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On the partition of energy between elastic waves in a semi-infinite solid

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Expressions are derived for the power radiated in the compressional, shear and surface waves set up by a circular disk vibrating normally to the free surface of a semi-infinite isotropic solid. The total radiated power is also calculated independently by integrating the displacement velocity over the area of the source.

The theory is extended to a general type of multi-element radiator in the form of an array of elements on the circumference of a circle. The calculation of the total power here involves a 'mutual admittance' function, a table of which is given for the case when the Poisson's ratio of the medium is equal to $\frac{1}{4}$. The theory is applied to a three-element radiator of a type used in a recent geophysical investigation, and it is shown that the efficiency of radiation in the compressional mode can be varied between wide limits by varying the distance between the elements.

Finally, an approach is suggested for problems in which the most suitable idealized boundary condition is one of known displacement under the radiator, the stress being zero elsewhere on the free surface. It is shown that the stress under the radiator satisfies an integral equation whose kernel is derived from the mutual admittance function.

1. Introduction and summary

In a recent paper (Miller & Pursey 1954; subsequently referred to as 'paper I'), the writers investigated the field generated by a circular disk of finite radius vibrating normally to the free surface of a semi-infinite isotropic solid medium. Explicit expressions were derived for the field at infinity at points within and on the surface of the solid, and methods were given for the numerical evaluation of integrals representing the radiation impedance, defined as the ratio of stress to mean displacement velocity under the disk.

The field in this problem may be regarded as comprising three parts, a compressional (or irrotational) wave, a shear (or rotational) wave and a surface wave. The purpose of the present paper is to determine the partition of energy between the three types of wave, both for this problem and in the case of a general type of multi-element radiator.

The paper is arranged as follows. In § 2 the power in each of the three waves issuing from a single source is evaluated by integrating the time average of the intensity of the wave over the corresponding wave surface at infinity, and in § 3 the sum of the power components so obtained is shown to correspond with the total power radiated by the source.

In §§ 4 and 5 the theory is extended to the case of a multiple radiator in the form of an array of radiating elements situated on the circumference of a circle. Evaluation of the double integrals representing the power components is now complicated by the variation of intensity with azimuth; the integration of the intensity with respect to the azimuth angle can, however, be performed analytically, with the result that numerical integration is necessary only with respect to the polar angle. In order to determine the power radiated by a multi-element source it is necessary to take account of the effect of the interaction between pairs of elements. Constant stress is assumed over the region of the surface beneath each radiating element, and the corresponding displacements are found by summation of contributions from all elements of the array. For this purpose it is convenient to work in terms of admittance, rather than impedance, since the self- and mutual admittances may then be summed directly to obtain the effective admittance of the array and hence the radiated power for a given stress. For small radiators the mutual admittance may be expressed, to a first approximation, as the product of a simple function of the source size and an integral expression which depends only on the distance between pairs of elements and the Poisson's ratio of the medium. The value of this integral has been calculated for a wide range of spacings for the case in which Poisson's ratio is equal to $\frac{1}{4}$, and the results are presented in table 1. An asymptotic formula, valid when the distance between the pair of elements is large, is derived in § 6.

The general theory of multiple radiators is applied in § 7 to the particular case of a three-element source of a type used in a recent geophysical experiment (Evison 1951), in which the elements are situated at the vertices of an equilateral triangle. In order to illustrate the application of the theory to the efficient design of radiators, results are obtained both in the case when the distance between the elements is approximately as in the experiment and when the spacing is chosen so as to minimize the power radiated in the surface wave.

In § 8 an approach is suggested for the solution of problems in which the displacement under the radiator is constant or, more generally, an arbitrary function of the distance from the centre of the radiator. In the simple case of a single circular radiator it is shown that the resulting stress distribution satisfies an integral equation whose kernel is derived from the mutual admittance function introduced in § 5.

In the analysis of paper I it was often convenient to measure distances in units of the compressional or shear wavelengths; this practice has not been followed in the present paper, and formulae quoted from the earlier paper have accordingly been converted to unnormalized form. It should also be observed that symbols denoting displacements and stress components represent peak values of these quantities.

So far as is practicable, results are obtained for general values of the Poisson's ratio σ or, equivalently, of the parameter μ defined by equation (7). For purposes of numerical illustration we take $\sigma = \frac{1}{4}$ and, correspondingly, $\mu = \sqrt{3}$.

2. Single-element sources: integration of the intensity at infinity

Compressional and shear waves

We consider first the case of a single circular disk of radius a vibrating normally on the surface of a semi-infinite isotropic solid, the stress beneath the disk at time t being given by $P_0 e^{i\omega t}$, where P_0 and ω are positive constants. In order to derive expressions for the intensity of the compressional and shear waves, whose wave surfaces are spherical in form, we shall use a system of spherical polar co-ordinates (R, θ, ϕ) with the centre of the disk at time $t = 0$ as origin and such that the polar angle θ is zero at points within the solid on the axis of the disk.

Denoting the radial and transverse components of the displacement u by u_R and u_θ respectively we have from equations (116) and (117) of paper I

$$u_R \sim -\frac{a^2 P_0}{2c_{44}} \frac{e^{i(\omega t - k_1 R)}}{R} \Theta_1(\theta), \tag{1}$$

$$u_\theta \sim -\frac{ia^2 \mu^3 P_0}{2c_{44}} \frac{e^{i(\omega t - k_2 R)}}{R} \Theta_2(\theta), \tag{2}$$

for large R and small a, where

$$\Theta_1(\theta) = \frac{\cos\theta(\mu^2 - 2\sin^2\theta)}{F_0(\sin\theta)}, \tag{3}$$

$$\Theta_2(\theta) = \frac{\sin 2\theta \sqrt{(\mu^2 \sin^2\theta - 1)}}{F_0(\mu \sin\theta)}, \tag{4}$$

$$F_0(\zeta) = (2\zeta^2 - \mu^2)^2 - 4\zeta^2 \sqrt{(\zeta^2 - \mu^2)} \sqrt{(\zeta^2 - 1)}, \tag{5}$$

$$k_1 = \omega \sqrt{(\rho/c_{11})}, \quad k_2 = \omega \sqrt{(\rho/c_{44})}, \tag{6}$$

and

$$\mu = \sqrt{(c_{11}/c_{44})} = k_2/k_1 = \sqrt{\{2(1-\sigma)/(1-2\sigma)\}}, \tag{7}$$

c_{11} and c_{44} being the compressional and shear elastic constants, σ the Poisson's ratio and ρ the density of the medium.

