sigma \approx 2 \sin \varphi \tag{29}$$

The σ extremes of Rossby waves are stationary with respect to s along the curve $\sigma s = -\cos^2 \varphi_s$, but, because they are different for different γ (no matter how large γ is), they represent a lesser order of condensation than the inertial frequencies. For the interesting case of large γ, the stationary points are

$$s = -\gamma \sin \varphi_S \cos \varphi_S \qquad \sigma = \cot \varphi_S/\gamma \tag{30}$$

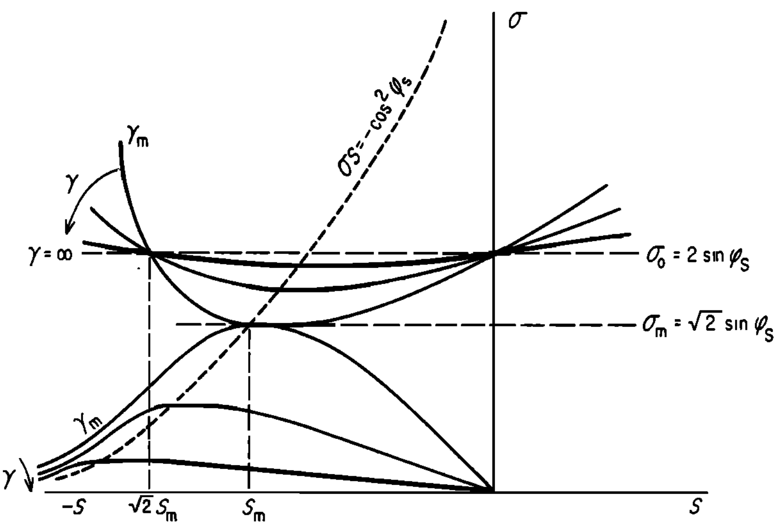

Fig. 1. The planetary-gravity wave dispersion s (σ) according to (27) for a fixed φ_s. Positive s corresponds to west–east propagation with phase velocity σ/s; positive slope $\partial\sigma/\partial s$ to west–east group velocity. Planetary (or Rossby) wave solutions are separated from gravity wave solutions according to whether $\sigma < \sigma_m$ or $\sigma > \sigma_m$. Rossby phase velocities are always east–west. The curve $\sigma s = -\cos^2\varphi_s$ separates positive and negative group velocities.

and they correspond to the Rossby-wave cutoff frequency described by *Longuet-Higgins* [1965].[4] Frequencies are very small, suitably long series of observations are lacking, and we shall therefore concentrate on (29).

AIRY SOLUTIONS

Airy solution to the spheroidal wave equation. The Airy solution of the spheroidal equation (24) can be obtained formally in the manner described by *Erdélyi* [1956, p. 91]. We first define a new independent variable ξ, which vanishes at φ_s:

$$\tfrac{2}{3}\xi^{3/2} = \int_{\varphi}^{\varphi_s} (\sin^2 \varphi_S - \sin^2 \varphi)^{1/2}\, d\varphi$$

(This expression is for northern hemisphere φ; similar formulas hold for negative φ.) After setting $V = f(\mu)\ g(\xi)$, with $f = (d\xi/d\mu)^{-1/2}$, we obtain the following form of the spheroidal equation:

$$\frac{d^2 g}{d\xi^2} + \left\{\gamma^2 \xi + \left(\frac{d\xi}{d\mu}\right)^{-2}\left[\frac{1}{f}\frac{d^2 f}{d\mu^2} + \frac{s^2(\sin^2 \varphi_S - \sin^2 \varphi)}{\sin^2 \varphi_S}\right]\right\} g = 0$$

Erdélyi shows that this has an asymptotic Airy solution for large γ^2. In terms of V, it is

$$V = \left(\frac{d\xi}{d\mu}\right)^{-1/2} g(\xi) = \left(\frac{d\xi}{d\mu}\right)^{-1/2} Ai(-\gamma^{2/3}\xi)\left[1 + O\left(\frac{1}{\gamma}\right)\right] \tag{31}$$

and is not restricted to small values of ξ. (We ignore the $Bi\ (-\gamma^{2/3}\ \xi)$ solution because it increases with ξ; see the appendix.)

In the vicinity of φ_s we have

$$\xi = -(\sin 2\varphi_S)^{1/3}(\varphi - \varphi_S)[1 + \tfrac{1}{5}(\cot 2\varphi_S)(\varphi - \varphi_S) + \cdots]$$

$$\frac{d\xi}{d\mu} = -(\sin 2\varphi_S)^{1/3} \cos \varphi[1 + \tfrac{2}{5}(\cot 2\varphi_S)(\varphi - \varphi_S) + \cdots]$$

The argument of the Airy function can be approximated there by

$$-\gamma^{2/3}\xi \approx (\gamma^2 \sin 2\varphi_S)^{1/3}(\varphi - \varphi_S) = (\varphi - \varphi_S)/L \tag{32}$$

where

$$L = (\gamma^2 \sin 2\varphi_S)^{-1/3} \tag{33}$$

is the Airy latitude scale. Values of L in degrees of latitude are listed in Table 2. The inapplicability of the theory to $r = 0$ is apparent.

Airy solution to the primitive wave equation. Although (31) with its error of order γ^{-1} is formally correct for the spheroidal equation (24), we are really interested in the solution of the primitive equation (22). Near φ_s ($\beta^2 = 0$), the

[4] *Longuet-Higgins* [1968] has shown that as $\gamma \to \infty$, one of the two lowest latitudinal modes in the solutions of (22) represents an equatorially trapped Kelvin wave, with $\sigma \approx 2\ s/\gamma$, $s > 0$. V is small in this wave, and the right side of (22) is not negligible. See, also, *Rattray* [1964] and *Matsuno* [1966].

TABLE 2. Airy Scale L in Degrees of Latitude for Turning Latitudes φ_s and vertical mode number $r = 10^{-5/2}$, $\gamma \geq 1$. For $r = 0$, $\gamma_0^2 = 22$.

	φ_s							
r	10°	20°	30°	40°	50°	60°	70°	80°
0	29.2	23.6	21.4	20.5	20.5	21.4	23.6	29.2
1	1.76	1.43	1.29	1.23	1.23	1.29	1.43	1.76
2	1.11	0.90	0.81	0.78	0.78	0.81	0.90	1.11
5	0.60	0.49	0.44	0.42	0.42	0.44	0.49	0.60
10	0.38	0.31	0.28	0.27	0.27	0.28	0.31	0.38
20	0.24	0.19	0.18	0.17	0.17	0.18	0.19	0.24

first bracket on the right side of (22) is approximately equal to $-\sigma \sin \varphi_s [\sin^2 \varphi_s + 2\, s/\sigma\, \gamma^2]^{-1}$. For $\sigma \approx 2 \sin \varphi_s$ and $|s| < \gamma^2$ this is essentially equal to -2. Under these conditions, the right side of (22) is of order $L \approx \gamma^{-2/3}$ times the left side. The γ^{-1} accuracy of (31) is therefore swamped by the error inherent in (24) (at least for $\sigma \approx 2 \sin \varphi_s$), and we are justified in simplifying (31). In so doing we abandon the validity of (31) for large ξ and consider only small deviations of φ from φ_s: $\varphi - \varphi_s = O(\mathrm{L})$. Solution (31) then reduces to the simple form

$$V = Ai\left(\frac{\varphi - \varphi_S}{L}\right) + O(L)$$

We have here the $\gamma^{-2/3}$ approximation to equation 31, which is a solution of the $\gamma^{-2/3}$ approximation (24) to the primitive wave equation (22). Clearly, it is more satisfactory to proceed directly and systematically from the primitive equation, particularly when we later require solutions to order $\gamma^{-4/3}$.

