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Over-reflection of horizontally propagating gravity waves by a vertical shear layer

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The problem of horizontally propagating gravity waves impinging on a vertical shear layer which forms the boundary between uniform currents is isomorphic to an acoustic problem solved by Miles and by Ribner. The consequences of over-reflection for a spatially growing instability are pointed out. The long wave reflection problem is solved for free surface and internal waves on a thin layer in an otherwise deep fluid. Over-reflection is found possible for Froude numbers in excess of two; infinite over-reflection will occur for certain angles of impingement.

I. INTRODUCTION

Landau[1] considered the stability of a supersonic discrete shear layer between two uniform parallel flows and found that it was stable to small disturbances for Mach numbers in excess of $2^{3/2}$, based on the velocity difference. He also mentioned that experiments showed jets of such Mach numbers to be unstable. The problem of stability of a laterally limited flow, such as a jet, and the relationship between spatial and temporal growth of disturbances in such flows may in part be explained by considering the process of over-reflection of disturbances. Miles[2] and Ribner[3] considered the reflection and transmission of sound waves by a discrete compressible shear layer. The equations that apply are analogous to the equations that describe the propagation of long gravity waves on a shallow layer; surface waves in shallow water and internal gravity waves on a thin layer overlaying a deep fluid. The results of Miles and of Ribner can be directly applied to the corresponding gravity wave problem.

Recent work by Acheson[4] and Lindzen[5] demonstrates the importance of over-reflection in geophysical flows. It is therefore of interest to consider the problem of over-reflection of horizontally propagating gravity waves by a vertical shear layer, and interpret the consequences as they apply to spatial development of disturbances.

The following analysis contributes nothing new mathematically, but may help to demonstrate the consequences of over-reflection.

II. LONG WAVES ON SHALLOW LAYER

Consider long waves compared with layer depth and propagating horizontally. Let the mean layer depth be H; the layer is bounded either above or below by a rigid surface and on the other horizontal boundary by a very deep fluid layer of different, ρ. The propagation speed of small amplitude waves in a fluid at rest is then given by $C^2 = gH = g'H(\rho' - \rho)/\rho$. The density difference between the two layers is $(\rho' - \rho)$. Now consider the mean flow consisting of two uniform flows of velocity U_1 and U_2, respectively, with a shear layer at $y = 0$ separating the two flows. g is the acceleration of gravity.

Consider waves coming in from $y = -\infty$ impinging on the shear layer. With a suitable choice of coordinate system, the problem is reduced to a steady state problem. The small perturbation equations (see Ref. 6, p. 387) are then:

$$Uu_x + g'h_x = 0, \tag{1}$$

$$Uv_x + g'h_y = 0, \tag{2}$$

$$Uh_x + H(u_x + v_y) = 0, \tag{3}$$

where h is the disturbance in layer depth, u and v are horizontal velocity components, $U = U_1$ or U_2 for y negative and positive, respectively.

Consider solutions of the form

$$(u, v, h) = (\hat{u}, \hat{v}, \hat{h})\, f(x \pm By), \tag{4}$$

where $\hat{u}, \hat{v}, \hat{h}$, and $\hat{B}$ are constants, and f is an arbitrary function. Substituting, one finds

$$\hat{u} = -g'\hat{h}/U, \tag{5}$$

$$\hat{v} = \pm Bg'\hat{h}/U, \tag{6}$$

$$B^2 = 1 - \mathrm{F}^2 = 1 - U^2/(g'H), \tag{7}$$

where F is the Froude number.

III. IMPINGING, REFLECTED, AND TRANSMITTED WAVES

Take the impinging, reflected, and transmitted waves to be, respectively:

$$h_1(x, y) = Af(x - B_1 y), \tag{8}$$

$$h_r(x, y) = Rf(x + B_1 y), \tag{9}$$

$$h_t(x, y) = Tf(x - B_2 y). \tag{10}$$

The subscripts 1 and 2 refer to the two sides of the shear layer, as before, while the subscripts r and t refer to the reflected and transmitted waves, respectively. To satisfy the radiation condition, namely, that the reflected and transmitted waves carry energy away from the shear layer, the sign for B will be defined the same as the sign of U. (This simple rule was introduced by Ribner.)[3] The waves fronts are shown in Fig. 1.

IV. CONDITIONS AT INTERFACE

Let the equation for the lateral deflection of the interface be $y = \eta(x)$. The linearized boundary conditions at $y = 0$ are then that h remain continuous across the inter-

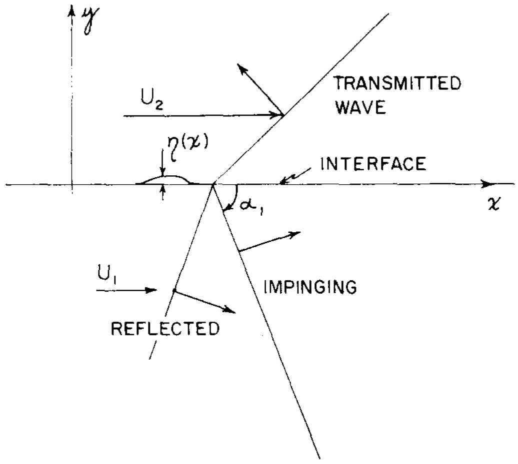

FIG. 1. Shear layer, impinging, 1, reflected, R, and transmitted, T, waves.