The mean intensities of the compressional and shear waves, defined in each case as the time average of the power radiated per unit area of the appropriate wave surface, are given by

$$\Upsilon_C = -\tfrac{1}{2}\dot{u}_R \widehat{RR}^*, \tag{8}$$

$$\Upsilon_{Sh} = -\tfrac{1}{2}\dot{u}_\theta \widehat{R\theta}^*, \tag{9}$$

respectively, the asterisks denoting complex conjugates.

The radial and transverse stress components across the wave surface, $\widehat{RR}$ and $\widehat{R\theta}$, are related to the displacement components by the formulae

$$\widehat{RR} = c_{12} \nabla \cdot \mathbf{u} + 2c_{44} \frac{\partial u_R}{\partial R}, \tag{10}$$

$$\widehat{R\theta} = c_{44} \left\{ R \frac{\partial}{\partial R} \left(\frac{u_\theta}{R} \right) + \frac{1}{R} \frac{\partial u_R}{\partial \theta} \right\}, \tag{11}$$

where, in the usual notation, $c_{12} = c_{11} - 2c_{44}$, and it follows from (1) and (2) that

$$\widehat{RR} \sim \frac{ia^2 k_1 \mu^2 P_0 \Theta_1(\theta)}{2R} e^{i(\omega t - k_1 R)}, \tag{12}$$

$$\widehat{R\theta} \sim -\frac{a^2 k_1 \mu^4 P_0 \Theta_2(\theta)}{2R} e^{i(\omega t - k_2 R)}. \tag{13}$$

If we denote the frequency of vibration by ν and the velocity of compressional waves in the medium by V_C, so that

$$\nu = \omega/2\pi, \quad V_C = \omega/k_1,$$

we find that
$$\Upsilon_C \sim \frac{\pi^2\nu^2a^4\mu^4P_0^2}{2\rho V_C^3}\frac{\{\Theta_1(\theta)\}^2}{R^2}, \tag{14}$$
$$\Upsilon_{Sh} \sim \frac{\pi^2\nu^2a^4\mu^9P_0^2}{2\rho V_C^3}\frac{\Theta_2(\theta)\,\Theta_2^*(\theta)}{R^2}, \tag{15}$$

since the function $\Theta_1(\theta)$ is purely real.

Finally, let W_C and W_{Sh} denote the quantities of power radiated in the compressional and shear waves, respectively. Then, integrating the intensities over a hemisphere of large radius R, we obtain

$$W_C = \frac{\pi^3\nu^2a^4\mu^4P_0^2}{\rho V_C^3}\int_0^{\frac{1}{2}\pi}\{\Theta_1(\theta)\}^2\sin\theta\,\mathrm{d}\theta, \tag{16}$$

$$W_{Sh} = \frac{\pi^3\nu^2a^4\mu^9P_0^2}{\rho V_C^3}\int_0^{\frac{1}{2}\pi}\Theta_2(\theta)\,\Theta_2^*(\theta)\sin\theta\,\mathrm{d}\theta. \tag{17}$$

These integrals have been evaluated numerically for the case $\mu = \surd 3$, corresponding to a Poisson's ratio of $\frac{1}{4}$, and the results are as follows:

$$W_C = 0{\cdot}333\frac{\pi^3\nu^2a^4P_0^2}{\rho V_C^3}, \tag{18}$$

$$W_{Sh} = 1{\cdot}246\frac{\pi^3\nu^2a^4P_0^2}{\rho V_C^3}. \tag{19}$$

Surface wave

To determine the intensity of the surface wave we employ a cylindrical system $(r,\ \phi,\ z)$ in which the positive z axis coincides with the line $\theta = 0$ in the above spherical co-ordinate system. The mean intensity is then given by

$$\Upsilon_{Su} = -\tfrac{1}{2}\dot{u}_z\widehat{zr}^* - \tfrac{1}{2}\dot{u}_r\widehat{rr}^*. \tag{20}$$

General expressions† for the surface wave displacement components may be obtained from the definite integrals representing the overall displacement components (formulae (72) and (73) of paper I) by determining in each case the contribution to the value of the integral arising from a pole of the integrand. The pole is situated at a point $\zeta = -p$ on the negative real axis, where p, defined as the positive root of the equation $F_0(\zeta) = 0$, is a function of the Poisson's ratio of the medium. The field components are given asymptotically by

$$u_z \sim \frac{a^2\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}P_0}{c_{44}F_0'(p)}\sqrt{\left(\frac{\pi k_1 p(p^2-1)}{2r}\right)}\,\mathrm{e}^{\mathrm{i}(\omega t-k_1pr)}\{2p^2\,\mathrm{e}_\mu-(2p^2-\mu^2)\,\mathrm{e}_1\}, \tag{21}$$

$$u_r \sim \frac{a^2\,\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}P_0}{c_{44}F_0'(p)}\sqrt{\left(\frac{\pi k_1 p^3}{2r}\right)}\,\mathrm{e}^{\mathrm{i}(\omega t-k_1pr)}\{2\surd(p^2-1)\,\surd(p^2-\mu^2)\,\mathrm{e}_\mu-(2p^2-\mu^2)\,\mathrm{e}_1\}, \tag{22}$$

where $\mathrm{e}_\mu = \exp\{-k_1z\surd(p^2-\mu^2)\},\quad \mathrm{e}_1 = \exp\{-k_1z\surd(p^2-1)\}.$

† In paper I (formulae (119) to (122)) expressions involving numerical coefficients were given for the displacement components of the surface wave in the particular cases $\mu = 2$ and $\mu = \surd 3$.

