On Kurtosis and Occurrence Probability of Freak Waves

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ABSTRACT

Based on a weakly non-Gaussian theory, the occurrence probability of freak waves is formulated in terms of the number of waves in a time series and the surface elevation kurtosis. Finite kurtosis gives rise to a significant enhancement of freak wave generation in comparison with the linear narrowbanded wave theory. For a fixed number of waves, the estimated amplification ratio of freak wave occurrence due to the deviation from the Gaussian theory is 50%–300%. The results of the theory are compared with laboratory and field data.

1. Introduction

In the last decade freak waves have become an important topic in engineering and science and are sometimes featured as a single and steep crest causing severe damage to offshore structures and ships. Freak wave studies started in the late 1980s (Dean 1990) and the high-order nonlinear effects on the freak waves were discussed in the early 1990s (e.g., Yasuda et al. 1992; Yasuda and Mori 1994). As a result of many research efforts, the occurrence of freak waves, their mechanisms, and detailed dynamic properties are now becoming clearer (e.g., Trulsen and Dysthe 1997; Lavrenov 1998; Osborne et al. 2000; Onorato et al. 2001; Haver 2001; Mori et al. 2002). The state of the art on freak waves was summarized at two recent rogue wave conferences (Olagnon and Athanassoulis 2000; Olagnon 2004). It was concluded that the third-order nonlinear interactions enhance freak wave appearance and are the primary cause of freak wave generation in a general wave field except for the case of strong wave–current interaction or wave diffraction behind the islands.

Numerical and experimental studies have demonstrated that freaklike waves can be generated frequently in a two-dimensional wave flume without current, refraction, or diffraction (Stansberg 1990; Yasuda et al. 1992; Trulsen and Dysthe 1997; Onorato et al. 2001). Moreover, the numerical studies clearly indicate that a freak wave having a single, steep crest can be generated by the third-order nonlinear interactions in deep water (Yasuda et al. 1992). Also, the theoretical background of freak wave generation has become more clear (Osborne et al. 2000), but the quantitative occurrence probabilities in the ocean remain uncertain. In addition, it is still questionable how to characterize the dominant statistical properties of the freak wave occurrence in terms of nonlinear parameters, spectral shape, water depth, and so on.

Nevertheless, although there is no doubt that the third-order nonlinear interactions are related to the steep wave generation in the random wave train, the theoretical background of the relationship between the freak wave generation and the third-order nonlinear interactions is not well established. Freak wave generation is sometimes discussed in the context of the Benjamin–Feir instability in deep-water waves because of the similarity of the steep wave profile itself (Yasuda et al. 1992; Onorato et al. 2001). Over the last two decades, Benjamin–Feir-type instability of the deep-water gravity waves has been studied by many researchers using the nonlinear Schrödinger type of equations (Yuen and Lake 1982; Caponi et al. 1982; Dysthe 1979), mode-coupling equations (Stiassnie and Shemer 1987), pseudospectral methods (Yasuda and Mori 1997), and

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experiments (Su 1982). However, there is a disparity between the periodic wave instabilities and random wave behavior, because the broad-banded spectra and random phase approximation are essentially describing the ocean waves in nature (i.e., Yasuda and Mori 1997; Mori 2003). Thus, the energy transfer of random waves due to four-wave interactions has been studied for describing spectral evolutions (Hasselmann 1962; Krasitskii 1990). By means of a series of numerical investigations, Yuen and Ferguson (1978) stated that the instability is confined within an initially unstable range and becomes weak if the spectral bandwidth broadens. Alber (1978) mathematically demonstrated that for a random sea the Benjamin–Feir instability vanishes if the wave spectrum is sufficiently broad. Therefore, there is a discrepancy between the nonlinear behavior of periodic waves and random waves.

Recently, Janssen (2003) investigated the freak wave occurrence as a consequence of four-wave interactions including the effects of nonresonant four-wave interactions. He found that the homogeneous nonlinear interactions give rise to deviations from the Gaussian distribution for the surface elevation on the basis of the Monte Carlo simulations of the Zakharov equation. Surprisingly, inhomogeneities only play a minor role in the evolution of the wave spectrum. He also formulated the analytical relationship between spectral shape and the kurtosis of the surface elevation. These results have the potential to unify previous freak wave studies covering nonlinear interactions, spectral profiles to nonlinear statistics, and so on.

The purpose of this study is to investigate the relationship between kurtosis and the occurrence probability of freak waves through nonlinear four-wave interactions. First, for a nonlinear stochastic wave field the relationship between high-order moments including kurtosis of the surface elevation and a nonlinear transfer function is derived. Second, the wave height and maximum wave height distributions are formulated as a simple function of kurtosis by non-Gaussian theory. Third, the wave height distribution is compared with laboratory experiments and the occurrence probabilities of freak waves are compared with field observations. Fourth, the dependence of the occurrence of freak waves on the number of waves and kurtosis will be analyzed and discussed.

2. High-order moments in the nonlinear stochastic wave field

a. General theory

Our starting point is the Zakharov equation (Zakharov 1968), which is a deterministic nonlinear evolution equation for surface gravity waves in deep water. Let us consider the potential flow of an ideal fluid of infinite depth. Coordinates are chosen in such a way that the undisturbed surface of the fluid coincides with the x–y plane. The z axis is pointed upward, and the acceleration of gravity g is pointed in the negative z direction. The surface elevation η may be written in terms of a Fourier expansion as

$$\eta = \int_{-\infty}^{\infty} d\mathbf{k}\,[a(\mathbf{k}) + a^*(-\mathbf{k})]e^{i\mathbf{k}.\mathbf{x}}, \tag{1}$$

where $a(\mathbf{k}, t) = \sqrt{(\omega/2g)}B(\mathbf{k}, t)$ and $B(\mathbf{k}, t)$ is the normal variable. Here, $\mathbf{k}$ is the wavenumber vector, k is its absolute value, and $\omega = \sqrt{gk}$ denotes the dispersion relation of deep-water, gravity waves. Alternatively, one may write for η

$$\eta = \int_{-\infty}^{\infty} d\mathbf{k}\, a(\mathbf{k})e^{i\mathbf{k}.\mathbf{x}} + \text{c.c}, \tag{2}$$

where c.c. denotes the complex conjugate.

Zakharov (1968) obtained from the Hamilton equations an approximate evolution equation for the amplitude of the free surface gravity waves that contained the third-order nonresonant and resonant four-wave interactions. To eliminate the effects of bound waves, he applied on B a canonical transformation of the type

$$B = B(b, b^*), \tag{3}$$

where b is the normal variable of the free gravity waves. The evolution equation for b, called the Zakharov equation, becomes

$$\frac{\partial b_1}{\partial t} + i\omega_1 b_1 = -i\int d\mathbf{k}_{2,3,4}\, T_{1,2,3,4} b_2^* b_3 b_4 \delta_{1+2-3-4}, \tag{4}$$

where, for brevity, we have introduced the notation $b_1 = b(\mathbf{k}_1)$, and so on, and the nonlinear transfer function $T_{1,2,3,4}$ as found by Krasitskii (1990) enjoys a number of symmetries that guarantee that the Zakharov equation is Hamiltonian and conserves wave energy. In Janssen (2003) it was shown that in the context of the deep-water version of the Zakharov equation extreme surface gravity waves are generated by nonlinear focusing in a random wave field. This process also causes the Benjamin–Feir instability of a uniform wave train. As a consequence, for deep-water waves a considerable enhancement of the probability for extreme waves is found. However, when going to shallow waters, the effect of nonlinear focusing is greatly reduced. In fact for a narrowband wave train (Whitham 1974) it can be shown that the nonlinear transfer coefficient $T_{1,2,3,4}$

vanishes when the dimensionless water depth kh = 1.36. For smaller dimensionless depth, nonlinear effects will lead to defocusing and, consequently, a considerable reduction of extreme events. In this paper we concentrate only on the deep-water case by restricting ourselves to $kh > 2$.