This derivation has been performed in the appendix. Writing $U(\varphi) = U_0 + LU_1 + \cdots$, similarly for V and P, and combining the formulas obtained with the definitions in the preceding section, we find the following kinematic relations, each correct to an error of order L:

$$u = q(n^{1/2} \cos \psi)(Ai(\eta)) \cos (s\lambda - \omega t) \tag{34}$$

$$v = q(n^{1/2} \cos \psi)(Ai(\eta)) \sin (s\lambda - \omega t) \tag{35}$$

$$w = q\,\frac{h}{\pi r a L}\,(n^{-1/2} \sin \psi)\left(\alpha L Ai - \frac{dAi}{d\eta}\right) \sin (s\lambda - \omega t) \tag{36}$$

$$p = qC_r\left(\frac{\cos \varphi_0 C_r}{\sin^2 \varphi_0 \Omega a}\right)^{1/3}(n^{1/2} \cos \psi)\left(\alpha L Ai - \frac{dAi}{d\eta}\right) \cos (s\lambda - \omega t) \tag{37}$$

$$b = -q\bar{N}\left(\frac{\cos \varphi_0 C_r}{\sin^2 \varphi_0 \Omega a}\right)^{1/3}(n^{3/2} \sin \psi)\left(\alpha L Ai - \frac{dAi}{d\eta}\right) \cos (s\lambda - \omega t) \tag{38}$$

q is an arbitrary velocity amplitude. The WKBJ treatment has been used for the z dependence, with $n(z)$ written for $N(z)/\bar{N}$, and $\psi(z)$ defined by the phase integral (17). $\alpha = s/\cos \varphi_0$ is the local west–east wave number, in cycles per earth circumference. $Ai(\eta)$ is the Airy function, with the argument

$$\eta = \frac{\varphi - \varphi_0}{L} + (\alpha L)^2 - \frac{\sigma - \sigma_0}{2L \cos \varphi_0} \tag{39}$$

$$\sigma_0 = 2 \sin \varphi_0 \qquad L = (\gamma^2 \sin 2\varphi_0)^{-1/3}$$

where φ_0 is some reference latitude.

Expedition space. For definiteness, we interpret the results in terms of an operational problem. Frequency spectra of measurements at various latitudes permit us to contour energy density in φ, σ space (deferring the question of stationarity.) A choice of 2 sin φ, σ coordinates conveniently places the turning latitudes near the 45° line in Figure 2. Consider distances of $O(L)$ relative to the position φ_0 of the reference ship, and frequencies of $O(L)$ relative to the reference frequency $\sigma_0 = 2 \sin \varphi_0$.

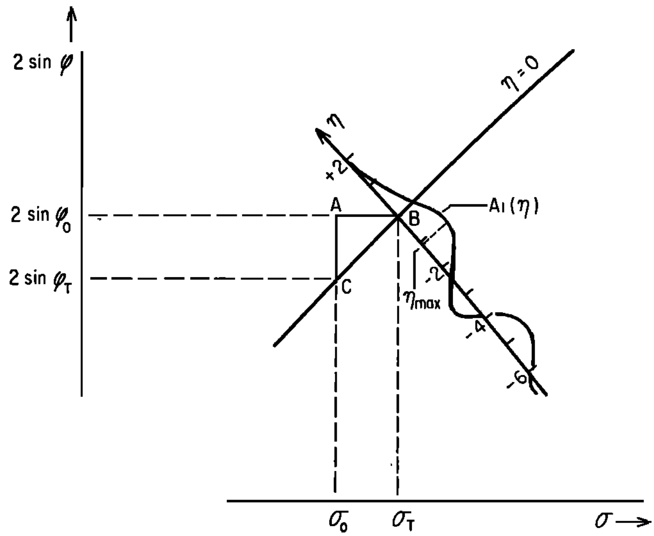

Fig. 2. Airy fine structure in local 'expedition space.' $AB = AC = 2L\,(\alpha L)^2 \cos \varphi_0$ and $\varphi_0 - \varphi_T = L\,(\alpha L)^2$.

The reference ship measures a horizontal velocity spectrum

$$F(\sigma; \varphi_0, \alpha, \gamma) \sim Ai^2(\eta) \qquad \eta = -\frac{\sigma - \sigma_0}{2L \cos \varphi_0} + \alpha^2 L^2 \tag{40}$$

which has a 'turning frequency' ($\eta = 0$) at

$$\sigma = \sigma_T = \sigma_0 + 2L(\alpha L)^2 \cos \varphi_0 \tag{41}$$

and a first maximum ($\eta = \eta_{\max} = -1.019$) at

$$\sigma = \sigma_{\max} = \sigma_0 + 2L \cos \varphi_0(\alpha^2 L^2 + 1.019) \tag{42}$$

A ship at φ_0 exploring neighboring latitudes for the intensity at the fixed frequency $\sigma_0 = 2 \sin \varphi_0$ finds

$$F(\varphi; \sigma_0, \alpha, \gamma) \sim Ai^2(\eta) \qquad \eta = \frac{\varphi - \varphi_0}{L} + \alpha^2 L^2 \tag{43}$$

with the turning latitude at

$$\varphi = \varphi_T = \varphi_0 - L(\alpha L)^2 \tag{44}$$

and a first maximum at

$$\varphi = \varphi_{max} = \varphi_0 - L(\alpha L)^2 - 1.019L \tag{45}$$

Dispersion near the turning point. Equation 41 can be recognized as an approximation of (27) for σ near 2 sin φ_0. The dispersion $\partial\sigma/\partial\alpha$ vanishes for $\alpha = 0$ (compared with $\alpha = -\frac{1}{2} \cot \varphi_0$ for (27)), but in all events the dispersion becomes very small for all α as the vertical wave number increases.

The inertia waves (equations 34–38) are trapped vertically between the top and bottom of the oceans, and, because $Bi(\eta)$ was discarded, they do not propagate north–south. *Eckart* [1960] and *Blandford* [1966] have described the horizontal propagation of wave packets along ray paths that are tangent to the local group velocity $(\partial\omega/\partial\alpha, \partial\omega/\partial\beta, \partial\omega/\partial\gamma)$. This is directed zonally when $\partial\omega/\partial\beta = 0$ (which in the case of (24) means $\beta = 0$), and the rays are refracted back to low latitudes. The Airy solutions given here describe part of the structure of such waves in the vicinity of their turning latitude. A more thorough analysis would connect the present solutions with the ray-path solution equatorward of the turning latitude, and, as suggested by Blandford, allow one to explore the effect of sources equatorward of φ_0. A recent paper by *Jacobs* [1967] performs this connection, for both Rossby waves and gravity waves, but does so in the context of the equatorial beta-plane model. An unpublished thesis by Blyth Hughes (Cambridge University, 1964) treats these problems in spherical geometry. Hughes concentrates on the inertia-gravity waves, as we do, but gives most attention to the special case $s = 0$, which we do not.

Magnitudes of pressure and vertical displacement. The factor $[\cos \varphi_0/\sin^2 \varphi_0)\ (C_r/\Omega a)]^{1/3} \sim 0.2\ r^{-1/3}$ in (37) and (38) represents a considerable reduction of p and b below the value typical of a nonrotating hydrostatic internal gravity wave with the same value of u. Thus, for $\varphi_0 = 40°$, $r = 10$, and $u = 1$ cm sec^{-1}, we find $p =$ (pressure perturbation $\div\ \rho_0$) $= 2.5$ cm^2 sec^{-2}, corresponding to 0.0025 cm of water pressure. The external mode and atmospheric oscillations are likely to produce pressure fluctuations exceeding 1 cm of water pressure. The corresponding vertical displacements (in the interior) would be $w/\omega = w/(2\Omega \sin \varphi_0)$ — about ± 40 cm.