The relevant stress components are given in terms of the displacement components u_z and u_r by

$$\widehat{zr} = c_{44}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right), \quad \widehat{rr} = c_{12}\nabla\cdot\mathbf{u}+2c_{44}\frac{\partial u_r}{\partial r}.$$

Substituting in the left-hand members of these equations the asymptotic expressions obtained for the derivatives by differentiation of (21) and (22) and using the relation $F_0(p) = 0$, we find that

$$\widehat{zr} \sim -\frac{a^2P_0}{F_0'(p)}\sqrt{\left(\frac{\pi k_1^3 p^3(p^2-1)}{2r}\right)}\,2(2p^2-\mu^2)\{\mathrm{e}_\mu-\mathrm{e}_1\}\,\mathrm{e}^{\mathrm{i}(\omega t-k_1pr+\frac{1}{4}\pi)},$$

$$\widehat{rr} \sim \frac{a^2P_0}{F_0'(p)}\sqrt{\left(\frac{\pi k_1^3 p}{2r}\right)}\,(2p^2-\mu^2)\{(2p^2-\mu^2)\,\mathrm{e}_\mu-(2p^2+\mu^2-2)\,\mathrm{e}_1\}\,\mathrm{e}^{\mathrm{i}(\omega t-k_1pr-\frac{1}{4}\pi)}.$$

We thus have for the mean intensity of the surface wave

$$\Upsilon_{Su} = \frac{\omega k_1^2 a^4 P_0^2}{2c_{44}r}X(k_1z) = \frac{2\pi^2\nu^2a^4\mu^2P_0^2k_1}{\rho V_C^3 r}X(k_1z), \tag{23}$$

where

$$X(k_1z) = \frac{\pi(2p^2-\mu^2)}{4\{F_0'(p)\}^2}[4p^2(p^2-1)\{2p^2\,\mathrm{e}_\mu-(2p^2-\mu^2)\,\mathrm{e}_1\}(\mathrm{e}_\mu-\mathrm{e}_1)$$
$$+(2p^2-\mu^2)\{(2p^2-\mu^2)\,\mathrm{e}_\mu-2p^2\,\mathrm{e}_1\}\{(2p^2-\mu^2)\,\mathrm{e}_\mu-(2p^2+\mu^2-2)\,\mathrm{e}_1\}], \tag{24}$$

and, integrating over a cylinder of large radius r, we find for the power radiated in the surface wave

$$W_{Su} = \frac{4\pi^3\nu^2a^4\mu^2P_0^2}{\rho V_C^3}\int_0^\infty X(k_1z)\,\mathrm{d}(k_1z). \tag{25}$$

The integrand in (25) is readily expressed as a sum of multiples of the exponential functions e_μ^2, $\mathrm{e}_\mu\mathrm{e}_1$ and e_1^2, and once numerical values have been assigned to the coefficients and exponents there is no difficulty in evaluating the integral. When p has been determined, the value of $F_0'(p)$ is most easily obtained from the formula

$$F_0'(p) = 8p\left\{(2p^2-\mu^2)-\frac{(2p^2-\mu^2)^2}{4p^2}-\frac{2p^4(2p^2-\mu^2-1)}{(2p^2-\mu^2)^2}\right\}. \tag{26}$$

In the case $\mu = \sqrt{3}$ we have

$$p = \tfrac{3}{2}\sqrt{(1+1/\sqrt{3})} = 1{\cdot}8839, \quad F_0'(p) = -8\sqrt{3}p = -26{\cdot}104;$$

hence $$X(k_1z) = 1{\cdot}5383\,\mathrm{e}_\mu^2-3{\cdot}1192\,\mathrm{e}_\mu\,\mathrm{e}_1+1{\cdot}8131\,\mathrm{e}_1^2,$$

where $$\mathrm{e}_\mu = \exp(-0{\cdot}7410k_1z), \quad \mathrm{e}_1 = \exp(-1{\cdot}5966k_1z),$$

and substituting this expression in (25) we obtain

$$W_{Su} = 3{\cdot}257\frac{\pi^3\nu^2a^4P_0^2}{\rho V_C^3}. \tag{27}$$

Finally, the total power W radiated by the source is found by addition of equations (18), (19) and (27) to be

$$W = W_C+W_{Sh}+W_{Su} = 4{\cdot}836\frac{\pi^3\nu^2a^4P_0^2}{\rho V_C^3}. \tag{28}$$

3. Single-element sources: total radiated energy by the admittance method

An alternative method of calculating the total power radiated by the source is based upon the formula

$$W = \tfrac{1}{2}(P_0 A)^2 G_R, \tag{29}$$

where A is the area of the source and G_R is the real part of its radiation admittance Y_R, defined† as the (complex) ratio of mean displacement velocity over the portion of the free surface under the source to the applied force. From equation (129) of paper I we have for the radiation admittance

$$Y_R = -\frac{2\mathrm{i}\omega\mu^2}{\pi c_{44}} \int_0^\infty \frac{\sqrt{(\zeta^2-1)}\,\{J_1(\zeta k_1 a)\}^2\,\mathrm{d}\zeta}{\zeta k_1 a^2 F_0(\zeta)}, \tag{30}$$

the contour of integration being indented so as to pass above the branch-points and pole of the integrand on the positive real axis.

The radiation conductance G_R evidently derives from the imaginary part of this integral, to which only the range $0<\zeta<\mu$ and the indentation above the pole at $\zeta = p$ make any contribution. We may therefore write

$$G_R = \frac{2\omega\mu^2}{\pi c_{44}} \mathscr{I} \int_0^\lambda \frac{\sqrt{(\zeta^2-1)}\,\{J_1(\zeta k_1 a)\}^2\,\mathrm{d}\zeta}{\zeta k_1 a^2 F_0(\zeta)},$$

where λ is an arbitrary positive number exceeding p.

Hence G_R can be expressed as a power series in a^2, and for small values of a

$$G_R = \frac{\omega\mu^2 k_1}{2\pi c_{44}} \mathscr{I} \int_0^\lambda \frac{\sqrt{(\zeta^2-1)}\,\zeta\,\mathrm{d}\zeta}{F_0(\zeta)} + O(a^2).$$

Numerical integration gives, in the case $\mu = \sqrt{3}$,

$$\mathscr{I} \int_0^\lambda \frac{\sqrt{(\zeta^2-1)}\,\zeta\,\mathrm{d}\zeta}{F_0(\zeta)} = 0{\cdot}5374$$

(cf. the entry against $x = 0$ in table 1), whence

$$W = 4{\cdot}836\,\frac{\pi^3\nu^2 a^4 P_0^2}{\rho V_C^3}, \tag{31}$$

in agreement with (28) above.