Thus, the nonlinear term in Eq. (4) will generate deviations from the normal, Gaussian probability distribution function (pdf) for the surface elevation. It is of interest to determine these deviations because it gives us information on the occurrence of extreme sea states. However, because the nonlinearity is of third order in amplitude, the skewness of the free waves vanishes, while the fourth cumulant is finite. On the other hand, the bound waves contribute to both the skewness and the kurtosis of the surface elevation pdf. To determine the effects of the bound waves on the pdf of a random, nonlinear wave field, one needs to utilize the canonical transformation Eq. (3). This results in complicated expressions for skewness and kurtosis because one has to go to third order in amplitude. The problem is simplified considerably when the narrowband approximation is adopted, because than one can simply use the Stokes solution for the surface elevation. For example, in the narrowband approximation it is easy to show (see section 2b) that for a "typical" steepness of the order 0.1 bound waves contribute less than 2% to the value of the kurtosis, so we can safely ignore their contributions.

Here, we are interested in obtaining the high-order moments of the surface elevation for a homogeneous random sea. The homogeneity and stationarity conditions imply

$$\langle b_1 b_2^* \rangle = N_1 \delta(\mathbf{k}_1 - \mathbf{k}_2), \quad \text{and} \quad \langle b_1 b_2 \rangle = 0, \tag{5}$$

where we have introduced the usual action density $N(\mathbf{k})$ and the angle brackets denote an ensemble average. Because for narrowband-spectra bound waves only have a small effect on the statistics of the surface gravity waves, we have to good approximation

$$\langle a_1 a_2^* \rangle \simeq \frac{\omega}{2g} N_1 \delta(\mathbf{k}_1 - \mathbf{k}_2) \quad \text{and} \quad \langle a_1 a_2 \rangle = 0. \tag{6}$$

Envelope A and phase ϕ are now defined as

$$\frac{1}{2} A e^{i\phi} = \int_{-\infty}^{\infty} d\mathbf{k}\, a e^{i\mathbf{k}.\mathbf{x}}; \tag{7}$$

hence,

$$\eta = A \cos\phi, \tag{8}$$

where for a narrowband wave train A and ϕ are slowly varying functions in time and space. Now we introduce the auxiliary variable ζ in such a way that the random variables η and ζ are not correlated in the linear wave field, $\langle \eta\zeta \rangle = 0$. Thus,

$$\zeta = A \sin\phi, \tag{9}$$

or

$$\zeta = -i\left(\int_{-\infty}^{\infty} d\mathbf{k}\, a e^{i\mathbf{k}.\mathbf{x}} - \text{c.c.}\right). \tag{10}$$

Assuming zero mean for η, the second-order moment μ_2 is given by

$$\mu_2 = \langle \eta^2 \rangle = m_0. \tag{11}$$

According to Eq. (29) of Janssen (2003), the fourth moment $\langle \eta^4 \rangle$ and kurtosis μ_4 can be obtained in terms of the action density N and the nonlinear transfer function $T_{1,2,3,4}$. The result is

$$\kappa_{40} = \frac{\langle \eta^4 \rangle}{m_0^2} - 3 \tag{12}$$

$$= \mu_4 - 3 \tag{13}$$

$$= \frac{12}{g^2 m_0^2} \int d\mathbf{k}_{1,2,3,4}\, T_{1,2,3,4} \sqrt{\omega_1 \omega_2 \omega_3 \omega_4}\, \delta_{1+2-3-4} \times R_r(\Delta\omega, t) N_1 N_2 N_3, \tag{14}$$

where κ_{40} is the fourth-order cumulant of the surface elevation η and is equivalent to $\mu_4 - 3$, where μ_4 is the normalized fourth-order moment, kurtosis of the surface elevation. The transfer function $R_r = [1 - \cos(\Delta\omega t)]/\Delta\omega \rightarrow \mathcal{P}/\Delta\omega$ for large time t, where $\Delta\omega = \omega_1 + \omega_2 - \omega_3 - \omega_4$ and $\mathcal{P}$ denotes the principal value of the integral to avoid singularity in the integral.

For the cross correlation between η and ζ, using Eq. (10) it immediately follows that for homogeneous ocean waves indeed there is no correlation between η and ζ. Then,

$$\langle \eta\zeta \rangle = -i\left\langle \int_{-\infty}^{\infty} d\mathbf{k}_{1,2}\, (a_1 e^{i\mathbf{k}_1.\mathbf{x}} + \text{c.c})(a_2 e^{i\mathbf{k}_2.\mathbf{x}} - \text{c.c}) \right\rangle. \tag{15}$$

Using the second equation of Eq. (4), this becomes

$$\langle \eta\zeta \rangle = -i \int_{-\infty}^{\infty} d\mathbf{k}_{1,2} [-\langle a_1 a_2^* \rangle e^{i(\mathbf{k}_1 - \mathbf{k}_2).\mathbf{x}} + \langle a_2 a_1^* \rangle e^{i(\mathbf{k}_2 - \mathbf{k}_1).\mathbf{x}}] = 0!, \tag{16}$$

which vanishes because of the first equation in Eq. (4). The second cumulant we need is termed κ_{22} and is defined as

$$\kappa_{22} = \frac{\langle \eta^2 \zeta^2 \rangle}{m_0^2} - 1, \tag{17}$$

where it is noted that $\langle \zeta^2 \rangle = \langle \eta^2 \rangle$. Evaluating the rhs of Eq. (17) using the definitions of η and ζ, one finds

$$\kappa_{22} = \frac{1}{3} \kappa_{40}; \tag{18}$$

hence κ_{22} is, as expected, precisely ⅓ as large as κ_{40}.

b. Statistics of a narrowband wave train

In practical applications the frequency spectrum is widely used rather than the action density. We define the wavenumber spectrum as

$$F(\mathbf{k}) = \frac{\omega}{g} N(\mathbf{k}) \tag{19}$$

and the frequency spectrum as

$$E(\omega, \theta) d\omega d\theta = F(\mathbf{k}) d\mathbf{k}. \tag{20}$$

Then, from Eq. (14), we obtain for long time the following relationship between κ_{40} and the frequency spectrum:

$$\kappa_{40} = \frac{12g}{m_0^2} \mathcal{P} \int d\theta_{1,2,3}\, d\omega_{1,2,3}\, T_{1,2,3,4} \sqrt{\frac{\omega_4}{\omega_1 \omega_2 \omega_3}} \frac{E_1 E_2 E_3}{\Delta\omega}, \tag{21}$$

where $\omega_4 = \sqrt{g|\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3|}$. Equation (21) is valid for arbitrary two-dimensional frequency spectra. However, for operational purposes the evaluation of a six-dimensional integral is far too time consuming and in order to make progress we will make the simplifying assumption of the so-called narrowband approximation. This means that we concentrate on almost unidirectional waves that have a sharply peaked frequency spectrum.

