of the daily magnetic activity index, *aa*. In one, for 1870-72, no peak was found. In the other, for 1876-78, a peak at 164 days was found, which may be compared with the peak at ~150 days found in the New England data for 1877-79. A spectrum of the number of magnetic storms with $aa > 60$ in 1892-94 shows no peak in the region of interest.

Figure 1*b*, *c* show the spectra in the vicinity of 160 days for the periods 1736-39 and 1787-90. The spectrum from the earlier period shows a peak at ~154 days as well as one at ~182 days. For the later interval, the 150-day peak is either absent or is present only as a weak shoulder on the strong 180-day peak. The relative amplitudes of the approximately semi-annual peak (180 days) are consistent with Swedish data, over the period 1721-1943[6] (for recent discussions of the annual variation, see refs 19, 20).

These results indicate that the 155-day periodicity is only infrequently prominent. The presence of a peak near this period in data going back to the sixteenth century, however, demonstrates that it is a persistent feature of solar activity over intervals of centuries, and confirms the report of the presence of the peak during the Maunder Minimum[13]. The finding that the period is sporadic is consistent with the results of the past three solar cycles, in which the peak is not observed clearly in whole-disk behaviour for the nineteenth cycle (see Fig. 1 of ref. 12), and in which it is weaker in the twentieth than in the twenty-first cycle[10]. The reality of this erratic peak seems to be established, both by the criteria used in the analyses and by its occurrence in a variety of observational parameters. A final comment may be made on the position of the ~150-day peak. I have assumed that peaks between 146 and 160 days are related because of resolution problems in data and the method of analysis. Table 1 indicates a slow drift over the centuries in the position of this peak. The occurrence of such fuzzy periodicities is not unusual or unexpected (see, for example, ref. 21). If real, this drift may reflect a basic change in a so-far unknown solar property. □

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The energy spectrum of knots and links

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KNOTTED and linked structures arise in such disparate fields as plasma physics, polymer physics, molecular biology and cosmic string theory. It is important to be able to characterize and classify such structures. Early attempts to do so[1] were stimulated by Kelvin's[2] recognition of the invariance of knotted and linked vortex tubes in fluid flow governed by the classical Euler equations of motion. The techniques of fluid mechanics are still very natural for the investigation of certain problems that are essentially topological in character. Here I use these techniques to establish the existence of a new type of topological invariant for knots and links. Any knot or link may be characterized by an 'energy spectrum'—a set of positive real numbers determined solely by its topology. The lowest energy provides a possible measure of knot or link complexity.

Continuous deformation of a knotted structure may be achieved by embedding the structure in an incompressible fluid medium moving with continuous velocity $\mathbf{v}(\mathbf{x}, t)$ (where $\nabla \cdot \mathbf{v} = 0$), convecting and distorting the structure in the process. This flow induces a continuous, time-dependent, orientatable and volume-preserving mapping $\mathbf{x} \to \mathbf{X}(\mathbf{x}, t)$ of the medium onto itself, where $\mathbf{X}(\mathbf{x}, t)$ represents the position at time t of the fluid particle that passes through position $\mathbf{x}$ at time $t = 0$.

Let $\mathbf{B}(\mathbf{x}, t)$ be any solenoidal vector field ($\nabla \cdot \mathbf{B} = 0$) convecting with the fluid under the condition that the flux of $\mathbf{B}$ through any closed circuit moving with the fluid is conserved; such a field may be described as 'frozen' in the fluid, and it satisfies the frozen-field equation[3]

$$\partial \mathbf{B}/\partial t = \nabla \times (\mathbf{v} \times \mathbf{B}) \qquad (1)$$

The solution can be obtained in the form

$$B_i(\mathbf{X}, t) = B_j(\mathbf{x}, 0)\partial X_i/\partial x_j \qquad (2)$$

a result that encapsulates both the convection of the field from position $\mathbf{x}$ to $\mathbf{X}$ and its simultaneous rotation and distortion by the deformation tensor $\partial X_i/\partial x_j$. Equation (2) establishes a topological equivalence between the initial field $\mathbf{B}_0(\mathbf{x}) = \mathbf{B}(\mathbf{x}, 0)$ and the field $\mathbf{B}(\mathbf{X}, t)$; in particular, all knots and links in the field-line structure are conserved under the deformation described by equation (2).