4. Multi-element sources: integration of the intensity at infinity

Let us now consider an array of n identical point sources, spaced at equal intervals on the circumference of a circle of radius b. The distance from a general point P,

† The radiation admittance defined in this way differs from the reciprocal of the radiation impedance defined in paper I in that 'applied force' is here substituted for 'stress'. The change has been made in order that the corresponding definition of mutual admittance introduced in § 5 may have the required symmetry.

whose spherical polar co-ordinates are (R, θ, ϕ), to a point $M = (b, \frac{1}{2}\pi, \alpha)$ on the circle (cf. figure 1) is given by

$$PM = R\left[1-\frac{2b}{R}\sin\theta\cos(\phi-\alpha)+\left(\frac{b}{R}\right)^2\right]^{\frac{1}{2}}$$

$$= R\left[1-\frac{b}{R}\sin\theta\cos(\phi-\alpha)+O\left\{\left(\frac{b}{R}\right)^2\right\}\right],$$

when $R \gg b$.

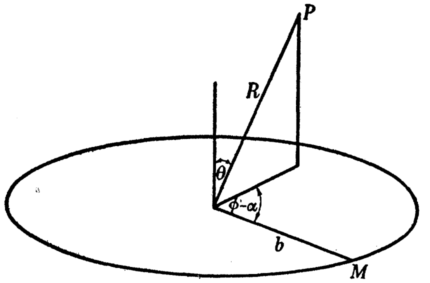

FIGURE 1

If the mth radiating element is situated at the point $(b, \frac{1}{2}\pi, 2\pi m/n)$, the radial displacement u_R^m at P due to this element is

$$u_R^m \sim \frac{a^2 P_0}{2c_{44}} \frac{e^{i(\omega t - k_1 R)}}{R} \Theta_1(\theta) \exp\left\{ik_1 b \sin\theta \cos\left(\phi - \frac{2\pi m}{n}\right)\right\}, \tag{32}$$

and the resultant displacement u_R due to the n elements is given by

$$u_R = \sum_{m=1}^{n} u_R^m. \tag{33}$$

Substituting (32) in (33) and differentiating, we obtain

$$|\dot{u}_R| \sim \frac{a^2 \omega P_0}{2c_{44}} \frac{\Theta_1(\theta)}{R} \left|\sum_{m=1}^{n} \exp\left\{ik_1 b \sin\theta \cos\left(\phi - \frac{2\pi m}{n}\right)\right\}\right|, \tag{34}$$

and from equations (10), (32) and (33)

$$|\widehat{RR}| \sim \frac{c_{11} a^2 k_1 P_0}{2c_{44}} \frac{\Theta_1(\theta)}{R} \left|\sum_{m=1}^{n} \exp\left\{ik_1 b \sin\theta \cos\left(\phi - \frac{2\pi m}{n}\right)\right\}\right|. \tag{35}$$

The resultant intensity Υ_C is therefore given asymptotically by

$$\Upsilon_C \sim \frac{c_{11} a^4 \omega k_1 P_0^2}{8c_{44}^2} \frac{\{\Theta_1(\theta)\}^2}{R^2} \left|\sum_{m=1}^{n} \exp\left\{ik_1 b \sin\theta \cos\left(\phi - \frac{2\pi m}{n}\right)\right\}\right|^2$$

$$= \frac{\pi^2 \nu^2 a^4 \mu^4 P_0^2}{2\rho V_C^3} \frac{\{\Theta_1(\theta)\}^2}{R^2} \sum_{l=1}^{n} \sum_{m=1}^{n} \cos\left[2k_1 b \sin\theta \sin\left\{\phi - \frac{\pi(l+m)}{n}\right\} \sin\frac{\pi(l-m)}{n}\right]. \tag{36}$$

Similarly, we find for the intensities of the shear and surface waves

$$\Upsilon_{Sh} \sim \frac{\pi^2 \nu^2 a^4 \mu^9 P_0^2}{2\rho V_C^3} \frac{|\Theta_2(\theta)|^2}{R^2} \sum_{l=1}^{n} \sum_{m=1}^{n} \cos\left[2\mu k_1 b \sin\theta \sin\left\{\phi - \frac{\pi(l+m)}{n}\right\} \sin\frac{\pi(l-m)}{n}\right], \tag{37}$$

$$\Upsilon_{Su} \sim \frac{2\pi^2 \nu^2 a^4 \mu^2 P_0^2 k_1}{\rho V_C^3} \frac{X(k_1 z)}{r} \sum_{l=1}^{n} \sum_{m=1}^{n} \cos\left[2pk_1 b \sin\left\{\phi - \frac{\pi(l+m)}{n}\right\} \sin\frac{\pi(l-m)}{n}\right]. \tag{38}$$

By integrating the intensities over a hemisphere of radius R in the cases of the compressional and shear waves, and over a cylinder of radius r in the case of the surface wave, we derive expressions for the power radiated in each of the three waves. The integrations with respect to ϕ may be performed analytically and the results are

$$W_C = \frac{\pi^3\nu^2a^4\mu^4P_0^2}{\rho V_C^3}\int_0^{\frac{1}{2}\pi}\{\Theta_1(\theta)\}^2\sum_{l=1}^{n}\sum_{m=1}^{n}J_0\left\{2k_1b\sin\theta\sin\frac{\pi(l-m)}{n}\right\}\sin\theta\,\mathrm{d}\theta, \tag{39}$$

$$W_{Sh} = \frac{\pi^3\nu^2a^4\mu^9P_0^2}{\rho V_C^3}\int_0^{\frac{1}{2}\pi}|\,\Theta_2(\theta)\,|^2\sum_{l=1}^{n}\sum_{m=1}^{n}J_0\left\{2\mu k_1b\sin\theta\sin\frac{\pi(l-m)}{n}\right\}\sin\theta\,\mathrm{d}\theta, \tag{40}$$

$$W_{Su} = \frac{4\pi^3\nu^2a^4\mu^2P_0^2}{\rho V_C^3}\int_0^{\infty}X(k_1z)\,\mathrm{d}(k_1z)\sum_{l=1}^{n}\sum_{m=1}^{n}J_0\left\{2pk_1b\sin\frac{\pi(l-m)}{n}\right\}. \tag{41}$$

We observe that as $b\to 0$ the Bessel functions J_0 tend to unity, and in this case the total power radiated is therefore equal to n^2 times the power from a single element acting independently (cf. equations (16), (17) and (25)). This is consistent with the fact that the power from a single element varies as a^4, that is, as the square of the radiating area.