The traditional approximation. The Coriolis terms $-\ 2\Omega w \cos \varphi$ and $+\ 2\ \Omega u \cos \varphi$ were omitted on the right side of the original perturbation equations (1 and 3) without discussion. In his book, *Eckart* [1960] refers to this as the 'traditional approximation' and has indicated (pp. 130–135) that the omission of these terms may be serious when $\omega \sim 2\ \Omega \sin \varphi$. If the detailed z, η, and λ dependence is ignored in our solutions (34)–(38), the order of magnitude of the $2\ \Omega \cos \varphi$ terms relative to the terms we have retained is (at $\varphi = \pi/4$)

$$\frac{2\Omega w \cos \varphi}{L 2\Omega u \sin \varphi} \sim \frac{2 \cos \varphi}{\sin \varphi} \left[r\pi \left(\frac{a}{h}\right)\left(\frac{\Omega}{\bar{N}}\right)^4 2 \sin^2 2\varphi \right]^{1/3} \sim \tfrac{1}{2} r^{1/3}$$

$$\frac{2\Omega u \cos \varphi}{b} \sim \left[r\pi \left(\frac{a}{h}\right)\left(\frac{\Omega}{\bar{N}}\right)^4 2 \sin^2 2\varphi \right]^{1/3} \sim \tfrac{1}{4} r^{1/3}$$

The neglect is most serious at large r. Evidently $r \sim O(10)$ represents an upper limit to the vertical modal number for which our neglect of these terms is reasonable. (Note, however, that these terms are orthogonal in z to the retained terms.) It is relevant to note that, in subsequent sections of this paper, our comparison of observed spectra and vertical coherence with the present theory leads to $r \sim O(10)$ as a typical vertical mode; i.e., the long-term statistics of the observed horizontal currents do not seem to require appreciable energies in modes $r \gg 10$. The eigenvalue problem arising from the complete equations (retaining the two $2\Omega \cos \varphi$ terms and $\partial w/\partial t$) is nonseparable in φ and z and has not yet been solved. (The thesis by Hughes, referred to previously, probably represents the most determined attack to date.) We are encouraged to speculate that perhaps a major effect of the $2\Omega \cos \varphi$ terms is to eliminate, at large r, the concentration of horizontal kinetic energy at the turning latitude predicted by the present solutions.

BAND STRUCTURE

Fofonoff-Webster measurements. Table 3 and Figure 3 give the energy per harmonic for two sets of current records. Both spectra show 10-db peaks within a few per cent of the local inertial frequency. The 30°20′ measurements, however, show side peaks comparable in magnitude to the main peak. The 28°07′ measurements, at almost twice the resolution, have a well separated main peak; the band at 1.01 cpd is presumably associated with diurnal tide constituents[5] (see Table 4).

TABLE 3. Summary of Measurements

	Webster (unpublished)	*Day and Webster* [1965]	*Hendershott* [1964]
Depth of measurement, m	120	100	0
Duration, days	43	106	9–18
φ_0	39°20′N	28°07′N	30°05′N
σ_0	1.268 cpd	0.942 cpd	1.0027 (K_1)
Estimated peak	1.29 ± .01 cpd	0.96 ± .005 cpd	29°36′N ± 15′

On the same frequency scale we have plotted some Airy amplitudes (these need to be squared for comparison) for selected values of r, α, using the parameters in Table 5. (These values apply strictly to the 39°20′ observations, but they are close enough for the other cases.) An increase in α shifts the Airy pattern toward higher frequencies, and an increase in r ($\sim L^{-3/2}$) leads to a compression of the pattern, in accordance with

$$\sigma - \sigma_0 = 2\alpha^2 L^3 - 2L\eta \cos \varphi_0 \tag{46}$$

We may attempt to fit the measured peak frequency in Table 3 to the principal Airy peak ($\eta_{\max} = -1.019$) by appropriate values of r, α:

[5] It is puzzling that this peak lies one harmonic above the K_1 frequency, whereas the 0.93 cpd agrees with O_1. Observations at shallow depth (50 m at 28°07′N) yield a well defined peak at K_1 frequency.

	39°20′N	28°07′N
$r = 5$	$\alpha = 120$ (0 to 176)	$\alpha = 63$ (26 to 85)
$r = 10$	$\alpha = 308$ (176 to 394)	$\alpha = 175$ (129 to 210)

(Parenthetical α values indicate sensitivity to $\pm$ shifts in peak frequency by half the harmonic interval.) The results are most uncertain; they indicate a typical value for α of order (100) (east–west wavelength 400 km). *Webster* [1968*b*] reports a phase difference of 34° (coherence 0.82) between the above station at 39°20′N and a second station 3 km to the east, indicating that a wavelength is roughly 10 times the station separation and that, accordingly, α = order (1000).

For a definitive study of the Airy structure one needs to sample the spectrum at least twice per interval between adjoining Airy extremes, or roughly at 0.005

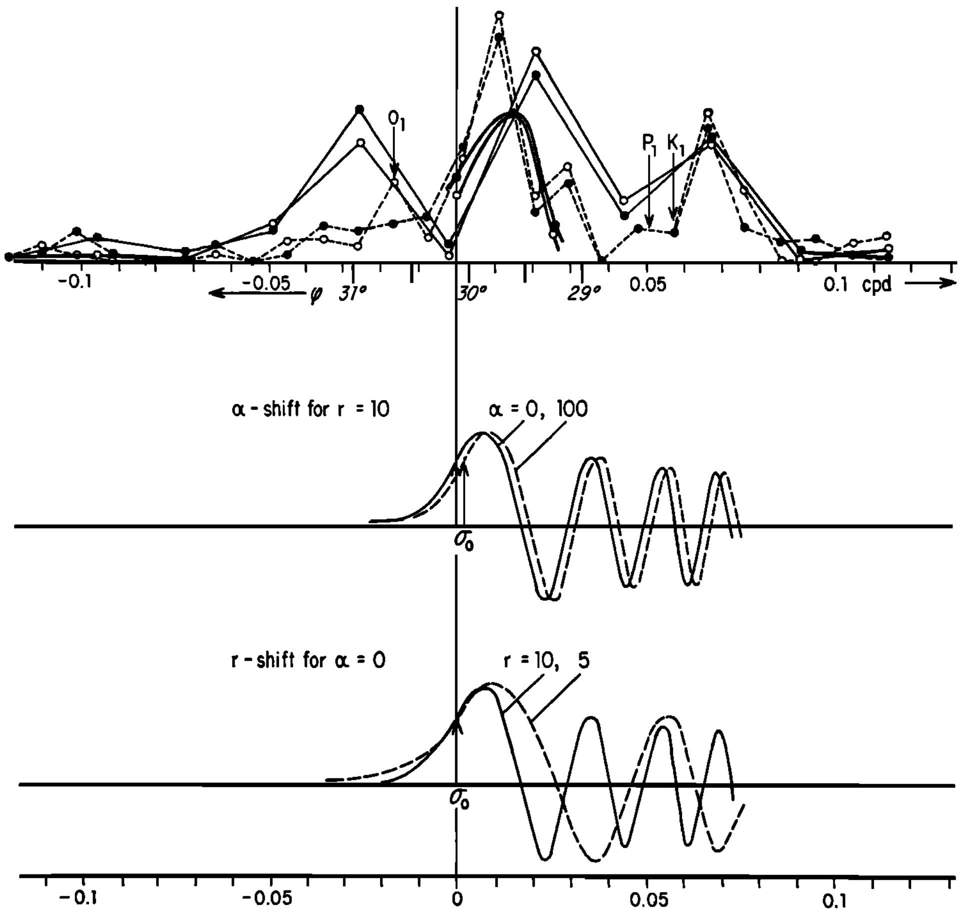

Fig. 3. The upper panel shows *observed* spectra (arbitrary scale) of $u(t)$ (open circles) and $v(t)$ (solid circles). Light lines refer to Webster's frequency spectra $F(\sigma_0)$ at fixed latitudes: φ_0 = 39°20′, σ_0 = 1.268 cpd (solid); φ_0 = 28°07′, σ_0 = 0.942 (dashed). The frequency scale gives departures from σ_0 in cpd. Heavy curves refer to Hendershott's measurements of the latitude dependence $F(\varphi)$ of intensity at a fixed frequency σ_0 = 1.0027 cpd. Lower panels give Airy amplitude spectra at φ_0 = 39°20′ for stated values of r, α. Bottom scale can be interpreted either as a relative frequency $\sigma - \sigma_0$ in cpd, or as a relative latitude $- 2 \cos \varphi_0 (\varphi - \varphi_0)$ in radians.