To exploit the properties of equations (1) and (2) in the search for topological invariants of a knot K, it is first necessary to construct a tubular neighbourhood $\mathcal{T}_K$ of the knot, and a knot field $\mathbf{B}_K(\mathbf{x})$ confined to this neighbourhood, each field line being a satellite of K. Care is needed, however, in considering the twist of the field in $\mathcal{T}_K$.

Consider a knot K of length L_0 in a configuration for which a plane projection exhibits a minimum number of crossings. The knot may be deformed inextensibly to lie entirely in the plane $z = 0$ except for smooth indentations into the half-space $z < 0$ at each of the underpasses. By a finite number of reflections of underpasses in the plane $z = 0$, a related unknotted curve C may be formed, which may be continuously deformed to a circle C_0 of radius $R = L_0/2\pi$.

Conversely, one may start with a circle, C_0, $x^2 + y^2 = R^2$, and define a tubular neighbourhood $\mathcal{T}_0$ of C_0, in cylindrical polar coordinates (r, φ, z), by

$$(r - R)^2 + z^2 < (\varepsilon R)^2 \qquad (3)$$

where $\varepsilon < 1$; the limiting situation when $\varepsilon \to 0$ is of particular interest. The area of cross-section of $\mathcal{T}_0$ is $A_0 = \pi\varepsilon^2 R^2$, and its volume is $V = 2\pi^2\varepsilon^2 R^3$. This tube can be made into a flux tube by defining a field $\mathbf{B}_0(\mathbf{x})$ that is zero outside $\mathcal{T}_0$, and which has the form

$$\mathbf{B}_0(\mathbf{x}) = (0, 2\pi r\Phi/V, 0) \qquad (4)$$

inside $\mathcal{T}_0$, where Φ is constant (the flux of $\mathbf{B}_0(\mathbf{x})$ across any meridian section of the tube). This field is invariant under any axisymmetric volume-preserving rearrangement of field lines; this follows from equation (1), which for axisymmetric convection of a toroidal field becomes $D(B_\varphi/r)/Dt=0$, where D/Dt is the Lagrangian derivative $(=\partial/\partial t+\mathbf{v}\cdot\nabla)$.

The knot field $\mathbf{B}_K(\mathbf{x})$ is obtained in three steps. (1) The flux tube is cut at any section $\varphi=$ constant, twisted through angle $2\pi h_0$, the twist being uniformly distributed with respect to the angle φ, and reconnected (Fig. 1). If $h_0=+1$, then every pair of $\mathbf{B}$ lines is now simply linked, and the associated helicity[4,5] of the field, is obtained by integrating over the flux elements $d\Phi_1$

$$\mathcal{H}_0=2\int_0^{\Phi}\Phi_1\, d\Phi_1=\Phi^2 \qquad (5)$$

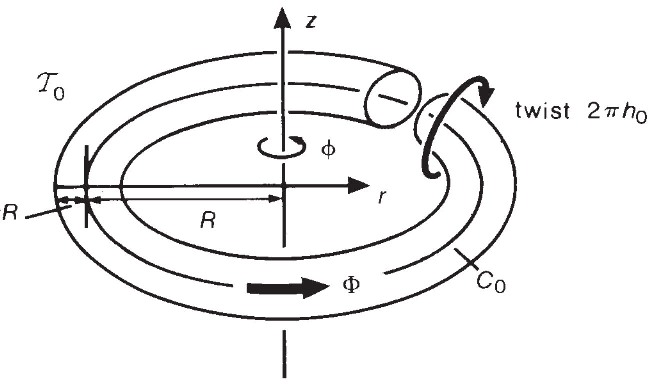

FIG. 1 The flux tube T_0 (equation (3)) is a torus of circular section centred on the circle C_0, and carrying flux Φ. Twist is introduced by cutting the tube at a section $\varphi=$constant, twisting through an angle $2\pi h_0$, and reconnecting. The helicity thus generated is $h_0\Phi^2$.