In the narrowband approximation the spectrum is mainly concentrated at $\omega = \omega_0$ and $\theta = \theta_0$ and falls off rapidly, much faster than the other terms in the integrand of Eq. (21). In that event, we can approximate the transfer coefficient $T_{1,2,3,4}$ by its narrowband value k_0^3. In addition, the angular frequency ω_4 becomes independent of θ:

$$\omega_4 = (|\omega_1^2 + \omega_2^2 - \omega_3^2|)^{1/2}. \tag{22}$$

In fact, apart from the spectra there is no θ dependence. This therefore gives a considerable simplification. We introduce the one-dimensional frequency spectrum

$$E(\omega) = \int d\theta\, E(\omega, \theta); \tag{23}$$

then

$$\kappa_{40} = \frac{12 g k_0^3}{m_0^2} \mathcal{P} \int d\omega_1\, d\omega_2\, d\omega_3 \times \sqrt{\frac{\omega_4}{\omega_1 \omega_2 \omega_3}} \frac{E(\omega_1) E(\omega_2) E(\omega_3)}{\Delta\omega}, \tag{24}$$

so in the narrowband approximation only the evaluation of a three-dimensional integral is required. This is operationally feasible, but in practice, the resolution of the frequency spectrum is too coarse to give an accurate evaluation of the singular integral.

A further simplification may be achieved as follows. Approximate the one-dimensional spectrum by a Gaussian function:

$$E(\omega) = \frac{m_0}{\sigma_\omega \sqrt{2\pi}} e^{-(1/2)\nu^2}, \tag{25}$$

with

$$\nu = \frac{\omega - \omega_0}{\sigma_\omega}, \tag{26}$$

and where m_0 is the surface elevation variance. Clearly, for small bandwidth there is a small parameter; namely,

$$\Delta = \frac{\sigma_\omega}{\omega_0}, \tag{27}$$

and in Eq. (24) all relevant frequencies and so on are expanded in terms of Δ. The resulting expression for the kurtosis becomes

$$\kappa_{40} = \frac{24\epsilon^2}{\Delta^2} \mathcal{P} \int \frac{d\nu_1\, d\nu_2\, d\nu_3}{(2\pi)^{3/2}} \frac{e^{-(1/2)(\nu_1^2 + \nu_2^2 + \nu_3^2)}}{(\nu_1 + \nu_2 - \nu_3)^2 - \nu_1^2 - \nu_2^2 + \nu_3^2}, \tag{28}$$

where we have introduced the steepness parameter $\epsilon = k_0 \sqrt{m_0}$. Equation (28) shows the important result that the kurtosis depends on the ratio of two small parameters, namely the integral steepness of the waves and the relative width of the frequency spectrum. The wave steepness reflects, of course, the importance of nonlinearity while the relative width represents the importance of dispersion (but in a spectral sense). The work of Benjamin and Feir (1967) and Alber and Saffman (1978) has shown that these parameters play a key role in the evolution of deep-water gravity waves. Nonlinearity counteracts dispersion in such a way that focusing of wave energy may occur, resulting in extreme wave events and as a consequence in large deviations from the normal distribution of the surface elevation. To measure the relative importance of nonlinearity and dispersion, Janssen (2003) introduced the Benjamin–Feir index (BF index: BFI), defined as

$$\mathrm{BFI} = \frac{\epsilon}{\Delta}\sqrt{2}. \tag{29}$$

The $\sqrt{2}$ factor is included for historical reasons as according to Alber and Saffman (1978) a random wave train becomes unstable if BFI > 1.

In the appendix we have evaluated the three-dimensional integral exactly, and introducing the BF index one finds that the kurtosis depends on the square of BFI:

$$\kappa_{40} = \frac{\pi}{\sqrt{3}}\mathrm{BFI}^2. \tag{30}$$

Note that for a narrowband, weakly nonlinear wave train the BF index is formally of $O(1)$. Therefore, the resonant and nonresonant interactions give rise to a much larger contribution to the kurtosis than the bound waves, as the latter contribution is only of the order of the square of the steepness. This follows immediately from the well-known expression for the surface elevation of a steady, narrowband wave train, which is correct to third order in amplitude:

$$\eta = \alpha\left(1 + \frac{1}{8}k_0^2\alpha^2\right)\cos\theta + \frac{1}{2}k_0\alpha^2\cos 2\theta + \frac{3}{8}k_0^2\alpha^3\cos 3\theta, \tag{31}$$

where α is connected to the free wave normal variable b through $\alpha = b\sqrt{\omega/2g}$. Since in lowest order b, and hence α, obeys a linear equation, it is justified to assume that the first-order wave train $\eta = \alpha\cos\theta$ obeys the Gaussian statistics. Hence, θ is uniformly distributed and α obeys a Rayleigh distribution with width $m_0^{1/2}$ (Longuet-Higgins 1957; Srokosz and Longuet-Higgins 1986), and the statistical properties of a narrowband wave train may now readily be obtained. The Stokes wave model [Eq. (31)] predicts wave moments of the form

$$E(\eta^n) = \int_0^\infty \int_0^{2\pi} d\alpha\, d\theta\, \eta^n(\alpha, \theta)F(\alpha, \theta), \tag{32}$$

where $F(\alpha, \theta)$ is the joint probability density of α and θ. For example, in lowest significant order, the skewness becomes

$$\mu_3 = \frac{\langle\eta^3\rangle}{\langle\eta^2\rangle^{3/2}} = 3\epsilon, \tag{33}$$

where $\epsilon = k_0 m_0^{1/2}$ is a measure of the spectral steepness, while the kurtosis becomes

$$\kappa_{40} = 24\epsilon^2. \tag{34}$$

Similar results were obtained by Vinje (1989). Comparing Eq. (34) with Eq. (30) it is evident that for a narrowband wave train the contribution of the (non)resonant waves dominates the one from the bound waves when the relative width Δ satisfies the inequality $\Delta^2 < \pi/12\sqrt{3} \simeq 0.15$. In practice this condition is easily achieved.

3. A simple non-Gaussian wave height distribution for a nonlinear random wave field

a. *Wave height distribution*

1) MATHEMATICAL FORMULATION

In the previous section we have obtained for homogeneously random waves a clear relationship between spectral shape and kurtosis through the resonant and nonresonant four-wave interactions of the Zakharov equation. Following a central limit theorem, linear, dispersive random waves have a Gaussian pdf for the surface elevation. Finite amplitude effects result, however, in deviations from the normal distribution, as measured by a finite skewness and kurtosis. Forristall (2000) investigated the influence of second-order nonlinearity on wave crest distributions. However, for narrowband wave trains it will be shown that the wave height distribution only depends on the kurtosis. Therefore, we shall formulate the relationship between wave height distribution and kurtosis to examine analytically the effects of kurtosis on freak wave occurrence.

We assume that waves to be analyzed here are unidirectional with narrowbanded spectra and satisfy the stationary and ergodic hypothesis. Let $\eta(t)$ be the sea surface elevation as a function of time t and $\zeta(t)$ be an auxiliary variable such that $\eta(t)$ and $\zeta(t)$ are not correlated. Assuming both $\eta(t)$ and $\zeta(t)$ are real zero-mean functions with variance σ, we have

$$Z(t) = \eta(t) + i\zeta(t) = A(t)e^{i\phi(t)}, \tag{35}$$

$$A(t) = \sqrt{\eta^2(t) + \zeta^2(t)}, \quad \text{and} \tag{36}$$

$$\phi(t) = \tan^{-1}\left[\frac{\zeta(t)}{\eta(t)}\right], \tag{37}$$

where A is the envelope of the wave train and ϕ is the phase. Mori and Yasuda (2002) investigated the wave height distribution as a function of kurtosis and skewness using the joint probability density function of $\eta(t)$ and $\zeta(t)$ for a narrowbanded weakly nonlinear wave train. We will follow this approach closely. For weakly nonlinear waves deviations from the normal distribution are small. In those circumstances the pdf of the surface elevation can be described by the Edgeworth distribution. As there is no correlation between $\eta(t)$ and $\zeta(t)$, the joint probability density function of $\eta(t)$ and $\zeta(t)$ becomes

$$p(\eta,\zeta) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(\eta^2+\zeta^2)\right]\left[1+\frac{1}{3!}\sum_{n=0}^{3}\frac{3!}{(3-n)!n!}\kappa_{(3-n)n}H_{3-n}(\eta)H_n(\zeta)+\frac{1}{4!}\sum_{n=0}^{4}\frac{4!}{(4-n)!n!}\kappa_{(4-n)n}H_{4-n}(\eta)H_n(\zeta)\right], \tag{38}$$

where H_n is the nth-order Hermite polynomial:

$$H_n(x) = (-1)^n \exp\left(+\frac{1}{2}x^2\right)\frac{d^n}{dx^n}\exp\left(-\frac{1}{2}x^2\right). \tag{39}$$

All variables will be normalized by the variance of the surface elevation $\sigma = m_0^{1/2}$ (where m_0 is the zeroth moment of the wave spectrum) and have zero mean.