TABLE 4. Principal Diurnal Tide Constituents

	Rel. Amp.	Period, hours	Frequency, cpd	$\varphi_0 = \arcsin \frac{1}{2}\sigma$
O_1	0.377	25.80	0.9295	27°43′
P_1	0.176	24.07	0.9971	29°54′
K_1	0.362	23.94	1.0027	30°05′

cpd resolution. This calls for a record length of 200 days (Tukey's 'quefrency'). To obtain 20 degrees of freedom at this resolution ($\pm\sqrt{1/20}$ relative uncertainty) requires 2000 days! For the moment, this appears technically unfeasible.

A frequency resolution of 0.005 cpd is comparable to a 'latitude resolution' of ¼ degree. Can we substitute closely spaced vessels for very long records? In general, the answer is 'no,' because to take advantage of close spacing we need to confine our measurements to a comparably narrow frequency band, and this, again, requires long records. Inspection of Figure 2 will convince the reader that frequency smearing will obliterate latitudinal fine structures and vice versa.

Hendershott's measurements. There is a way out, however, and this forms the basis of Hendershott's pioneering effort in 1964. If a spectral line is naturally present and sufficiently strong to dominate a selected frequency band, the variation from station to station is presumably associated with this one line, and there is no further need for refining the bandpass filter. Hendershott selected K_1 (see Table 4). The vessel latitudes and associated frequencies were

φ	29°15′	29°36′	30°06′
$2 \sin \varphi$	0.977	0.988	1.003

The northern vessel is nearly at the reference latitude $\varphi_0 = \varphi(K_1) = 30°05'$, and the others are to the south, giving an effective resolution of about 0.01 cpd. A strong neighbor, O_1, has its cutoff latitude too far south to seriously interfere. At the same time, the effective bandwidth of K_1 (associated with cycle per year splitting) is only 0.003 cpd and is sufficiently narrow not to blur the Airy fine structure of 0.02 cpd.

TABLE 5. Parameters for $\varphi_0 = 39°20'$

Assumed vertical mode	$r = 5$		$r = 10$	
$\gamma_r(14, 16)$	1580		3160	
$L(33)$	0.00740 = 0.424°		0.00468 = 0.268°	
α (assumed)	0	100	0	100
EW wavelength, km	∞	400	∞	400
Frequencies in cpd at Turning Point and First Three Airy Maxima				
$\eta = 0$	1.268	1.274	1.268	1.270
$\eta = -1.019$	1.287	1.293	1.275	1.277
$\eta = -3.248$	1.305	1.311	1.292	1.294
$\eta = -4.820$	1.323	1.329	1.303	1.305

Hendershott's measurements are plotted in Figure 3 against a latitude scale $-2 \cos \varphi_0 (\varphi - \varphi_0)$, which is comparable to the frequency scale $\sigma - \sigma_0$ used for Webster's spectra. An agreement with the Airy structure can be achieved by selecting

$$r = 5 \qquad \alpha = 32 \text{ (0 to 103)}$$
$$r = 10 \qquad \alpha = 172 \text{ (0 to 258)}$$

which is not altogether out of line with Webster's measurement and agrees with Hendershott's estimates[6] and previous rough estimates by *Cox and Sandstrom* [1962].

There is a serious difficulty, however, because at each of Hendershott's three stations the observed energy is not contained within a harmonic interval containing K_1 but is spread into neighboring harmonics, much like Webster's data. Hendershott ascribes this spreading to 'Doppler smearing' by variable currents; the plotted values in Figure 3 are the mean-squared velocities found in five neighboring harmonics, covering 0.86 to 1.14 cpd. If, in fact, there is such smearing, the whole point of a sharp input line is lost, and we are back where we started.

Webster and Fofonoff [1967] have measured currents for a number of days at the 'tidal latitudes' in the Sargasso Sea: 28°50′N (depths 55, 620, 3240 meters), 29°11′N (55, *617* meters), and 29°30′N (55 meters). Only the italicized record shows an indication of inertial current. This result could imply an Airy peak centered at the midstation, which is too narrow to reach the outer stations (not unlike the Hendershott results) and has r values that discriminate against 55-meter depth. *Webster* [1968*b*] interprets these data as an intermittent signal that happened to have been received only at the central station. Such intermittent disturbances of local inertial frequency have been observed at non-tidal latitudes, and this interpretation implies that, even at tidal latitudes, the tides are not a predominant source of inertial motion.

Sample superposition spectra. The observed spectra are presumed to be the result of superposition of the various r modes, each with some given distribution of energy as a function of α. We assume statistical homogeneity in longitude and random phase among all modes. Let $q_r^2\ (r, \alpha, \sigma)\ d\alpha\ d\sigma$ designate the mean square amplitude of the discrete mode r in the interval $d\alpha\ d\sigma$ centered on α, σ. It then follows from (40) and (41) that

$$F(\sigma; z, \varphi) = \frac{n(z)}{2 \cos \varphi} \sum_r \cos^2 \psi_r(z) \int_{-\infty}^{\infty} q_r^2(\alpha) Ai^2(\eta)\, d\alpha \tag{47}$$

$$\eta = -\frac{\sigma - \sigma_0}{2L_r \cos \varphi} + \alpha^2 L_r^2 \qquad \sigma_0 = 2 \sin \varphi$$

is the spectrum of $v(t)$ or $u(t)$ at a fixed point λ, z, φ. In a two-dimensional diagram with coordinates $x = (\sigma - \sigma_0) \div 2L \cos \varphi$, $y = \alpha L$, curves of equal values of the Airy coordinate η would appear as parabolas, $x = y^2 - \eta$. Figure 4 portrays a surface whose elevation is equal to $Ai^2\ (y^2 - x)$. At point φ, z a cut in this diagram at constant αL (e.g., as shown for $\alpha L = 0$ and $\alpha L = 2$) gives the spectrum $F(\sigma)$ for a single pair α, r. As remarked earlier, larger α values shift the spectrum toward higher frequencies, and larger r (smaller L) values compress the frequency scale.

[6] Hendershott computed $F(\eta)$ from a quantitative r, α distribution he obtained from a sea-bottom scattering model. In a way, this is further than we shall go. It is difficult to estimate how sensitive Hendershott's conclusion is to the model parameters. We prefer an approach that is independent of any assumed model.