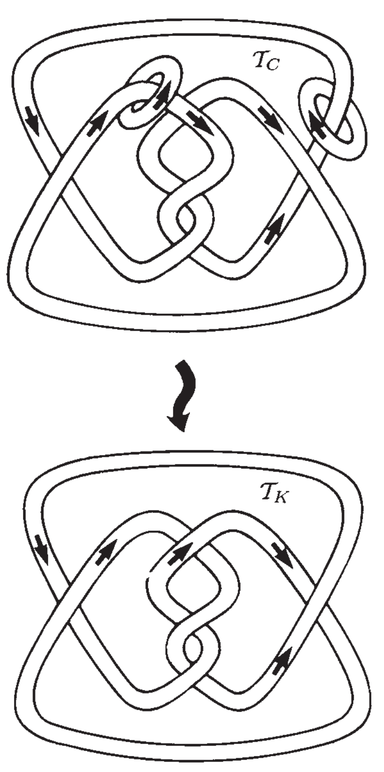

FIG. 2 The unknotted tube $\mathcal{T}_C$ is converted to the knotted tube $\mathcal{T}_K$ by switching a number of crossings. In the example shown here, two positive crossings are switched to become negative crossings; each switch is equivalent to the insertion of a small loop that cancels the field on one side and makes it reappear on the other. The change in helicity associated with the switches in this case is $-4\Phi^2$, so that $h=h_0-4$.

More generally, the helicity is

$$\mathcal{H}_0=h_0\Phi^2 \qquad (6)$$

(2) The tube is now deformed by a continuous incompressible flow field $\mathbf{v}_1(\mathbf{x},t)$ $(0\leq t\leq T)$, with associated mapping $\mathbf{x}\to\mathbf{X}_1(\mathbf{x},T)$, so that $C_0\to C$ and $\mathcal{T}_0\to\mathcal{T}_C$, a tubular neighbourhood of the curve C defined above; both Φ and V, and also the helicity $\mathcal{H}_0$ are conserved in this process. Here, ε may be assumed sufficiently small that $\mathcal{T}_C$ does not intersect itself. (3) The field is reflected in the relevant indentations about $z=0$ so that the flux tube $\mathcal{T}_C$ becomes a flux tube $\mathcal{T}_K$ containing the knot K (Fig. 2). Again, Φ and V are conserved. The example of Fig. 2 shows that the helicity changes by $\pm2\Phi^2$ for every reflection that converts a negative (positive) crossing to a positive (negative) one; hence $\mathcal{H}_0\to\mathcal{H}=h\Phi^2$ where

$$h=h_0+2(N_+-N_-) \qquad (7)$$

where N_+ and N_- are the numbers of such crossings required to yield the knot K. There may well be different ways of achieving this end (different sets of reflections may yield the same knot) but because h_0 is arbitrary, any desired value of h, positive or negative or zero, may be achieved, and this freedom corresponds to the freedom in specifying the twist of the field in the flux tube $\mathcal{T}_K$.

The choice $h_0=-2(N_+-N_-)$ makes $h=0$, that is, the field helicity is zero; this corresponds to 'zero-framing' of the knot as discussed in a physical context by Witten[6]; I adopt this natural choice here. The topology of the field $\mathbf{B}_K(\mathbf{x})$ thus constructed is then uniquely determined.

The knot field is allowed to 'relax' by a procedure that has proved effective in considering the existence and stability of magnetostatic equilibria[7]. For any field $\mathbf{B}(\mathbf{x},t)$ that is non-zero only in some bounded volume, the field energy may be defined as