In keeping Eqs. (9) and (10) in mind, the pdf of the envelope A now follows immediately from an integration of the joint probability distribution:

$$p(A,\phi) = Ap(\eta,\zeta) \tag{40}$$

over ϕ; hence,

$$p(A) = \int_0^{2\pi} d\phi\, p(A,\phi). \tag{41}$$

Performing the integration over ϕ it is found that the first term of Eq. (38) gives the usual Rayleigh distribution $A\exp(-A^2/2)$, while the terms involving the skewness κ_{30}, and so on all integrate to zero because they are odd functions of ϕ. The third term does give contributions and as a result we find

$$p(A) = Ae^{-(1/2)A^2}\left[1+\frac{1}{4}(\kappa_{40}+\kappa_{22})\left(1-A^2+\frac{1}{8}A^4\right)\right], \tag{42}$$

where we have used $\kappa_{40} = \kappa_{04}$, a relation that can easily be verified. Last, using $\kappa_{22} = \kappa_{40}/3$ the final result for the narrowband approximation of the pdf of the envelope becomes

$$p(A) = Ae^{-(1/2)A^2}\left[1+\frac{1}{3}\kappa_{40}\left(1-A^2+\frac{1}{8}A^4\right)\right]. \tag{43}$$

It is emphasized that, as expected, the pdf for the envelope does not contain contributions that are linear in the skewness. However, as follows from Eqs. (33) and (34), the skewness is, relative to the kurtosis, a large quantity. Quadratic terms in skewness could give an equally important contribution to the pdf of A as the kurtosis terms. This was pointed out by Mori and Yasuda (2002) and it required the extension of the Edgeworth distribution to sixth order. Nevertheless, this additional term, proportional to μ_3^2, gives for narrowband wave trains only a small contribution to the pdf for the same reason as the bound wave contribution to the kurtosis may be neglected.

From the results of Eq. (43) interesting consequences for the distribution of maximum wave heights may be derived. In the narrowband approximation wave height H is equal to $2A$ and hence the wave height pdf becomes

$$p(H) = \frac{1}{4}He^{-(1/8)H^2}[1+\kappa_{40}A_H(H)], \tag{44}$$

where

$$A_H(H) = \frac{1}{384}(H^4 - 32H^2 + 128). \tag{45}$$

The exceedance probability $P_H(H)$ for wave height then follows from an integration of Eq. (45) from H to ∞:

$$P_H(H) = e^{-(1/8)H^2}[1+\kappa_{40}B_H(H)], \tag{46}$$

where

$$B_H(H) = \frac{1}{384}H^2(H^2-16). \tag{47}$$

2) Comparison of the theory with laboratory data

The validity of the theory is examined by means of a comparison with experimental data. The laboratory experiment was conducted in a glass channel that was 65 m long, 1 m wide, 2 m high, and was filled to a depth of about 1.0 m. Water surface displacements were measured with 12 capacitance type wave gauges. Measurements with a sampling frequency of 32 Hz were performed for over a period of 330 s. The number of waves per wave train was about 350–450. The initial spectra were given by Wallops-type spectra with bandwidths of m = 5, 10, 30, 60, and 100, and peak frequency of f_p = 1 Hz and a dimensionless water depth of k_ph = 3.99

(with h the water depth), so that the waves were deep-water waves. Initial phases of the waves were given by uniformly distributed random numbers. The details of the experimental setup and conditions are given in Mori and Yasuda (2002).

The comparison of exceedance probability of wave heights is shown in Fig. 1. The filled circles (●) denote experimental data, the Rayleigh distribution is represented by the dotted line, Eq. (46) corresponds to the solid line, and the wave height distribution including skewness effects proposed by Mori and Yasuda (2002) (denoted as ER, Edgeworth–Rayleigh, in the figure) corresponds to the dashed line. For simplicity we refer to Eq. (46) as modified ER (MER) hereinafter. Because of the nonlinear effects, the exceedance probability obtained from the laboratory data departs for large wave height from the Rayleigh distribution. Both the MER and ER distributions for the exceedance probability of wave heights follow this separation in large-amplitude regions. Surprisingly, the MER distribution shows better agreement with the laboratory data than the ER distribution, although the corrections to the Rayleigh distribution only stem from the effects of finite kurtosis. This same conclusion holds for larger values of the kurtosis (Fig. 1b). Thus, Eq. (46) can be regarded as a theory capable of predicting the height distribution for large-amplitude waves in a narrowband, weakly nonlinear wave field.

It is noted that the MER distribution has a much simpler form than the ER distribution (Mori and Yasuda 2002), as only the effects of finite kurtosis are retained. As argued before, for a narrowband wave train there is no need to include the effects of bound waves on the wave height distribution; hence, the effects of skewness can be discarded. Also, Mori and Yasuda (2002) thought that the cross-correlation term κ_{22} was small, while it, in fact, makes a considerable contribution. This last point explains why the MER gives better agreement with data than does the ER.

b. *Maximum wave height distribution and freak wave occurrence*

1) MATHEMATICAL FORMULATION

There are several possibilities to categorize a freak wave. We use the most simple freak wave definition, which defines a freak wave as one having a maximum wave height H_{max} exceeding 2 times the significant wave height $H_{1/3}$ of the wave train. Hence, in the context of the above freak wave definition, the pdf of the maximum wave height is necessary.

The pdf of the maximum wave height p_m in wave trains can be obtained once the pdf of the wave height

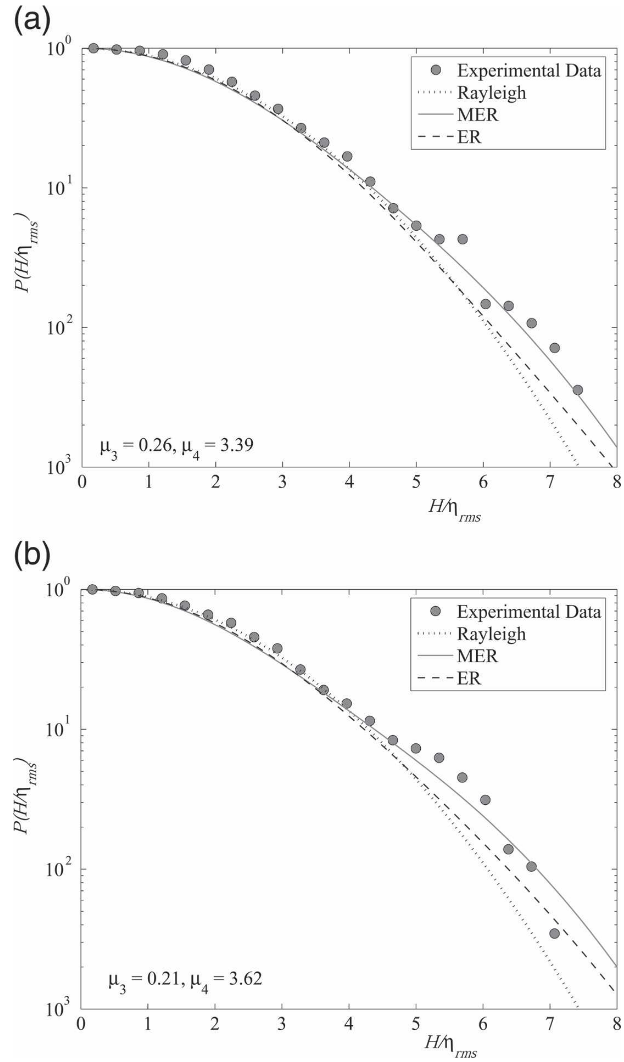

FIG. 1. Comparison of wave height distribution from laboratory data and theory [filled circle (●), laboratory data; solid line, Eq. (46); dashed line, Mori and Yasuda (2002); dotted line, Rayleigh distribution]: (a) $\mu_3 = 0.26$, $\mu_4 = 3.39$ and (b) $\mu_3 = 0.21$, $\mu_4 = 3.62$.