As $b\to\infty$, the only terms which do not vanish are those for which $l = m$, and the results correspond, as we should expect, with the case in which n radiating elements act independently.

Variation of the parameter b alters the relative amounts of power in the three waves, and this may be put to practical advantage if it is desired to increase or reduce the proportion of power radiated in a particular wave (cf. § 7).

5. Multi-element sources: total radiated power by the admittance method

The direct calculation of the total power radiated by a multiple source proceeds from the formula

$$W = \tfrac{1}{2}P_0^2\sum_{i,j}A_iA_jG_{ij}, \tag{42}$$

where A_i denotes the area of the ith element and G_{ij} is the real part of the mutual admittance Y_{ij}, defined as the ratio of mean displacement velocity over the jth element to applied force at the ith element. When $i=j$, G_{ij} and Y_{ij} are identical with the radiation conductance G_R and radiation admittance Y_R respectively, introduced in § 3.

Let a_i and a_j denote the radii of the ith and jth elements respectively, and suppose a_j to be so small that the mean value of the displacement velocity over the jth element may be approximated by its value at the centre. Then, from equation (125) of paper I, we have

$$Y_{ij}\sim -\frac{\mathrm{i}\omega\mu^2}{\pi a_i c_{44}}\int_0^{\infty}\frac{\surd(\zeta^2-1)\,J_1(\zeta k_1a_i)\,J_0(\zeta k_1s_{ij})\,\mathrm{d}\zeta}{F_0(\zeta)}, \tag{43}$$

where s_{ij} is the distance between the centres of the ith and jth elements.

If a_i is also small we may make the approximation

$$Y_{ij} \sim \left[\frac{\partial}{\partial a_i}(a_i Y_{ij})\right]_{a_i=0} = -\frac{\mathrm{i}\omega k_1 \mu^2}{2\pi c_{44}} \int_0^\infty \frac{\sqrt{(\zeta^2-1)}\, J_0(\zeta k_1 s_{ij})\, \zeta\, \mathrm{d}\zeta}{F_0(\zeta)}$$

$$= -\frac{2\pi \mathrm{i} \nu^2 \mu^4}{\rho V_C^3} \int_0^\infty \frac{\sqrt{(\zeta^2-1)}\, J_0(\zeta k_1 s_{ij})\, \zeta\, \mathrm{d}\zeta}{F_0(\zeta)}, \tag{44}$$

provided $s_{ij} > 0$. Thus, for small values of a_i and a_j, the mutual admittance depends only on the distance s_{ij} between the elements and on the elastic properties of the medium.

TABLE 1

x	$\mathscr{R}I(x, \sqrt{3})$	δ^2	$\mathscr{I}I(x, \sqrt{3})$	δ^2	$\mathscr{I}\{xI(x, \sqrt{3})\}$	δ^2	x	$\mathscr{R}I(x, \sqrt{3})$	δ^2	$\mathscr{I}I(x, \sqrt{3})$	δ^2
0.0	+0·5374	−81	∞	—	+0·2500	−113	**2·5**	−0·1125	+36	+0·1308	−55
0·1	·5336	77	+2·1810	—	·2181	96	**2·6**	·0958	27	·1348	53
0·2	·5221	73	0·8831	—	·1766	75	**2·7**	·0764	20	·1335	51
0·3	·5033	69	·4253	—	·1276	53	**2·8**	·0550	10	·1271	46
0·4	·4776	62	·1834	—	·0733	− 26	**2·9**	·0326	+ 2	·1161	41
0·5	+0·4457	−54	+0·0328	—	+0·0164	0	**3·0**	−0·0100	− 6	+0·1010	−34
0·6	·4084	47	−0·0675	—	−0·0405	+ 26	**3·1**	+0·0120	13	·0825	25
0·7	·3664	35	·1353	—	·0948	51	**3·2**	·0327	21	·0615	17
0·8	·3209	25	·1800	+183	·1440	74	**3·3**	·0513	26	·0388	− 8
0·9	·2729	14	·2064	146	·1858	94	**3·4**	·0673	31	+0·0153	0
1·0	+0·2235	− 3	−0·2182	+122	−0·2182	+110	**3·5**	+0·0802	−35	−0·0082	+ 9
1·1	·1738	+ 8	·2178	100	—	—	**3·6**	·0896	36	·0308	17
1·2	·1249	20	·2074	81	—	—	**3·7**	·0954	38	·0517	24
1·3	·0780	29	·1889	63	—	—	**3·8**	·0974	37	·0702	30
1·4	+0·0340	37	·1641	47	—	—	**3·9**	·0957	35	·0857	36
1·5	−0·0063	+45	−0·1346	+ 33	—	—	**4·0**	+0·0905	−32	−0·0976	+38
1·6	·0421	52	·1018	17	—	—	**4·1**	·0821	28	·1057	41
1·7	·0727	56	·0673	+ 4	—	—	**4·2**	·0709	23	·1097	41
1·8	·0977	59	−0·0324	− 10	—	—	**4·3**	·0574	18	·1096	40
1·9	·1168	60	+0·0015	22	—	—	**4·4**	·0421	12	·1055	38
2·0	−0·1299	+59	+0·0332	− 31	—	—	**4·5**	+0·0256	− 5	−0·0976	+34
2·1	·1371	58	·0618	40	—	—	**4·6**	+0·0086	+ 1	·0863	30
2·2	·1385	54	·0864	46	—	—	**4·7**	−0·0083	7	·0720	24
2·3	·1345	49	·1064	51	—	—	**4·8**	·0245	13	·0553	18
2·4	·1256	42	·1213	54	—	—	**4·9**	·0394	16	·0368	+ 9
2·5	−0·1125	+36	+0·1308	− 55	—	—	**5·0**	−0·0527	+20	−0·0174	0