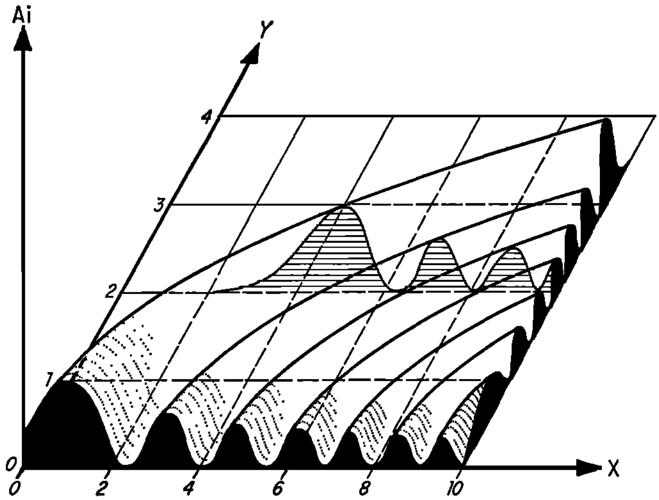

Fig. 4. A perspective view of the surface $Ai^2(\eta)$ where $\eta = y^2 - x$. The horizontal coordinates denote $x = (\sigma - \sigma_0) \div (2L \cos \varphi_0)$ and $y = \alpha L$. The three shaded areas show cuts through a volume bounded by $Ai^2(\eta)$ at $y = 0$, $y = 2$, and $x = 10$. Silhouettes of the Airy ridges are shown by the quasi-parabolic curved lines.

The integration in (47) is parallel to the y axis of Figure 4. As a simple example let

$$q_r^2(\alpha) = q_r^2(-\alpha) = Q_r^2/\bar{\alpha}_r \qquad \text{for} \quad 0.5\bar{\alpha}_r < \alpha < 1.5\bar{\alpha}_r \tag{48}$$

and zero otherwise. The resulting integral has been evaluated numerically and is displayed in Figure 5 for selected values of $\bar{\alpha}_r L$. Evidently, $\bar{\alpha}L$ values around 1 are most effective in producing a power spectrum with a single dominant peak; for smaller $\bar{\alpha}L$ values the Airy line structure is retained in the composite spectrum, whereas larger $\bar{\alpha}L$ values subdue all features. Taking $\bar{\alpha}L = 1$ gives $\alpha = 140, 200$ for $r = 5, 10$ at 40° latitude, which is consistent with our estimates based on Webster's spectra.

The reader can visualize further smudging associated with the superposition of r modes (weighted according to $Q_r^2 \cos^2 \psi_r(z)$), each mode being characterized by the appropriate stretching of the frequency scale. In general, this will tend to wipe out the fine structure at frequencies above the principal peak.

The u, v red shift. There is good agreement between the u spectra and v spectra (Figure 3). If there are differences, they are well below the present resolution, i.e. 0.01 cpd. We shall inquire what the expected differences, based on theoretical considerations, are.

The solutions (34) and (35) give u and v both proportional to $Ai(\eta)$, so that there are no differences to order L. In accordance with the expansion (A2) we write

$$U(\eta) = U_0 + LU_1 = -Ai + LU_1$$

$$V(\eta) = V_0 + LV_1 = Ai + LV_1$$

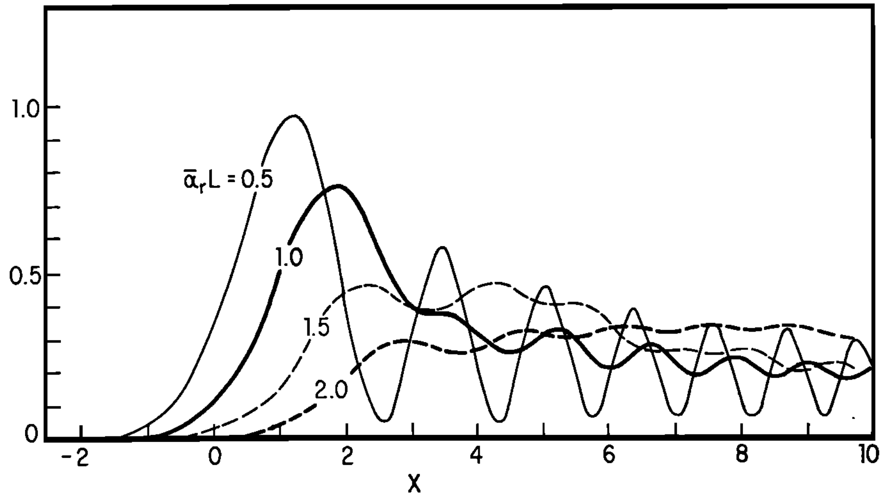

Fig. 5. The integrated spectra (equation 48) plotted against $x = (\sigma - \sigma_0) \div (2L \cos \varphi_0)$, for selected values of $\alpha_r L$. For convenience the spectra have been normalized by division with $Q_r^2 Ai^2(-1.019)$.

and designate by $\eta_u = \eta_{\max} + \delta_u$, $\eta_V = \eta_{\max} + \delta_v$ the Airy arguments for which U, V reach their maximum; $\eta_{\max} = -1.019$ is the argument for which U_0, V_0 have their principal maximum. In the vicinity of $\eta_{\max}$, we have

$$Ai(\eta_u) = Ai(\eta_{\max}) + \delta_u Ai'(\eta_{\max}) + \cdots$$
$$Ai'(\eta_u) = 0 + \delta_u \eta_{\max} Ai(\eta_{\max}) + \cdots$$

Thus

$$U'(\eta_u) = 0 = -\delta_u \eta_{\max} Ai(\eta_{\max}) + LU_1'(\eta_{\max}) + O(L\delta)$$
$$V'(\eta_v) = 0 = \delta_v \eta_{\max} Ai(\eta_{\max}) + LV_1'(\eta_{\max}) + O(L\delta)$$

and so

$$\delta_v - \delta_u = -\frac{L[U_1'(\eta_{\max}) + V_1'(\eta_{\max})]}{\eta_{\max} Ai(\eta_{\max})}$$

The expression for $U_1 + V_1$ is given in (A10). At a fixed latitude, $\delta_v - \delta_u = \eta_v - \eta_u$ is simply related to the separation in peak frequency $\delta_v - \delta_u$ according to (39), and we find, on differentiation,

$$\sigma_v - \sigma_u = 2L^2 \cos \varphi_0 \cot \varphi_0 [2\alpha L + (-\eta_{\max})^{-1}] \tag{49}$$

$\sigma_u - \sigma_v$ is positive for positive αL, thus giving a v versus u red shift. For sufficiently larger negative αL value the relation is reversed. The indicated shift is of the order $L^2 \sim 10^{-4}$ cpd, far too small to have been detected. Boundary influences are presumably far more important.

COHERENCE

Qualitative considerations. Observational evidence consists essentially of the experience (some of it extending over thirty years) that the resemblance between inertial currents measured at nearby stations is surprisingly poor, whether these stations are separated east–west, north–south, or up–down. We would expect that phenomena whose period is of the order of a day should have characteristic vertical scales comparable to the ocean's depth and horizontal scales comparable to the earth's radius. Somehow our intuition is violated.

Let us put the problem into somewhat more definite terms. Consider the contribution from multiple modes within some narrow frequency band centered at σ. In general, different modes have different wave numbers, and a cluster of neighboring modes gives rise to a distribution of wave numbers, from say $k_i - \frac{1}{2}(\delta k)_i$ to $k_i + \frac{1}{2}(\delta k)_i$. (The subscripts i refer to Cartesian components). It is then plausible (and will be demonstrated) that

$$(\delta k)_i(\delta x)_i = O(1) \tag{50}$$

defines a coherence scale $(\delta x)_i$. Certainly over distances much smaller than $|\delta x| = O(\delta k)^{-1}$ even the extreme modes hardly drift out of phase, and the record remains essentially undistorted. For a single mode, $\delta k = 0$, and the coherence distance is infinite. Thus, the coherence loss depends on the *spread* of wave numbers, as distinct from the problem of line broadening (discussed in the preceding section), which involves the *mean* wave numbers.