$p(H)$ and the exceedance probability of the wave height $P(H)$ are known (Goda 2000); thus,

$$p_m(H_{\max})dH_{\max} = N[1 - P(H_{\max})]^{N-1}p(H_{\max})dH_{\max}, \tag{48}$$

with N being the number of waves. For sufficiently large N one may use the approximation

$$\overset{N}{\underset{N\to\infty}{\lim}}[1 - P(H_{\max})]^N \simeq \overset{N}{\underset{N\to\infty}{\lim}} \exp[-NP(H_{\max})]. \tag{49}$$

Substituting Eq. (46) into Eq. (48) gives the pdf of the maximum wave height, p_m,

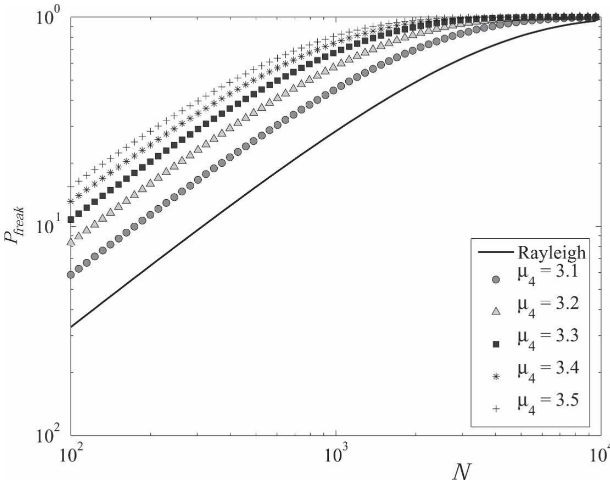

FIG. 2. Occurrence probability of freak wave as a function of the number of waves N and kurtosis μ_4.

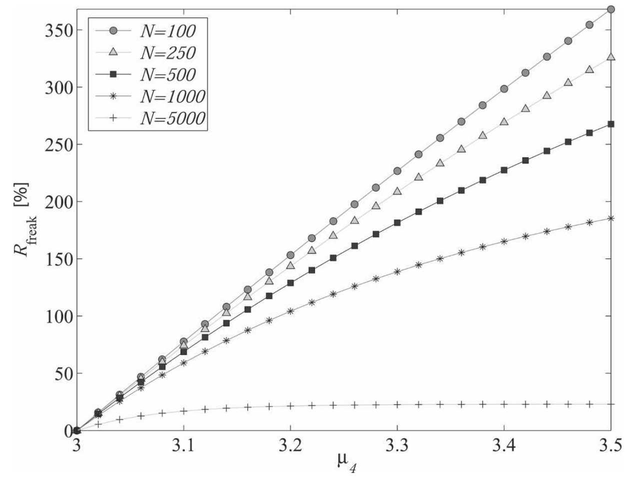

FIG. 3. Ratio of freak wave occurrence predicted by Eq. (51) to the Rayleigh theory.

$$P_m(H_{\max})dH_{\max} = \frac{N}{4} H_{\max} e^{-(H_{\max}^2/8)}[1 + \kappa_{40}A_H(H_{\max})] \times \exp\{Ne^{-(H_{\max}^2/8)} \times [1 + \kappa_{40}B_H(H_{\max})]\}dH_{\max}, \quad (50)$$

and the exceedance probability of maximum wave height P_m,

$$P_m(H_{\max}) = 1 - \exp\{-Ne^{-(H_{\max}^2/8)}[1 + \kappa_{40}B_H(H_{\max})]\}. \quad (51)$$

Equations (50) and (51) are evaluated as a function of N and κ_{40} (or μ_4). For $\kappa_{40} = 0$, the results are identical to the ones following from the Rayleigh distribution. For simplicity it will be assumed that $H_{1/3} = 4m_0^{1/2}$, although it is $H_{1/3} = 4.004m_0^{1/2}$ in an exact linear random wave theory. The freak wave condition in this study therefore becomes $H_{\max}/m_0^{1/2} > 8$, and we obtain from Eq. (51) the following simple formula to predict the occurrence probability of a freak wave as a function of N and κ_{40}:

$$P_{\text{freak}} = 1 - \exp[-\beta N(1 + 8\kappa_{40})], \quad (52)$$

where $\beta = e^{-8}$ is constant.

Using Eq. (52) it is seen that the effect of kurtosis already becomes of the same order as linear theory for $\kappa_{40} = 1/8$. This corresponds to $\mu_4 = 3.125$, and is not a strong nonlinear condition. Hence, both the effects of finite kurtosis and the number of waves N are important for determining the probability of maximum wave height in the nonlinear wave train.

Figure 2 shows for μ_4 increasing from 3.0 to 3.5 the comparison between linear (Rayleigh) theory and the present theory of the occurrence probability of a freak wave, P_{freak}, as a function of the number of waves N. For the case of $N = 100$, the occurrence probability of a freak wave predicted by linear theory is 3.3%, while it is 15.4% according to Eq. (51) with $\mu_4 = 3.5$, and for the case of $N = 1000$, the occurrence probability of the freak wave is 28.5% according to linear theory, while it is 81.3% according to Eq. (51) with $\mu_4 = 3.5$. The number of waves $N = 1000$ corresponds to a duration of about 3 h for the case of $T_{1/3} = 10$ s, which is not an unusual situation in stormy conditions. Alternatively, defining the threshold value of the occurrence probability of a freak wave as 50%, the expected number of waves that include at least one freak wave as a maximum wave is 2000 when predicted by linear theory, and becomes 500 when predicted by Eq. (51) with $\mu_4 = 3.5$. Thus, in a strong nonlinear field freak waves can occur several times more frequently than in a linear wave field.

Figure 3 shows the ratio R_{freak} of freak wave occurrence probability predicted by the present approach and Rayleigh theory as a function of kurtosis μ_4:

$$R_{\text{nonlinear}} = \frac{P_{\text{freak}}}{P_{\text{freak}|\kappa_{40}=0}} - 1. \quad (53)$$

For the case of a small number of waves, $N \leq 250$, the ratio R_{freak} depends linearly on μ_4. If μ_4 is 3.1 and $N \leq 500$, the occurrence probability of freak waves is 50% more than according to linear theory. On the other hand, the increment of R_{freak} decreases as the number of waves increases. This is because for a very large

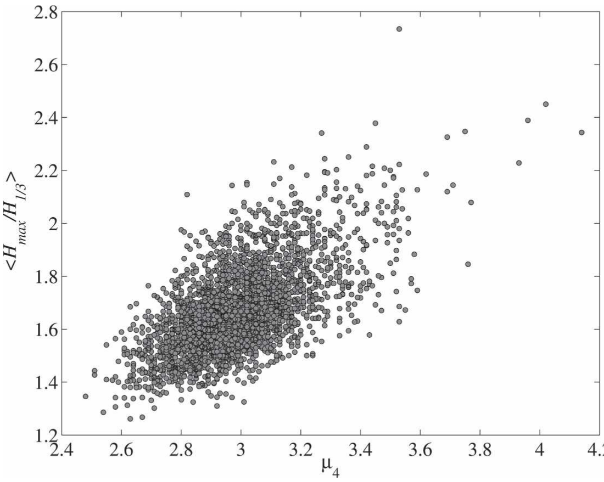

FIG. 4. Relationship between $H_{\max}/H_{1/3}$ and μ_4.

number of waves even in linear theory the maximum wave height almost always exceeds 2 times $H_{1/3}$.