If we now write

$$I(x, \mu) = -\mathrm{i} \int_0^\infty \frac{\sqrt{(\zeta^2-1)}\, J_0(\zeta x)\, \zeta\, \mathrm{d}\zeta}{F_0(\zeta)}, \tag{45}$$

and denote the function $\mathscr{R}I(x, \mu)$ by $f(x)$, we obtain

$$G_{ij} = \mathscr{R}Y_{ij} = \frac{2\pi \nu^2 \mu^4}{\rho V_C^3} f(k_1 s_{ij}). \tag{46}$$

Hence the total radiated power is given by

$$W = \frac{\pi\nu^2\mu^4P_0^2}{\rho V_C^3}\sum_{i,j} A_i A_j f(k_1 s_{ij})$$
$$= \frac{\pi^3\nu^2\mu^4P_0^2}{\rho V_C^3}\sum_{i,j} a_i^2 a_j^2 f(k_1 s_{ij}). \tag{47}$$

Table 1 gives values of $\mathscr{R}I(x, \surd 3)$ to four decimal places, together with its second differences, for the range $x = 0{\cdot}0\,(0{\cdot}1)\,5{\cdot}0$. For completeness, and because of their use in the calculation of energy supplied to a radiator by a driving system, values of $\mathscr{I}I(x, \surd 3)$ have also been given. An auxiliary table of $\mathscr{I}\{xI(x, \surd 3)\}$ for $x = 0{\cdot}0\ (0{\cdot}1)\ 1{\cdot}0$ is included to assist interpolation when x is small. Tabular values are accurate to within two units of the last figure given.

The asymptotic evaluation of $I(x, \mu)$ for large values of x is considered in § 6.

Numerical evaluation of the integral

The method used to evaluate the integral $I(x)$ was essentially that applied to integrals of a similar type in paper I (§ 7), to which the reader is referred for further details.

For the 'tail' of the integral an expansion of the form

$$\int_\lambda^\infty \frac{\surd(\zeta^2-1)\,J_0(\zeta x)\,\zeta\,d\zeta}{F_0(\zeta)} = C_0\int_\lambda^\infty J_0(\zeta x)\,d\zeta + C_1\int_\lambda^\infty \frac{J_0(\zeta x)}{\zeta^2}\,d\zeta + \dots \tag{48}$$

was obtained by expanding $\zeta\surd(\zeta^2-1)/F_0(\zeta)$ in inverse powers of ζ^2 and integrating term by term. The integrals which appear on the right-hand side of equation (48) are readily evaluated with the aid of recurrence formulae, using tables of Bessel functions and the integral of $J_0(x)$.

The behaviour of the function $I(x, \mu)$ as $x \to 0$ can be determined with the aid of the expansion (48). It is easily seen that, for any fixed positive λ,

$$\lim_{x\to 0}\left[x\int_\lambda^\infty \{\surd(\zeta^2-1)\,J_0(\zeta x)/F_0(\zeta)\}\,\zeta\,d\zeta\right] = C_0$$

and

$$\lim_{x\to 0}\left[x\int_0^\lambda \{\surd(\zeta^2-1)\,J_0(\zeta x)/F_0(\zeta)\}\,\zeta\,d\zeta\right] = 0,$$

whence

$$\lim_{x\to 0}\{xI(x,\mu)\} = -iC_0 = \frac{i}{2(\mu^2-1)}. \tag{49}$$

6. Asymptotic evaluation of the mutual impedance integral

The integral

$$I(x,\mu) = -i\int_0^\infty \frac{\surd(\zeta^2-1)\,J_0(\zeta x)\,\zeta\,d\zeta}{F_0(\zeta)}$$

is among those discussed by Lamb (1904) in determining the field at infinity on the free surface of a semi-infinite isotropic solid due to a point source vibrating normally to the surface. Lamb evaluated a term representing the contribution to the integral from the surface wave, which is of order $x^{-\frac{1}{2}}$, and indicated the order of magnitude of the contributions from the other waves, without, however, giving the dominant terms explicitly.

In order to determine the behaviour of $I(x,\mu)$ for large x, we adapt the methods applied in paper I to integrals of a more general type. The integral may be written

$$I(x,\mu) = -\frac{i}{2}\int_0^\infty \frac{\sqrt{(\zeta^2-1)}}{F_0(\zeta)}\{H_0^{(1)}(\zeta x)+H_0^{(2)}(\zeta x)\}\,\zeta\,d\zeta,$$

where $H_0^{(1)}(\zeta x)$ and $H_0^{(2)}(\zeta x)$ are the Hankel functions.

Using the relation $$H_0^{(2)}(\zeta x) = -H_0^{(1)}(e^{i\pi}\,\zeta x)$$

(see, for example, Watson 1922, p. 75), we obtain

$$I(x,\mu) = -\frac{i}{2}\int_{-\infty}^{\infty} \frac{\sqrt{(\zeta^2-1)}}{F_0(\zeta)}H_0^{(1)}(\zeta x)\,\zeta\,d\zeta, \tag{50}$$

where the new contour is indented to pass below the singularities of the integrand on the negative real axis and above those at the origin and on the positive real axis (cf. figure 1 of paper I).

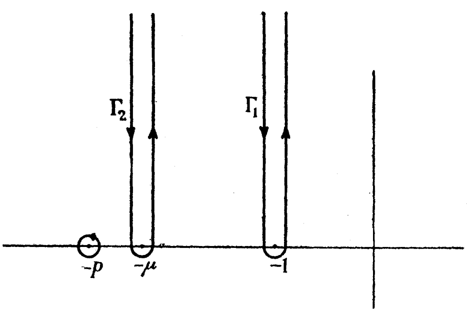

FIGURE 2. Modified contour of integration.