$$M(t)=\tfrac{1}{2}\int\mathbf{B}^2\, dV \qquad (8)$$

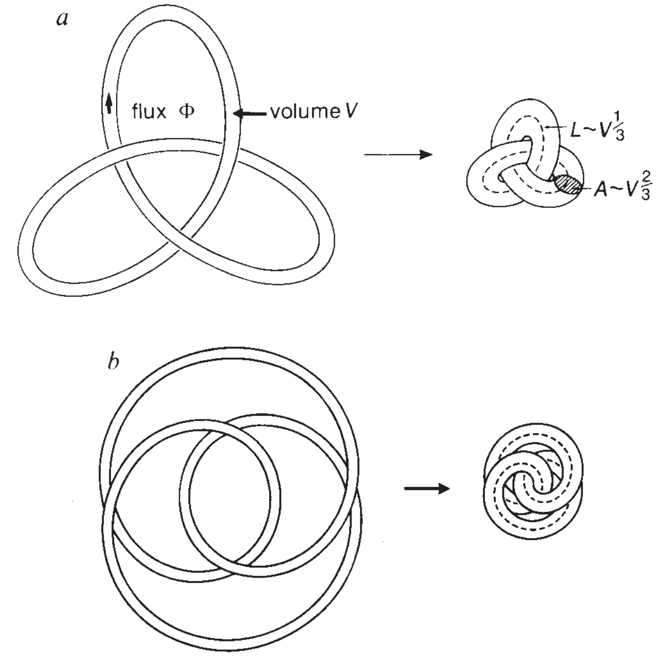

FIG. 3 Relaxation of a flux tube knotted in the form of a trefoil knot, with V and Φ conserved; relaxation is arrested when the axial length L is of the order of $V^{1/3}$ and the mean cross-sectional area A is of the order of $V^{2/3}$. The energy is then $M^E=m\Phi^2V^{1/3}$, where m is a dimensionless positive real number, a topological invariant of the knot. a), The presentation $K_{2,3}$ of the trefoil for which $m=m_0$; b), the presentation $K_{3,2}$ for which I conjecture that $m=m_1>m_0$.

which, by virtue of equation (1), satisfies

$$\frac{dM}{dt}=\int \mathbf{B}\cdot[\nabla\times(\mathbf{v}\times\mathbf{B})]\,dV=-\int \mathbf{v}\cdot[((\nabla\times\mathbf{B})\times\mathbf{B})]\,dV \quad (9)$$

I use the initial condition

$$\mathbf{B}(\mathbf{x},0)=\mathbf{B}_K(\mathbf{x}) \quad (10)$$

and assume that $\mathbf{v}(\mathbf{x}, t)$ is itself instantaneously determined by the equation

$$k\mathbf{v}=-\nabla p+(\nabla\times\mathbf{B})\times\mathbf{B} \quad (11)$$

where k is a positive constant and p is a 'pressure' field determined by the condition that $\nabla\cdot\mathbf{v}=0$ for all t. Both $\mathbf{v}$ and ∇p are at most of the order of $|\mathbf{x}|^{-3}$ as $|\mathbf{x}|\to\infty$. Combining equations (9) and (11)

$$\frac{dM}{dt}=-k\int \mathbf{v}^2\,dV \quad (12)$$

so that $M(t)$ is indeed monotonic decreasing if $\mathbf{v}$ is not identically zero.

It has been shown[8] that if the topology of $\mathbf{B}(\mathbf{x}, 0)$ is non-trivial, then the energy of any field obtainable by a transformation of the form of equation (2) has a positive lower bound. Hence, if the knot K is non-trivial, the energy function $M(t)$ is bounded below and, by virtue of equation (12), must tend to a limit M^E (>0), as $t\to\infty$. Here the superscript E indicates that this is an equilibrium state for which $\mathbf{v}=0$.

Consider how the decrease of energy described by equation (12) actually occurs. At the initial instant

$$L_0=2\pi(V/\pi\varepsilon^2)^{1/3} \quad \text{and} \quad A_0=V/L_0 \quad (13)$$

and, if it is assumed that $\varepsilon\ll 1$, then

$$M(0)\approx\tfrac{1}{2}(\Phi/A_0)^2V=8\Phi^2V^{-1/3}(\pi/\varepsilon)^{1/3} \quad (14)$$

As Φ and V are invariant during the relaxation process, the energy can decrease only through increase of the average cross-sectional area A of $\mathcal{T}_K$ and consequent decrease of (axial) tube length L. This process is illustrated for the trefoil knot in Fig. 3. It is evident that the process of energy reduction must come to a halt when different parts of the flux tube come in contact with each other. It is this topological barrier that implies the positive lower bound for $M(t)$ referred to above. The lower bound is attained when L and A are determined by V alone (in conjunction with the knot topology) that is, $L\sim V^{1/3}$, $A\sim V^{2/3}$. Then on dimensional grounds (compare to equation (14))

$$M^E=m\Phi^2V^{-1/3} \quad (15)$$

where m is a positive real number, determined only by the knot topology. By virtue of its construction, m is a topological invariant of the knot.