2) COMPARISON OF THE THEORY WITH FIELD DATA

It is very difficult to check the theory developed in this paper. The main difficulty is that the probability of maximum wave height depends on both μ_4 and N, which means that a huge amount of data, including spectral information such as the BF index, and statistical parameters such as μ_4, and so on, are required to verify the theory. Unfortunately, from the present *operational* observations not all of these parameters have been obtained and archived. Therefore, we will only try to check the dependence of the maximum wave on μ_4 qualitatively using the available field dataset.

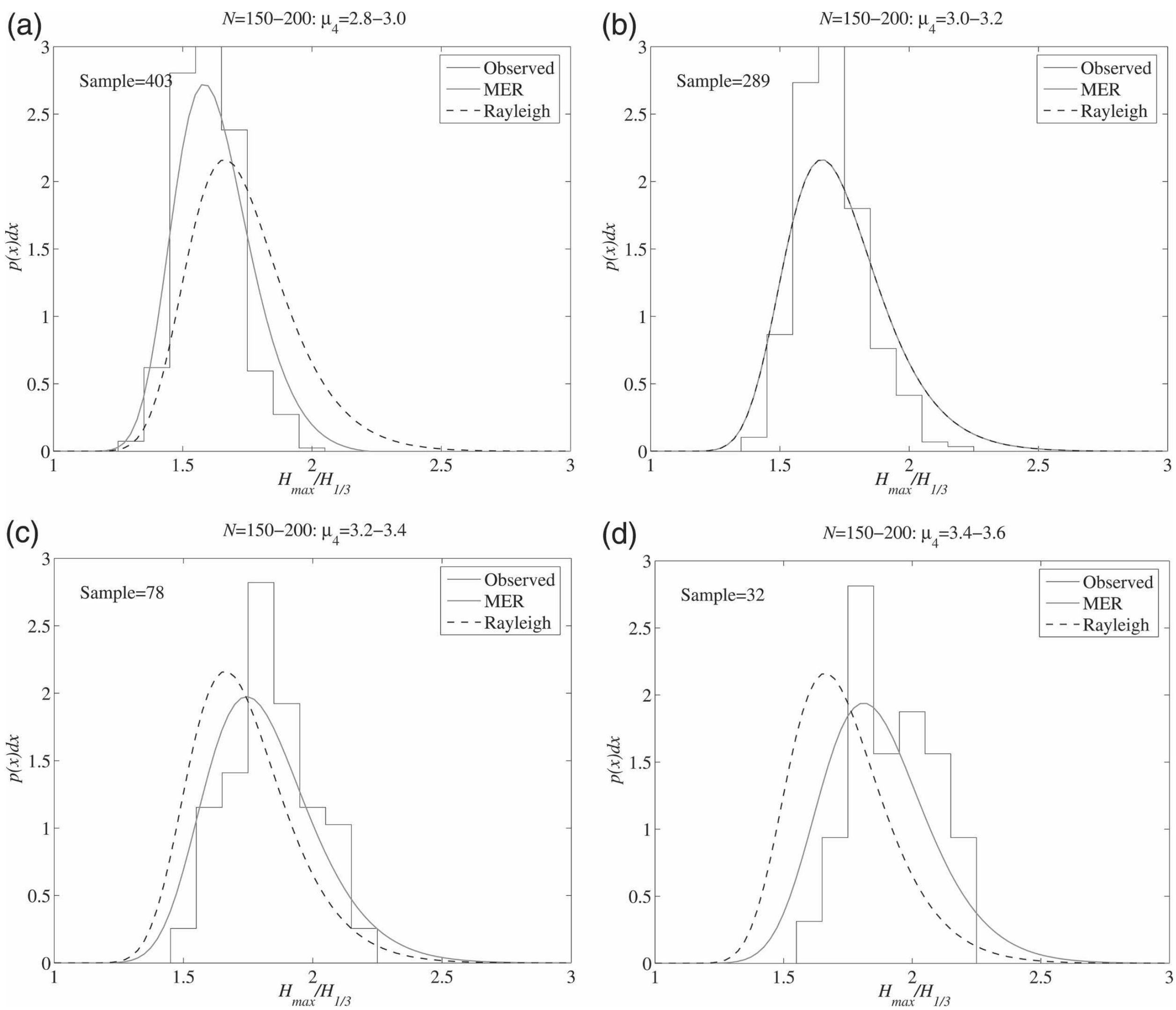

FIG. 5. Comparison of the maximum wave height distribution with observed data and theory (N = 150−200): (a) μ_4 = 2.8−3.0, (b) μ_4 = 3.03.2, (c) μ_4 = 3.2−3.4, and (d) μ_4 = 3.4−3.6.

The observed data were originally collected by the Tokyo Electric Power Company using an ultrasonic wave gauge at a depth of 30 m, off the coast of the Pacific Ocean. The length of each record was 20 min and the data were collected every hour from 1 March to the end of June in 2001. The wave statistics such as $H_{\max}$, $H_{1/3}$, $T_{1/3}$, N, μ_3, and μ_4 were operationally calculated and archived. Note that the water depth of 30 m is relatively shallow. Therefore, to eliminate shallow water effects,[1] the data are excluded if the dimensionless water depth $k_p h$ is less than 2.0 (it corresponds to $T_{1/3} \geq 8$ s). The total number of valid data points was about 2546. Figure 4 shows the direct comparison between $H_{\max}/H_{1/3}$ and μ_4. The linear correlation between $H_{\max}/H_{1/3}$ and μ_4 is only 0.73. However, it is well known that $H_{\max}/H_{1/3}$ not only depends on μ_4 but also on N. Thus, the data are stratified according to μ_4 and N and are compared with the theory. Figure 5 shows the direct comparison of the maximum wave height distribution between the observed data and theory for the N = 150–200 bin. The histogram shows the observed pdf of the maximum wave height, while the solid line and the dashed line indicate Eq. (50) and Rayleigh theory, respectively. The number of wave records in each category is indicated by the "sample" number. For a fixed number of waves, the maximum wave height distribution according to Rayleigh theory is constant, although the observed data show a clear dependence of the pdf on μ_4. The peak of the observed pdf is lower than that for Rayleigh theory for $\mu_4 < 3$ but becomes higher than for Rayleigh theory for $\mu_4 > 3$. The maximum wave height distribution predicted by Eq. (50) qualitatively agrees with the observed data, although it slightly underestimates it.

Next, we discuss the general behavior of the pdf of maximum wave height in the nonlinear wave field, by showing the ensemble-averaged $H_{\max}/H_{1/3}$ of each bin as a function of μ_4 and N in Fig. 6. The brackets $\langle\rangle$ indicate the ensemble-averaged value. Figure 6a is the observed data and Fig. 6b is the expected value of Eq. (50) through numerical integration. The dependence of $H_{\max}/H_{1/3}$ on N is weaker than expected from Eq. (50). This is because the length of the observed time series was fixed at 20 min, so we cannot discuss the dependence of $H_{\max}/H_{1/3}$ on the number of waves in detail. On the other hand, the dependence of$(H_{\max}/H_{1/3})$ on μ_4 is clear. The theoretically predicted $(H_{\max}/H_{1/3})$ is underestimated relative to the observed data but it agrees with the observed data in a qualitative sense. The observed $(H_{\max}/H_{1/3})$ monotonically increases for increasing μ_4, but for high values of kurtosis the theoretically estimated value of $(H_{\max}/H_{1/3})$ is lower.