An equivalent contour consists of a small circle enclosing the pole of $1/F_0(\zeta)$ at $\zeta=-p$ and two loops Γ_1 and Γ_2 encircling the points $\zeta=-1$ and $\zeta=-\mu$ respectively, as shown in figure 2. The value of the integral taken round the circle, which corresponds to the surface wave displacement and will be denoted by I_{Su}, is a multiple of the residue of the integrand at the pole $\zeta=-p$, and is given by

$$I_{Su} = -\frac{\pi p\sqrt{(p^2-1)}\,H_0^{(1)}(e^{i\pi}px)}{F_0'(-p)} = -\frac{\pi p\sqrt{(p^2-1)}\,H_0^{(2)}(px)}{F_0'(p)}. \tag{51}$$

In the case $\mu=\sqrt{3}$, $p=1{\cdot}8839$ and we have

$$I_{Su} = 0{\cdot}3620H_0^{(2)}(1{\cdot}8839x). \tag{52}$$

For the purpose of evaluating the integrals along Γ_1 and Γ_2, it is convenient to write

$$\frac{\sqrt{(\zeta^2-1)}}{F_0(\zeta)} = \Phi_1(\zeta)+\Phi_2(\zeta),$$

where

$$\Phi_1(\zeta) = \frac{(2\zeta^2-\mu^2)^2\sqrt{(\zeta^2-1)}}{(2\zeta^2-\mu^2)^4-16\zeta^4(\zeta^2-1)(\zeta^2-\mu^2)},$$

$$\Phi_2(\zeta) = \frac{4\zeta^2(\zeta^2-1)\sqrt{(\zeta^2-\mu^2)}}{(2\zeta^2-\mu^2)^4-16\zeta^4(\zeta^2-1)(\zeta^2-\mu^2)},$$

thus separating the integrand into two terms each of which possesses only one branch-point in the half-plane $\mathscr{R}\zeta < 0$. The loop integrals, which represent surface displacements corresponding to compressional and shear waves respectively, then become

$$I_C = -\frac{\mathrm{i}}{2}\int_{\Gamma_1} H_0^{(1)}(\zeta x)\,\Phi_1(\zeta)\,\zeta\,\mathrm{d}\zeta, \quad I_{Sh} = -\frac{\mathrm{i}}{2}\int_{\Gamma_2} H_0^{(1)}(\zeta x)\,\Phi_2(\zeta)\,\zeta\,\mathrm{d}\zeta,$$

and if the Hankel function is replaced by its asymptotic expansion

$$H_0^{(1)}(\zeta x) \sim \left(\frac{2}{\pi\zeta x}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}(\zeta x - \frac{1}{4}\pi)}\left(1 - \frac{\mathrm{i}}{8\zeta x} + \ldots\right),$$

the problem reduces to the evaluation of integrals of the form

$$-\frac{\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}}{(2\pi)^{\frac{1}{2}}}\int_{\Gamma_1}\frac{\mathrm{e}^{\mathrm{i}\zeta x}}{(\zeta x)^{m+\frac{1}{2}}}\Phi_1(\zeta)\,\zeta\,\mathrm{d}\zeta \quad \text{and} \quad -\frac{\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}}{(2\pi)^{\frac{1}{2}}}\int_{\Gamma_2}\frac{\mathrm{e}^{\mathrm{i}\zeta x}}{(\zeta x)^{m+\frac{1}{2}}}\Phi_2(\zeta)\,\zeta\,\mathrm{d}\zeta,$$

where m is zero or a positive integer.

To evaluate integrals of the first type, we deform Γ_1 in such a way that both its branches coincide with the line $\mathscr{R}\zeta = -1$ and make the substitution $\zeta = -1 + \mathrm{i}\eta_1$. For small values of η_1 we obtain

$$\Phi_1 = \pm\sqrt{(2\eta_1)}\frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{(2-\mu^2)^2}\{1 + O(\eta_1)\}, \tag{53}$$

where the positive sign is to be taken on the left-hand side of the loop Γ_1 and the negative sign on the right-hand side, to correspond with the assignment of principal values to the radicals $\sqrt{(\zeta^2-1)}$ and $\sqrt{(\zeta^2-\mu^2)}$ on the real ζ-axis (see § 3 of paper I).

Hence for the dominant term in the asymptotic expansion of I_C we obtain

$$I_C \sim -\frac{2\,\mathrm{e}^{-\mathrm{i}x}}{\sqrt{\pi}(2-\mu^2)^2}\int_0^\infty \mathrm{e}^{-\eta_1 x}\left(\frac{\eta_1}{x}\right)^{\frac{1}{2}}\mathrm{d}\eta_1 = -\frac{\mathrm{e}^{-\mathrm{i}x}}{(2-\mu^2)^2 x^2}. \tag{54}$$

The same method can be used to evaluate I_{Sh} and the corresponding results are

$$\Phi_2 = \pm\sqrt{(2\mu\eta_2)}\frac{4(\mu^2-1)\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\mu^6}\{1 + O(\eta_2)\}, \tag{55}$$

where $\mathrm{i}\eta_2 = \zeta + \mu$, and

$$I_{Sh} \sim -\frac{8(\mu^2-1)\,\mathrm{e}^{-\mathrm{i}\mu x}}{\sqrt{\pi}\,\mu^5}\int_0^\infty \mathrm{e}^{-\eta_2 x}\left(\frac{\eta_2}{x}\right)^{\frac{1}{2}}\mathrm{d}\eta_2 = -\frac{4(\mu^2-1)\,\mathrm{e}^{-\mathrm{i}\mu x}}{\mu^5 x^2}. \tag{56}$$

Summarizing, we have shown that

$$I(x,\mu) = I_{Su} + I_C + I_{Sh}, \tag{57}$$

where the terms on the right-hand side are given asymptotically by formulae (51), (54) and (56).

It should be noted that, owing to the smallness of the radii of convergence of the ascending series (53) and (55) for Φ_1 and Φ_2, the constants implicit in the error terms in (54) and (56) are rather large. The application of these formulae is therefore restricted to cases when x is very large.

7. Application to a three-element radiator

To illustrate the application of the theory presented in §§ 4 and 5, we consider the case of a radiator consisting of three elements situated at the vertices of an equilateral triangle, with $\mu = \surd 3$ and $k_1 s_{ij} = \frac{1}{2}$ (so that $k_1 b = \frac{1}{6}\surd 3$). This arrangement corresponds closely to that in a recent geophysical experiment described by Evison (1951). In this instance, expressions (39), (40) and (41) become

$$W_C = \frac{\pi^3 \nu^2 a^4 \mu^4 P_0^2}{\rho V_C^3} \int_0^{\frac{1}{2}\pi} \{\Theta_1(\theta)\}^2 \{3 + 6J_0(\tfrac{1}{2}\sin\theta)\} \sin\theta \, d\theta, \tag{58}$$

$$W_{Sh} = \frac{\pi^3 \nu^2 a^4 \mu^9 P_0^2}{\rho V_C^3} \int_0^{\frac{1}{2}\pi} |\,\Theta_2(\theta)\,|^2 \{3 + 6J_0(\tfrac{1}{2}\mu\sin\theta)\} \sin\theta \, d\theta, \tag{59}$$

$$W_{Su} = \frac{3\cdot257\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3} \{3 + 6J_0(0\cdot9419)\}. \tag{60}$$