Suppose the energy of baroclinic waves is largely contained in vertical mode numbers $r = 5$ to $r = 15$, or equivalently in wave numbers 5π to 15π radians per ocean depth. This is not inconsistent with the existing observations; further, an octave bandwidth ($\delta k/k = O(1)$) is typical of many geophysical processes. The vertical coherence distance is then of the order of ocean depth $\div 10\,\pi$, or 200 meters. The horizontal coherence is found to be of the order of the Airy scale, L_r, or 10 miles. The latter value is so very much smaller than the earth's radius because of the small phase velocity of high-order internal waves, and this, in turn, is related to the relatively weak density stratification in the oceans. An essential requirement in this estimate is that the various modes have no fixed phase relation with respect to one another. We shall return to this important point.

From these considerations, admittedly based on hindsight, we might even expect to find the short coherence distances that have been reported. The remainder of this section attempts to make these arguments quantitative.

Definitions. Let $f_n(t)$, $n = 1, 2$, designate any two stationary time series. The time average

$$\rho_{mn}(\tau) = \langle f_m(t) f_n(t - \tau)\rangle \tag{51}$$

is called the covariance of the two series, and

$$C_{mn}(\sigma) + iQ_{mn}(\sigma) = \int_{-\infty}^{\infty} \rho_{mn}(\tau) \exp(-i\sigma\tau)\, d\tau \tag{52}$$

are the associated co-spectra and quadrature spectra. For the special case of $m = n$ we have $Q_{nn}(\sigma) = 0$, and $C_{nn}(\sigma) = F_n(\sigma)$ is then known as the power spectrum of $f_n(t)$. The coherence, $R_{mn}(\sigma)$, and relative phase, $\phi_{mn}(\sigma)$ are defined by

$$C_{mn} + iQ_{mn} = (C_{mm}C_{nn})^{1/2}R_{mn} \exp i\phi_{mn} \tag{53}$$

Interpretation. We refer to the literature on time series [for example, *Jenkins,* 1961; *Haubrich,* 1965; *Bendat and Piersal,* 1966] for further discussion, but some brief remarks are perhaps in order as to why we regard $R_{mn}(\sigma)$ a measure of the resemblance between records m and n. If $f_2(t)$ is the result of any linear transformation of $f_1(t)$, plus a noise, e.g.,

$$f_2(t) = \int_0^\infty f_1(t - \tau) \text{ kernel } (\tau)\, d\tau + \text{noise } (t)_1 \tag{54}$$

then $[1 - R_{12}^2(\sigma)]\; C_{22}(\sigma)\; \delta\sigma$ measures the noise energy of $f_2(t)$ in the band $\sigma \pm \frac{1}{2}\,\delta\sigma$. If $R(\sigma) = 1$, then $f_2(t)$ is perfectly predictable from $f_1(t)$ by realizable linear devices (or their numerical equivalents). With increasing noise, the coherence drops, as do the resemblance and predictability. This is the most familiar way of looking at coherence and is satisfactory for one-dimensional linear processes, because then the convolution integral in (54) allows for a frequency-dependent attenuation and phase delay between records 1 and 2.

In the case of a multi-dimensional problem, such as ours, however, the process $f_n(t)$ in any *one* frequency band $\sigma \pm \frac{1}{2}\,\delta\sigma$ may be associated with two (or more) modes, each with its own attenuation and delay, and the linear transformation (54) is unable to cope with mode separation. By assumption, the modes bear no fixed phase relation with respect to one another, and each mode fades in and out with a time scale $(\delta\sigma)^{-1}$ of the reciprocal bandwidth. The 'instantaneous' phase difference $\phi_{12}(\sigma)$ alternates between values appropriate to each of the modes. Again coherence (and predictability) is lost, quite apart from any noise in the system. Under these circumstances the coherence scale turns out to be in accord with statement 50.

One can also regard each mode as comprised of standing plane waves with propagation vector k_i (for inertial waves k_i is nearly vertical). From this point of view, multimodes are regarded as superpositions of independent pencil beams, and, unless the modes are individually resolved, they lead effectively to beam broadening of the order $\delta k/k$ and attendant coherence losses. Again, the results are in accord with statement 50 [*Munk, Miller, Snodgrass, and Barber,* 1963].

The case of pg waves. Inertial solutions (u, v) are of the type

$$f_n(t) = f(t; \lambda_n, \eta_n, z_n) = Ai(\eta_n) \cos \kappa z_n \cos (s\lambda_n - \sigma t) \tag{55}$$

for stations $n = 1, 2$, where $\kappa(z)$ is a slowly varying (compared with h/r) vertical wave number (equations 12 and 17). We presume that all pertinent statistics are stationary in time (invariable with respect to translation of the t axis).

The situation differs with respect to the space axes. (1) We may assume east–west stationarity for λ displacements, within limits imposed by climatological and bathymetrical considerations. (2) We may assume quasi-stationarity for z displacements, provided that these displacements do not exceed the stratification scale height $(N^{-1}\cdot d\bar{N}/dz)^{-1}$, typically 1 km at abyssal depths. Alternatively, one could presume stationarity with regard to a 'stretched' z axis, $\zeta(z) = [N(z)/\bar{N}]z$. (3) Conditions are definitely not stationary in η near the turning latitude.

Two-mode coherence. A simple model is that of two modes of almost equal frequencies σ, σ' (both lying within the narrow band $\sigma \pm \frac{1}{2}\,\delta\sigma$), but significantly different wave numbers s, s'. The frequency doublet is a simple device for avoiding fixed phase relations between the modes. Later we shall show that the results are a useful guide to a many-mode distribution, provided that we interpret the wave number separation in the doublet as some sort of weighted spread in the multiplet.

East–west separation. Consider two stations separated only in longitude; then from (55)

$$f_n(t) \approx a \cos(s\lambda_n - \sigma t) + a' \cos(s'\lambda_n - \sigma' t)$$

for stations $n = 1, 2$. On computing the quantities (51), we find

$$\rho_{11}(\tau) = \rho_{22}(\tau) = \tfrac{1}{2}a^2 \cos \sigma\tau + \tfrac{1}{2}a'^2 \cos \sigma'\tau$$

$$\rho_{12}(\tau) = \tfrac{1}{2}a^2 \cos(\sigma\tau + s\,\delta\lambda) + \tfrac{1}{2}a'^2 \cos(\sigma'\tau + s'\,\delta\lambda)$$

with $\delta\lambda = \lambda_2 - \lambda_1$. We now ignore the slight difference in σ and σ', writing each as σ. For simplicity let $a = a'$; then

$$\begin{aligned} C_{11} &= C_{22} = a^2 \\ C_{12} + iQ_{12} &= \tfrac{1}{2}a^2[\exp(-is\,\delta\lambda) + \exp(-is'\,\delta\lambda)] \end{aligned} \tag{56}$$

It follows that

$$R_{12} = \cos \tfrac{1}{2}(\delta s \cdot \delta\lambda) \qquad \phi_{12} = \bar{s}\,\delta\lambda \tag{57}$$

with

$$\delta s = s - s' \qquad \bar{s} = \tfrac{1}{2}(s + s')$$

Near the origin, the coherence drops as $R_{12} = 1 - \frac{1}{8}\,(\delta s \cdot \delta\lambda)^2 + \cdots$, which is consistent with the general remarks leading to (50). The additional 'lobes' centered at $\delta s \cdot \delta\lambda = 2\pi, 4\pi, \cdots$ are related to the assumed doublet structure and are not generally significant. Thus, for a superposition of j incoherent modes, with frequencies σ_j all within $\sigma \pm \frac{1}{2}\,\delta\sigma$,

$$\begin{aligned} f_n(t) &= \sum_j a_j \cos(s_j\lambda_n - \sigma_j t) \\ C_{11} &= C_{22} = \tfrac{1}{2} \sum_j a_j^2 \\ C_{12} + iQ_{12} &= \tfrac{1}{2} \sum_j a_j^2 \exp(-is_j\,\delta\lambda) \end{aligned} \tag{58}$$

and

$$R_{12}^2 = \sum_j \sum_{j'} a_j^2 a_{j'}^2 \cos(s_j - s_{j'})\,\delta\lambda \Big/ \Big(\sum_j a_j^2\Big)^2 \tag{59}$$