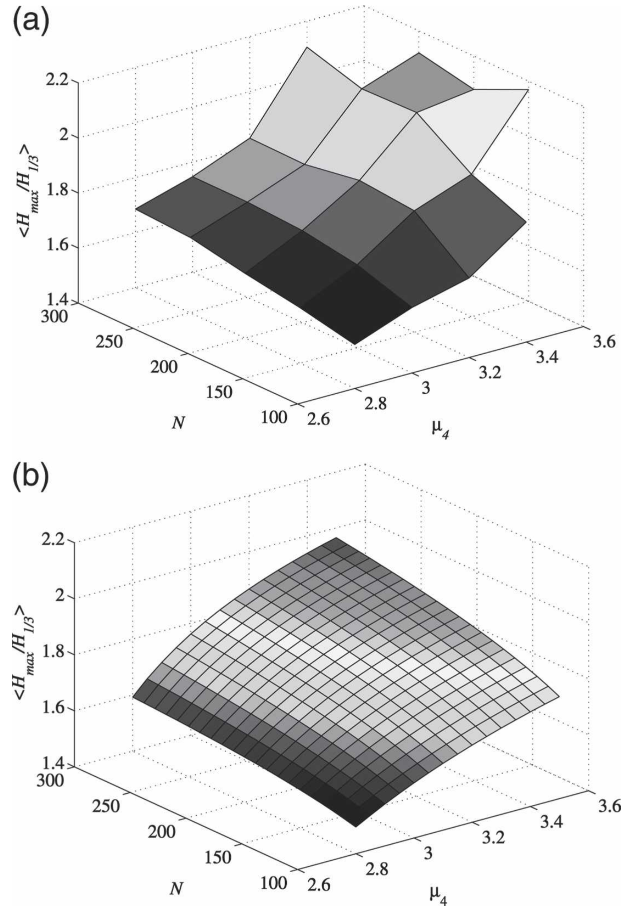

FIG. 6. Dependence of $\langle H_{\max}/H_{1/3}\rangle$ on μ_4 and N for (a) the observed data and (b) theory.

Figure 7 shows the comparison between observed data and the theory of freak wave occurrence frequency, P_{freak}. To eliminate statistical fluctuations, the observed data are excluded if the number of samples is less than 20. The observed P_{freak} clearly increases as μ_4 is increased. However, there is no clear dependence of P_{freak} on N where, according to the theory, there should be. The total number of wave trains is 2546 but this is still not sufficient to examine the validity of the theory completely. Hence, more data will be required to verify the theory quantitatively.

4. Conclusions

For a narrowband, random wave train we have shown that the kurtosis of the surface elevation is mainly determined by resonant and nonresonant wave–

[1] As remarked upon in section 2a, in the context of the nonlinear Schrödinger equation, there is only nonlinear focusing for $k_p h > 1.36$.

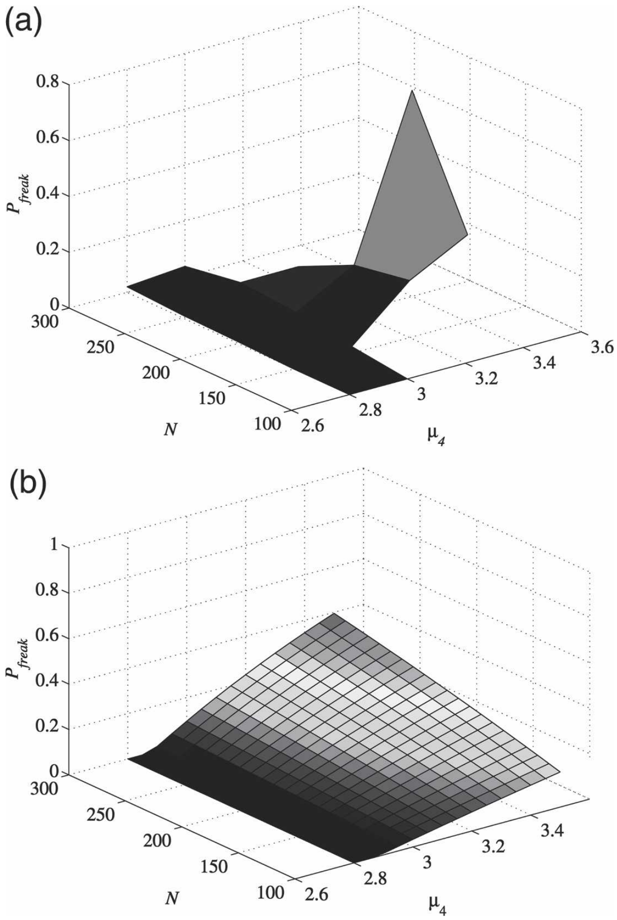

FIG. 7. Dependence of occurrence probability of a freak wave on μ_4 and N for (a) the observed data and (b) theory.

wave interactions, while bound waves make only a small contribution. Thus, the kurtosis and related high-order cumulants can be evaluated on the basis of Janssen's (2003) work. Second, we have shown that for a narrowband wave train the wave height and the maximum wave height probability distribution depend to a good approximation on the wave variance and the kurtosis. As a consequence, it is possible to formulate the freak wave occurrence probability in terms of the kurtosis and the number of waves in a time series. From the comparison with laboratory and field data, we conclude the following.

- The second-order cross-cumulant κ_{22} is ⅓ of the fourth cumulant, κ_{40}, of the surface elevation.
- The weakly non-Gaussian theory shows the dependence of the expected maximum wave height on kurtosis, which is supported by the observed data.
- The occurrence probability of freak waves is significantly enhanced by the kurtosis increase caused by four-wave interactions.

To check the validity of the approach developed here, in particular the dependence of freak wave occurrence on the kurtosis and the number of waves, systematic and continuous field measurements of freak waves including wave spectra and nonlinear statistics will be critically required.

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APPENDIX

Evaluation of the Singular Integral J

Starting from Eq. (28), we simplify the denominator in the integral and we introduce the BF index. This gives for the kurtosis

$$\kappa_{40} = 6J \times \mathrm{BFI}^2,$$

where

$$J = \mathcal{P}\int_{-\infty}^{\infty} \frac{dv_1\, dv_2\, dv_3}{(2\pi)^{3/2}} \frac{e^{-(1/2)(v_1^2+v_2^2+v_3^2)}}{(v_3 - v_1)(v_3 - v_2)}. \quad \text{(A1)}$$

Let us determine this integral, which, as will be seen, is not a trivial exercise. We introduce $t_i = v_i/\sqrt{2}$; hence,

$$J = \frac{1}{2\pi^{3/2}} \mathcal{P}\int_{-\infty}^{\infty} dt_1\, dt_2\, dt_3 \frac{e^{-(t_1^2+t_2^2+t_3^2)}}{(t_1 - t_3)(t_2 - t_3)}. \quad \text{(A2)}$$