There is no guarantee that the end state as $kt\to\infty$ is uniquely determined, and indeed it seems likely that for knots of any complexity, a variety of distinct asymptotic configurations may be attainable starting from different initial geometrical configurations of the same knot. In this case, different values of m will in general be attained for different end states. I denote these by m_i ($i=0,1,2,\ldots$) ordered so that $0<m_0\leq m_1\leq m_2\leq\ldots$. The sequence $\{m_0, m_1, m_2,\ldots\}$ may then be described as the energy spectrum of the knot, with m_0 the ground state energy. Generally speaking, a high value of m_0 will indicate a complex knot and indeed m_0 could reasonably be used as a measure of knot complexity.

The suggestion that there is in general a multiplicity of possible end states derives support from consideration of the two different forms of the trefoil knot (usually denoted $K_{2,3}$ and $K_{3,2}$) shown in Fig. 3. I conjecture that these may relax to distinct stable end states with distinct energy levels m_0 (for $K_{2,3}$) and m_1 (for $K_{3,2}$).

Very similar considerations apply to the problem of characterizing links of two or more components. Considering the case of two linked curves C_1 and C_2, flux tubes of equal volume V and carrying equal flux Φ around C_1 and C_2 can be constructed by the same procedure as for knots, and thus an initial link field $\mathbf{B}_L(\mathbf{x})$ can be generated for the relaxation problem. Again the asymptotic energy has the form of equation (15), where the real number m is a topological invariant for the link in question.

In the four physical contexts mentioned at the outset, the energy of knotted and linked structures has a central role in formulating a self-consistent theory. In polymer physics, for example, a tangled macromolecule will tend to relax towards the 'ground-state' configuration compatible with its topology. The result that the energy of a structure, if suitably defined, has a minimum value that is unambiguously related to its topology, seems to be new, and may suggest techniques whereby topologically distinct structures may be detected. Related concepts are discussed in recent papers by Freedman and He[9,10]. □

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Tunnelling evidence for predominantly electron–phonon coupling in superconducting $Ba_{1-x}K_xBiO_3$ and $Nd_{2-x}Ce_xCuO_{4-y}$

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AMONG the superconducting oxides, the cubic, copper-free $Ba_{1-x}K_xBiO_3$ (BKBO) and the electron-doped compound $Nd_{2-x}Ce_xCuO_{4-y}$ (NCCO) stand out as being somewhat different from the rest. Nevertheless, an understanding of the pairing mechanism in BKBO (transition temperature, $T_c\approx 30$ K) and NCCO ($T_c\approx 23$ K) may give important insights into the mechanisms of the higher-T_c superconductors. Here we report tunnelling spectroscopy measurements on BKBO and NCCO, using point-contact junctions that exhibit low leakage currents and sharp conductance peaks at the gap voltages $V=\pm\Delta/e$. Reasonably symmetric and reproducible structures are observed in the high-bias tunnelling conductances which are characteristic of phonon effects as seen in conventional superconductors. We have inverted the tunnelling data and obtained the Eliashberg functions, $\alpha^2F(\omega)$, where $F(\omega)$ is the phonon density of states at energy $\hbar\omega$. For BKBO, $\alpha^2F(\omega)$ bears a close resemblance to the available phonon density of state determined by inelastic neutron scattering, most importantly consistently reproducing the minima. The fact that the $\alpha^2F(\omega)$ are not identical for different junctions leads to some uncertainty, but the calculated values of T_c are in good agreement with experiment for both BKBO and NCCO. Also, there is a good match between the calculated values of the total