For closely spaced s modes of equal amplitudes, we can replace the summations by

$$R_{12}^2 = \frac{1}{4\Delta^2} \int_{s-\Delta}^{s+\Delta} \int_{s-\Delta}^{s+\Delta} \cos(s - s')\,\delta\lambda\,ds\,ds' = \frac{\sin^2(\Delta\,\delta\lambda)}{(\Delta\,\delta\lambda)^2}$$

The coherence drop near the origin is $1 - 1/6\ (\Delta\ \delta\lambda)^2 + \cdots$, similar to that of (57), but successive lobes are now greatly suppressed. For a tapered distribution of s modes, the lobes might be altogether absent. We conclude that the doublet model is an adequate representation of multiplets near the origin, with the doublet spacing at about half the multiplet spread.

Up–down separation. Next, we have measurements separated only vertically.

$$
\begin{aligned}
f_n(t) &= a \cos \sigma t \cos \kappa z_n + a \cos \sigma' t \cos \kappa' z_n \\
C_{mn} &= \tfrac{1}{2}a^2(\cos \kappa z_m \cos \kappa z_n + \cos \kappa' z_m \cos \kappa' z_n) \\
Q_{mn} &= 0
\end{aligned}
$$

and

$$
R_{12}^2 = 1 - \frac{\frac{1}{4}a^4[\sin 2\bar{\kappa}\bar{z} \sin \frac{1}{2}\ \delta\kappa\ \delta z + \sin \bar{\kappa}\ \delta z \sin\ \delta\kappa\ \bar{z}]^2}{C_{11}C_{22}} \tag{60}
$$

For the case of a single mode, $\delta\kappa = 0$, and we recover $R = 1$. For well separated modes, κ and $\delta\kappa$ are of the same order, and $R^2 = 1 - O\ (\delta\kappa \cdot \delta z)^2$, in accordance with our general expectation.

North–south separation. Finally, we have a latitudinal separation with stations at φ_1 and φ_2 and a mode doublet characterized by

$$
\eta_n = \frac{\varphi_n - \varphi_0}{L} + (\alpha L)^2 - \frac{\sigma - \sigma_0}{2L \cos \varphi_0}
$$

and similarly for η', α' L', σ' (but σ nearly equal to σ'). We find

$$
\begin{aligned}
f_n(t) &= a \cos \sigma t\ Ai(\eta_n) + a' \cos \sigma' t\ Ai(\eta_n') \\
C_{mn} &= \tfrac{1}{2}a^2[Ai(\eta_m)Ai(\eta_n) + Ai(\eta_m')Ai(\eta_n')] \\
Q_{mn} &= 0
\end{aligned}
$$

and

$$
R_{12}^2 = 1 - \frac{\frac{1}{4}a^4[Ai(\eta_1)Ai(\eta_2') - Ai(y_1')Ai(y_2)]^2}{C_{11}C_{22}} \tag{61}
$$

For a single mode, $\eta_n = \eta_n'$, and so $R^2 = 1$. For well separated modes and nearby stations, $R^2 = 1 - O\ (\delta\varphi/L)^2$. Accordingly, the coherence is determined by the Airy scale.

In all the foregoing derivations, the underlying assumption is that the modes bear no fixed phase relation with respect to one another. If the relative phases are fixed, the coherence between any two stations is unity, regardless of the complexity of the mode structure. (Formally, this follows by taking all σ_j in (48) to be identical.) Random phase relationships between modes result if the modes are generated by independent random processes, or they may be coherently generated and subsequently 'randomized' by transmission through a time-variable medium. For example, baroclinic tides are found to be incoherent with respect to one another and with respect to the tide-producing forces [*Radok, et al.,* 1967]. The baroclinic modes are believed to be generated by bottom scattering of the barotropic (or surface) tide. It can be demonstrated that minor fluctuations in the thermocline would soon destroy all existing phase relations.

Fofonoff-Webster measurements. There is little to go by beyond the general agreement that records at neighboring stations bear surprisingly little resemblance (see the introduction). The only quantitative measurements known to us are those reported by *Day and Webster* [1965], *Webster and Fofonoff* [1967], and *Webster* [1968*b*]. In the first report, no significant coherence was found between currents at 50 and 100 meters. In the third report, Webster notes a coherence of $R = 0.34$ between currents at 7- and 88-meter depths at one location, and $R = 0.67$ between the 88-meter current at this location, and 98 ± 10 meters at a second location[7] displaced by 3 km in an east–west direction.

To estimate the vertical coherence, we may take $N(50 \text{ m})/\Omega = 100$ cpd. Suppose the prominent modes extend from $r = 5$ to $r = 15$. Then from (12)

$$\delta\kappa = \frac{\gamma_{15} - \gamma_5}{2a} \frac{N(50 \text{ m})}{\Omega} \doteq \frac{300\ \delta r}{2a} \frac{N(50 \text{ m})}{\Omega} = 0.025 \frac{\text{radians}}{\text{meter}}$$

and coherence is significantly reduced for separations exceeding $\delta z = (\delta\kappa)^{-1} = 40$ meters. Therefore, there is at least no conflict here.

For an east–west separation of 3 km, the result (57) implies $\frac{1}{2}\delta k \cdot \delta x =$ arccos 0.67; hence, $\delta k \doteq 0.5$ km^{-1}. The measured phase difference of 34° suggests a wavelength of 10 station separations, or 30 km (somewhat larger than the Airy scale $L \approx 25$ km). The mean wave number is then 0.21 km^{-1}, and the relative spread $\delta k/\bar{k} \doteq 2$. It is possible that vertical separation of perhaps 10 meters may account for some coherence loss; hence, the spread is not quite so broad.

For a north–south separation, the expected coherence loss over a distance L (~ 0.3° latitude) is consistent with Hendershott's finding that measurements at 0.5° separation did not resemble one another.

PERSISTENCE

Fofonoff-Webster measurements. At any given station the inertial motion is characterized by a high degree of intermittency. Figure 4 of *Day and Webster* [1965] shows groups of days of high inertial activity at 28°07′N, apparently following severe storms and separated by times of relative calm. The 'events' are not very well defined, lying perhaps 5 db above the mean level. As an example, hurricane Daisy on October 6, 1962, was followed by 5 days of high activity centered on October 16 at 50-meter depth, and 10 days centered on October 18 at 100 meters. Figure 6 of *Webster* [1968] shows the result of (complex) demodulation at inertial frequency of current measurements at 39°20′N, 70°W, 7-meter depth. Much of the energy in this 40-day record is contained in the first week's high activity.

Model of random, distributed sources. This is essentially the view adapted in our paper. We assume a fair degree of spatial homogeneity, with a 'noise' centered near the inertial frequency within a bandwidth of the order of L cpd. The

[7] The absence of a surface float at the second location makes the precise depth of the current meter uncertain and subject to some variation in time.

persistence of a band-limited noise is roughly the reciprocal bandwidth, or of the order $L^{-1} \sim 100$ days. The reported persistence is of the order of a week.

Model of coherent, local generation. Suppose that inertial waves from a source at (0, 0) are observed at (x_i, t), $i = 1, 2, 3$. The energy is propagated along 'rays' of constant σ, according to

$$dx_i/dt = \partial\sigma/\partial k_i \tag{62}$$

and the solution of the three equations of (62) together with known dispersion $\sigma(k_i)$ determines $k_i(x_j, t)$ and $\sigma(x_j, t)$. If energy is fed into a limited range of values of k_i, σ, the duration at a point x_j may be quite limited and differ from that at a neighboring point x_j'. In this way one might be able to account for the observed intermittency in space and time. *Blandford* [1966] gives some discussion along this point of view.