The integrals have been evaluated by numerical quadrature, and we find that

$$W_C = 2\cdot945 \frac{\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3}, \quad W_{Sh} = 10\cdot415 \frac{\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3}, \quad W_{Su} = 25\cdot21 \frac{\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3}. \tag{61}$$

The total radiated power is therefore given by

$$W = W_C + W_{Sh} + W_{Su} = 38\cdot57 \frac{\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3}. \tag{62}$$

To calculate the total radiated power by the admittance method we use the expression (47) above. Thus

$$W = \frac{\pi^3 \nu^2 a^4 \mu^4 P_0^2}{\rho V_C^3} \{3f(0) + 6f(0\cdot5)\},$$

and with the aid of table 1 we find that

$$W = 38\cdot58 \frac{\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3}, \tag{63}$$

which is in close agreement with (62) above.

In the geophysical experiment referred to above the only useful power was that radiated by the compressional wave. Since the surface wave is responsible for most of the wastage of power, it is interesting to compare the result (61) with the corresponding result when the distance between elements is chosen so as to minimize the power in the surface wave.

The terms for which $l - m = \pm 1$ in (41) are minimized by choosing

$$2pk_1 b \sin\tfrac{1}{3}\pi = 3\cdot8317,$$

so that $k_1 b = 1\cdot1743$. In this case we obtain

$$W_C = 2\cdot254 \frac{\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3}, \quad W_{Sh} = 3\cdot173 \frac{\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3}, \quad W_{Su} = 1\cdot899 \frac{\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3}, \tag{64}$$

and, for the total power,

$$W = 7\cdot326 \frac{\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3}. \tag{65}$$

Calculating the total power also by the admittance method we have

$$W = \pi \frac{\pi^3 \nu^2 a^4 \mu^4 P_0^2}{\rho V_C^3} \{3f(0) + 6f(2\cdot 0339)\} = 7\cdot 328 \frac{\pi^3 \nu^2 a^4 P_0^2}{\rho V_C^3}, \tag{66}$$

in close agreement with (65).

Table 2 shows the percentages of power in the three waves for the various cases considered.

TABLE 2

	compressional wave (%)	shear wave (%)	surface wave (%)
single-element radiator	6·9	25·8	67·4
three-element radiator ($k_1 b = \frac{1}{6}\sqrt{3}$)	7·7	27·0	65·4
three-element radiator ($k_1 b = 1\cdot 1743$)	30·8	43·3	25·9

8. APPLICATION TO RADIATORS WITH CONSTANT DISPLACEMENT

We conclude by giving a further example of the application of the mutual admittance function defined in § 5.

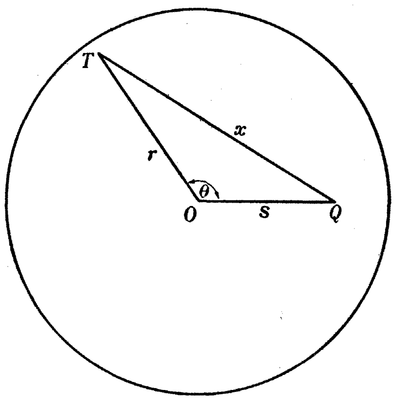

FIGURE 3

The difficulty of a rigorous treatment of a given physical problem, taking complete account of the elastic behaviour of both the semi-infinite medium and the source, compels us to make some assumption concerning the distribution of either stress or displacement over the area of the free surface beneath each radiating element. For simplicity we have assumed that the stress is constant, but elementary considerations may sometimes indicate the form of the displacement rather than the stress. We therefore consider the problem in which the displacement over a circular radiator is constant or, more generally, an arbitrary function of the radial distance. We shall show that the resulting stress distribution over the radiator can be represented as the solution of an integral equation.

With the notation of figure 3, the normal displacement at an arbitrary point Q due to the stress on an elementary area $\mathrm{d}A$ at a second point T is equal to

$$Y_{TQ} P(r)\, \mathrm{d}A/\mathrm{i}\omega, \tag{67}$$

where $P(r)$ is the stress at T (a function of r alone because of the symmetry of the problem) and Y_{TQ} is the mutual admittance for the points T and Q. Since we are here concerned with a relationship between two *points*, we use the limiting form (44) of the mutual admittance function.

From (44) and (45) we have

$$Y_{TQ} = \frac{2\pi\nu^2\mu^4}{\rho V_C^3} I(k_1 x, \mu),$$

and by integration of (67) it follows that

$$\frac{2\pi\nu^2\mu^4}{\mathrm{i}\omega\rho V_C^3}\int_0^{2\pi} \mathrm{d}\theta \int_0^a I(k_1 x, \mu)\, P(r)\, r\, \mathrm{d}r = D(s), \tag{68}$$

where $D(s) = (u_z)_Q$ is the given resultant displacement at Q and a is the radius of the element.

Some simplification results if we put

$$\frac{2\pi\nu^2\mu^4}{\mathrm{i}\omega\rho V_C^3}\int_0^{2\pi} I(k_1 x, \mu)\,\mathrm{d}\theta = -\frac{4\pi^2\nu^2\mu^4}{\omega\rho V_C^3}\int_0^\infty \frac{\surd(\zeta^2-1)\, J_0(k_1\zeta r)\, J_0(k_1\zeta s)\,\zeta\,\mathrm{d}\zeta}{F_0(\zeta)} = \phi(r, s; \mu),$$

whereupon equation (68) becomes

$$\int_0^a \phi(r, s; \mu)\, P(r)\, r\, \mathrm{d}r = D(s). \tag{69}$$

The problem is thus reduced to the solution of a Fredholm integral equation of the first kind for the stress $P(r)$.

The numerical solution of integral equations of this kind presents difficulties of a fundamental character (see Fox & Goodwin 1953), and the subject is beyond the scope of this paper. The result (69) is, however, of some intrinsic interest, and it is hoped that the approach followed in this section may suggest a possible line of development for future work in this field.

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