face and that the interface slope match the streamline slope, which requires

$$\eta_x U = v(x, 0). \tag{11}$$

Applying the last equation on both sides of the interface gives

$$1 - R = ZT,$$

where

$$Z = B_2 U_1^2/(B_1 U_2^2). \tag{12}$$

Continuity of h across the interface requires

$$1 + R = T. \tag{13}$$

Solving for R and T gives

$$R = (1 - Z)/(1 + Z), \tag{14}$$

$$T = 2/(1 + Z). \tag{15}$$

The angle between the interface and the impinging wave is

$$\alpha_1 = \sin^{-1}(1/F_1). \tag{16}$$

A can be expressed in terms of angles as

$$z = \sin 2\alpha_1 / \sin 2\alpha_2. \tag{17}$$

Note that T will be infinite if $(1 + Z)$ equals zero. This corresponds to a negative U_2 and a corresponding negative value of B_2, according to the convention adopted. When T is infinite, R will also be infinite, which states that for a finite amplitude of the impinging wave, the response will be infinite. For this case, considering Eqs. (13) and (14) as equations for R and T in terms of the unit amplitude of the impinging wave, the case of infinite values for R and T corresponds to the vanishing of the principal determinant of the two equations. This is also the condition for the existence of finite R and T for zero impinging wave amplitude, the condition for a nontrivial solution to the homogeneous problem for zero amplitude impinging wave. The condition $Z + 1 = 0$ is thus the condition for neutral stability of disturbances. Singular over-reflection corresponds to neutral stability. Ribner[3] has calculated the values of the reflection coefficient R as a function of the Froude number difference $F = F_2 - F_1$ and the angle of incidence as defined in Eq. (16).

The result is shown in Fig. 2, reproduced from Ribner with his permission and relabeled to fit the present variables. Note that for $F > 2$ over-reflection occurs for all angles of inpingement that render the flows in opposite directions when the impinging wave is steady in the coordinate system used. For $2 < F < 2^{3/2}$ singular over-reflection occurs for one angle of impingement for each value of F, while for $F > 2^{3/2}$, three impingement angles give over-reflection for each F.

V. REYNOLDS STRESSES

One way to explain why there are no unstable disturbances above a certain critical Froude number is to argue that for disturbances involving waves of finite amplitude, there will be energy and momentum radiated away from the shear layer by the disturbances, so that even a nongrowing disturbance is capable of transporting momentum away from the shear layer. The mean flow will therefore be modified when there are neutrally stable disturbances present. The energy arguments used at times to produce stability criteria therefore do not necessarily work when radiative disturbances are involved.

From Eqs. (5), and (6), and (7) one finds

$$\overline{uv} = g'^2 b \overline{h^2} / U^2 \tag{18}$$

for the Reynolds stresses. Integrating this over x gives the total momentum transport. The direction of momentum transport will be found to be outward, which seems to reduce the mean velocity difference across the shear layer, for both the reflected and transmitted waves.

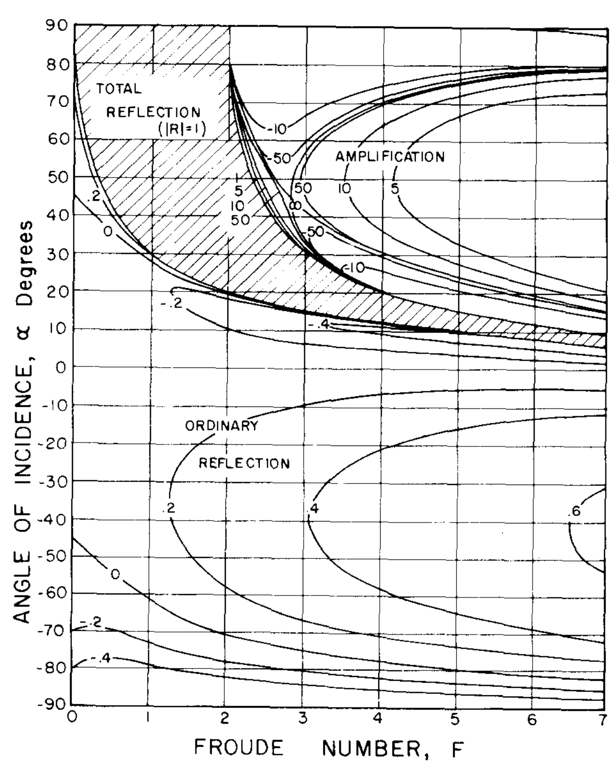

FIG. 2. Reflection coefficient R as function of impingement angle and Froude number difference, from Ribner.