If the source is concentrated, the wave trains associated with various k_i at a frequency σ bear some fixed phase relations, and the currents at x_j and x_j' are coherent (provided that they are above instrumental noise level).

Conclusion. Both models are lacking. The random, distributed sources cannot account for the reported intermittency, and the coherent, local generation cannot account for the observed loss of coherence between stations.

The truth must lie between these two extreme viewpoints. In order for our interpretation of the observed band and coherence structure to have any claim to reality, it is necessary that the observed motion at a point be the result of at least several independent generators.

APPENDIX

We consider motion near a *reference* latitude φ_0 with frequencies near the inertia frequency $\sigma \sim 2 \sin \varphi_0$. The spheroidal wave equation suggests

$$L = (\gamma^2 \sin 2\varphi_0)^{-1/3} \tag{A1}$$

as the appropriate scale for the latitudinal 'fine structure.' We set $\varphi = \varphi_0 + L\eta_1$, $d/d\eta_1 = O(1)$, and

$$\sigma = 2 \sin \varphi_0 + 2L\sigma_1$$

In the equations of motion (8), (9), and (11), we first insert the magnitudes $U = \mathcal{U}L^0 = \mathcal{U}$, $V = \mathcal{V}L^l$, $P = \mathcal{P}L^m$, and $s = \mathcal{S}L^n$, where $\mathcal{U}$, $\mathcal{V}$, $\mathcal{P}$ and $\mathcal{S}$ are of order unity and l, m, and n are integers to be determined. The principal terms, after using (A1), are

$$2 \sin \varphi_0 \mathcal{U} + 2 \sin \varphi_0 \mathcal{V} L^l = \mathcal{S}\mathcal{P}L^{m+n}$$

$$2 \sin \varphi_0 \mathcal{U} + 2 \sin \varphi_0 \mathcal{V} L^l = -\cos \varphi_0 \, (d\mathcal{P}/d\eta_1) \, L^{m-1}$$

$$\mathcal{S}\mathcal{U}L^{n+1} + \cos \varphi_0 \, (d\mathcal{V}/d\eta_1) \, L^l = \tfrac{1}{4} \cos \varphi_0 \mathcal{P} L^{m-2}$$

An examination of the possible balances (for the three possibilities $l \leq -1$, $l = 0$, and $l \geq +1$) shows that l must be zero; m and n can then be either $m = 2$,

$n \geq -1$ or $n = -1$, $m \geq 2$. These are allowed for by defining the series

$$\begin{aligned} U &= U_0 + LU_1 + \cdots \\ V &= V_0 + LV_1 + \cdots \\ P &= 4L^2(P_0 + LP_1 + \cdots) \\ sL &= O(L^0) \end{aligned} \tag{A2}$$

$\sin\varphi$ and $\cos\varphi$ are now expanded in $\varphi - \varphi_0 = L\eta_1$, and to sufficient order we have

$$(\sin\varphi_0 + L\sigma_1)(U_0 + LU_1) + (\sin\varphi_0 + L\eta_1\cos\varphi_0)(V_0 + LV_1) = 2L(sL)P_0$$

$$(\sin\varphi_0 + L\eta_1\cos\varphi_0)(U_0 + LU_1) + (\sin\varphi_0 + L\sigma_1)(V_0 + LV_1) = -2L\cos\varphi_0\frac{dP_0}{d\eta_1}$$

$$(sL)(U_0 + LU_1) + (\cos\varphi_0 - L\eta_1\sin\varphi_0)\left(\frac{dV_0}{d\eta_1} + L\frac{dV_1}{d\eta_1}\right)$$
$$= [\cos\varphi_0 + L(\sigma_1\cot\varphi_0 - 2\eta_1\sin\varphi_0)](P_0 + LP_1)$$

The zero-order relations are

$$U_0 = -V_0 \quad \text{(twice)} \tag{A3}$$

$$P_0 = \frac{dV_0}{d\eta_1} - \frac{sL}{\cos\varphi_0}V_0 \tag{A4}$$

The first-order relations from the first two equations are now

$$\sin\varphi_0(U_1 + V_1) + (\eta_1\cos\varphi_0 - \sigma_1)V_0 = 2(sL)P_0 \tag{A5}$$

$$\sin\varphi_0(U_1 + V_1) - (\eta_1\cos\varphi_0 - \sigma_1)V_0 = -2\cos\varphi_0\frac{dP_0}{d\eta_1} \tag{A6}$$

Subtracting these and combining with (A4) results in

$$\frac{d^2V_0}{d\eta_1^2} - \left[\eta_1 + \left(\frac{sL}{\cos\varphi_0}\right)^2 - \frac{\sigma_1}{\cos\varphi_0}\right]V_0 = 0$$

This relation can be written as the standard Airy equation

$$(d^2V_0/d\eta^2) - \eta V_0 = 0 \tag{A7}$$

if we define

$$\begin{aligned} \eta &= \eta_1 + \left(\frac{sL}{\cos\varphi_0}\right)^2 - \frac{\sigma_1}{\cos\varphi_0} \\ &= \frac{\varphi - \varphi_0}{L} + (\alpha L)^2 - \frac{\sigma - 2\sin\varphi_0}{2L\cos\varphi_0} \end{aligned} \tag{A8}$$

where $\alpha = s/\cos\varphi_0$. (A7) has the two solutions $Ai(\eta)$ and $Bi(\eta)$. The $Bi(\eta)$ solution increases exponentially with positive η. It is required to represent latitudinally traveling waves, for which $V_0 \sim Ai \pm (-1)^{1/2}Bi$, or to form standing waves that would satisfy boundary conditions of $V_0 = 0$ at two arbitrary values of η. Our philosophy here is to ignore Bi, based on the general hypothesis that the complicated shape of the real oceans makes a normal mode analysis unsatisfactory. [We also remark that for frequencies σ close to 2 sin φ_0, $(\alpha L)^2$ of order unity, and L small ($r \geq 1$), an observation point close to φ_0 need be only a short distance equatorward (about 1 degree of latitude) of any idealized poleward boundary before the $Bi(\eta)$ contribution in a normal mode analysis would contribute little to the solution at the observation point.] The solution without Bi ignores the passage of waves in a latitudinal direction across $\varphi \sim \varphi_0$. From the preliminary ray treatment given by *Eckart* [1960, p. 170] it can be seen that this occurs only for $\sigma > 2 \sin \varphi_0$, i.e. for our $\beta^2 > 0$. Our treatment, with its neglect of Bi, therefore ignores the occurrence of (presumably) transient situations in which the dominant spectral peak is not close to the inertia frequency 2 sin φ_0. It should, however, give a reasonable zero-order description of the statistically stationary conditions near $\sigma = 2 \sin \varphi_0$, although our neglect of currents and horizontal variations in h and N^2 must be kept in mind.

In addition to $U_0 = -V_0 = -Ai(\eta)$, we have

$$P = 4L^2P_0 = 4L^2[dV_0/d\eta - (\alpha L)V_0] \tag{A9}$$

We shall require the combination $U_1 + V_1$, which can be obtained by adding (A5) and (A6):

$$\begin{aligned} U_1 + V_1 &= \cot \varphi_0[2\alpha L(dV_0/d\eta) + (\eta + \alpha^2L^2)V_0] \\ &= \cot \varphi_0[2\alpha L(dAi(\eta)/d\eta) + (\eta + \alpha^2L^2)Ai(\eta)] \end{aligned} \tag{A1}$$

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