We perform the integration over t_1 and t_2 first; then

$$J = \frac{1}{2\sqrt{\pi}} \mathcal{P}\int_{-\infty}^{\infty} dt\, Z^2(t) e^{-t^2}, \quad \text{(A3)}$$

where

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \mathcal{P}\int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t - \zeta} \quad \text{(A4)}$$

is also called the plasma dispersion function. It is customary to express the function Z in terms of the error function with a complex argument. However, normally the residual at $t = \zeta$ is included. It is omitted here because only the principal value is required. Hence,

$$Z(\zeta) = i\sqrt{\pi} e^{-\zeta^2} \Phi(i\zeta), \quad \text{(A5)}$$

where

$$\Phi(i\zeta) = \frac{2}{\sqrt{\pi}} \int_0^{i\zeta} dt\, e^{-t^2} \tag{A6}$$

is the error function. (Note that in the usual plasma dispersion function an integration from $-\infty$ to 0 is added.) Elimination of Z thus gives for J

$$J = -\frac{\sqrt{\pi}}{2} \mathcal{P} \int_{-\infty}^{\infty} dt\, e^{-3t^2} \Phi^2(it). \tag{A7}$$

We now evaluate $\Phi^2(it)$:

$$\Phi^2(it) = \frac{4}{\pi} \int_0^{it} dx \int_0^{it} dy\, e^{-(x^2+y^2)}. \tag{A8}$$

We can perform one integration by introducing polar coordinates,

$$x = i\rho\cos\theta \quad \text{and} \quad y = i\rho\sin\theta, \tag{A9}$$

and the result becomes

$$\Phi^2(it) = 1 - \frac{4}{\pi} \int_0^{\pi/4} d\theta\, e^{t^2/\cos^2\theta}. \tag{A10}$$

Therefore,

$$J = \frac{2}{\sqrt{\pi}} \int_0^{\pi/4} d\theta \int_{-\infty}^{\infty} dt\, e^{-t^2(3-1/\cos^2\theta)} - \frac{\pi}{2\sqrt{3}}. \tag{A11}$$

Integration over t now gives

$$J = -\frac{\pi}{2\sqrt{3}} + 2\int_0^{\pi/4} \frac{\cos\theta}{\sqrt{3\cos^2\theta - 1}}. \tag{A12}$$

The remaining integral can be evaluated by means of the transformation $\sin\theta = z$ and equals $\pi/3\sqrt{3}$. The final result for J is

$$J = \frac{\pi}{6\sqrt{3}}. \tag{A13}$$

REFERENCES

Alber, I., 1978: The effects of randomness of the stability of two-dimensional surface wavetrains. *Proc. Roy. Soc. London,* **363A,** 525–546.

——, and P. Saffman, 1978: Stability of random nonlinear deep-water waves with finite bandwidth spectra. Tech. Rep. 31326-6035-RU-00, TRW Defense and Space System Group.

Benjamin, T., and J. Feir, 1967: The disintegration of wavetrains on deep water. Part 1. Theory. *J. Fluid Mech.,* **27,** 417–430.

Caponi, E., P. Saffman, and H. Yuen, 1982: Instability and confined chaos in a nonlinear dispersive wave system. *Phys. Fluids,* **25,** 2159–2166.

Dean, R., 1990: Freak waves: A possible explanation. *Water Wave Kinematics,* A Tørum and O. Gudmestad, Eds., Kluwer Academic, 609–612.

Dysthe, K., 1979: Note on modification to the nonlinear Schrödinger equation for application to deep water waves. *Proc. Roy. Soc. London,* **369A,** 105–114.

Forristall, G., 2000: Wave crest distributions: Observation and second-order theory. *J. Phys. Oceanogr.,* **30,** 1931–1942.

Goda, Y., 2000: *Random Seas and Design of Maritime Structures.* 2d ed. World Scientific, 464 pp.

Hasselmann, K., 1962: On the nonlinear energy transfer in gravity-wave spectrum. I. General theory. *J. Fluid Mech.,* **12,** 481–500.

Haver, S., 2001: Evidences of the existence of freak waves. *Rogue Waves,* M. Olagnon and G. Atanassoulis, Eds., 129–140.

Janssen, P., 2003: Nonlinear four-wave interactions and freak waves. *J. Phys. Oceanogr.,* **33,** 2001–2018.

Krasitskii, V., 1990: Canonical transformation in a theory of weakly nonlinear waves with a nondecay dispersion law. *Soviet Phys. JETP,* **71,** 921–927.

Lavrenov, I., 1998: The wave energy concentration at the Agulhas current off South Africa. *Nat. Hazard,* **17,** 117–127.

Longuet-Higgins, M., 1957: The statistical analysis of a random moving surface. *Philos. Trans. Roy. Soc. London,* **249A,** 321–387.

Mori, N., 2003: Effects of wave breaking on wave statistics for deep-water random wave train. *Ocean Eng.,* **30,** 205–220.

——, and T. Yasuda, 2002: A weakly non-Gaussian model of wave height distribution for random wave train. *Ocean Eng.,* **29,** 1219–1231.

——, P. Liu, and T. Yasuda, 2002: Analysis of freak wave measurements in the Sea of Japan. *Ocean Eng.,* **29,** 1399–1414.

Olagnon, M., Ed., 2004: *Rogue Waves 2004.* IFREMER, 200 pp.

——, and G. Athanassoulis, Eds., 2000: *Rogue Waves 2000.* IFREMER, 395 pp.

Onorato, M., A. Osborne, M. Serio, and S. Bertone, 2001: Freak waves in random oceanic sea states. *Phys. Rev. Lett.,* **86,** 5831–5834.

Osborne, A., M. Onorato, M. Serio, and S. Bertone, 2000: The nonlinear dynamics of rogue waves and holes in deep water gravity wave trains. *Phys. Lett.,* **275A,** 386–393.

Srokosz, M., and M. Longuet-Higgins, 1986: On the skewness of sea-surface elevation. *J. Fluid Mech.,* **164,** 487–497.

Stansberg, C., 1990: Extreme waves in laboratory generated irregular wave trains. *Water Wave Kinematics,* A. Tørum and O. Gudmestad, Eds., Kluwer Academic, 573–590.

Stiassnie, M., and L. Shemer, 1987: Energy computations for evolution of class I and II instabilities of Stokes waves. *J. Fluid Mech.,* **174,** 299–312.

Su, M., 1982: Three-dimensional deep-water waves. Part 1. Experimental measurement of skew and symmetric patterns. *J. Fluid Mech.,* **124,** 73–108.

Trulsen, K., and K. Dysthe, 1997: Freak waves—A three-dimensional wave simulation. *Proceedings of the 21st Symposium on Naval Hydrodynamics,* National Academy Press, 550–558.

Vinje, T., 1989: The statistical distribution of wave heights in a random seaway. *Appl. Ocean Res.,* **11,** 143–152.

Whitham, G., 1974: *Linear and Nonlinear Waves.* John Wiley and Sons, 656 pp.

Yasuda, T., and N. Mori, 1994: High order nonlinear effects on deep-water random wave trains. *Int. Symp. on Waves—Physical and Numerical Modelling,* Vol. 2. Vancouver, BC, Canada, International Association of Hydraulic Engineering and Research, 823–332.

——, and ——, 1997: Roles of sideband instability and mode coupling in forming a water wave chaos. *Wave Motion,* **26,** 163–185.

——, ——, and K. Ito, 1992: Freak waves in a unidirectional wave train and their kinematics. *Proc. 23d Int. Conf. on Coastal Engineering,* Vol. 1, Venice, Italy, American Society of Civil Engineers, 751–764.

Yuen, H., and W. Ferguson, 1978: Relationship between Benjamin–Feir instability and recurrence in the nonlinear schrödinger equation. *Phys. Fluids,* **21,** 1275–1278.

——, and B. Lake, 1982: Nonlinear dynamics of deep-water gravity waves. *Adv. Appl. Mech.,* **22,** 67–229.

Zakharov, V., 1968: Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Tech. Phys.,* **9,** 190–194.