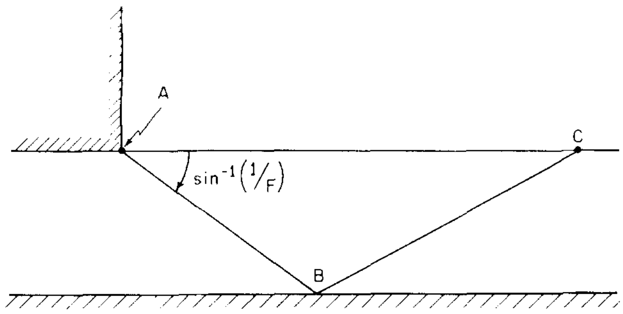

FIG. 3. Illustrating spatially growing disturbances in jet. Origin of disturbance (1), point of first impingement on opposite side (2), impingement of reflected wave (3), where a centered disturbance generated by nonlinear effects occurring at shear layer may originate and propagate upstream outside the jet to hit (1) and start the process over again.

The presence of a second shear layer or a rigid surface that reflects waves within a finite distance of the shear layer, so as to produce a series of amplified reflections and transmissions, with the disturbances growing with distance along the flow direction. This was pointed out by Mollo-Christensen[7] for the acoustic case.

VI. EFFECT OF SOLID BOUNDARY NEAR SHEAR LAYER

Consider a flow that is confined between parallel solid boundaries for negative values of x, and where the boundaries are missing for positive x, so that a free jet is formed between a region of fluid at rest and the half-jet present between the shear layer and the remaining solid boundary. The disturbances we considered before will, in the coordinate system fixed with respect to the beginning of the free shear layer, generally be unsteady. The flow is shown in Fig. 3. Consider a transient disturbance originating at the free shear layer at $x=0$. A wave will propagate away from the disturbance as a spreading circular wave front being swept downstream with the mean flow. The wave will propagate away from the disturbance as a spreading circular wave front being swept downstream with the mean flow. The wave will be reflected by the other layer and later impinge upon the shear layer. If the Froude number of the flow exceeds two, over-reflection will occur over a range of angles of impingement, and singular over-reflection will occur at either one or three angles, depending upon whether the Froude number exceeds $2^{3/2}$ or not.

The reflected wave will be reflected back toward the solid boundary from which the wave will again be reflected to impinge upon the shear layer to be over-reflected again and so on. The production of singular over-reflection obtained using linear theory will immediately make the wave amplitude too large to be treated by linear theory, but we may conclude that there may be a nearly stepwise spatial growth of disturbances with downstream distance, the length between steps being given, to first order, by the distance between successive reflections.

The next question is to ask where the initial disturbance may come from? Powell[8] considered the role of amplification of waves by shear layers when looking for the mechanism of sound production of supersonic jets. Although he did not specifically discuss over-reflection, he argued that finite amplitude waves may be reflected with amplification when they impinge on a shear layer. He then argued that the disturbance generated by impingement will possibly also contain a centered wave, propagating upstream outside the jet. When this wave reaches the beginning of the shear layer, another disturbance is started propagating inside the jet, to impinge, be reflected again, and produce another centered wave that produces a disturbance at the beginning of the shear layer, etc. Powell estimated the frequency of the wave produced by what could now be called resonant over-reflection by calculating travel time per period. For the present example one finds that the period of oscillation ΔT for a jet of width b, will be

$$\Delta T = b(\mathrm{F}+1)/(g'H)^{1/2}. \tag{19}$$

One may, from this, conclude that although a flow when analyzed as extending infinitely far in the positive and negative streamwise direction may prove neutrally stable, the introduction of a change in lateral boundary condition can cause spatial instability. This instability will not be revealed by analysis which considers the temporal development of disturbances of real, constant wavenumber disturbances, since they cannot describe the development of disturbances whose amplitude varies with the streamwise coordinate x.

These arguments may apply to geophysical flows along changing topography, where one may have free shear layers form as a stratified current reaches the end of a stretch of coast and continues away from the coast. Similar situations may also occur in the atmosphere, in the wake of an island, for example.

VII. CONCLUSIONS

The preceding generalization of the analysis of Landau, Miles, and Ribner may be useful as a demonstration of amplified reflection for the simplest kind of gravity waves, both of a free surface of shallow water and as internal waves on a shallow interface in deep water. The role of waves in momentum transport needs to be emphasized for geophysical flows, in particular. A disturbance that radiates to infinity, but is capable of maintaining constant amplitude along a shear layer, will tend to be called neutrally stable, but, such a disturbance will still be capable of exchanging momentum between the flow and its surroundings. Thus, disturbances that under some customary usage are called neutrally stable and are therefore disregarded, may be very important in modifying a flow. On the other hand, if growing disturbances that also radiate energy and momentum can be present, they will, of course, tend to dominate the exchange processes. Yet another important role of waves that are reflected with amplification is the possible occurrence of spatially growing disturbances, even if a parallel flow analysis only yields neutrally stable (and radiative) solutions. This shows the need for careful interpretation of parallel flow stability analysis where the solutions are radiative.

The present case of nondispersive waves was chosen because of its simplicity, since the boundary conditions at the discrete shear layer can be satisfied without having to add other types of solutions, as had to be done in most cases, such as for surface waves in deep water, internal-inertial waves in a rotating and continuously stratified flow, and other flows that are complicated by the presence of external fields and other kinds of body forces.

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