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Long wave interaction over varying topography

P. Milewski [1]

Department of Mathematics, University of Wisconsin, Madison, WI 53719, USA

Abstract

The propagation of long waves on the surface of a three-dimensional fluid domain bounded below by slowly varying topography is considered. There are two important limits: If the initial data can be written in terms of a discrete set of one-dimensional wavefronts, the resulting wave field is described by a set of variable coefficient Korteweg–de Vries (KdV) equations for each wave along its characteristic curve. Waves along different characteristics interact with each other yielding phase shifts that depend on the wave amplitudes, the angle between the rays and the local depth. If the initial data is modulated slowly in the direction parallel to the wavefronts, the wave field is described by variable coefficient Kadomtsev–Petviashvili (KP) equations along rays. For topography varying only in one direction, we calculate explicit results for the interaction between two sets of periodic or solitary waves and show the equivalence of a single nearly normally incident KdV wave and a normally incident KP wave. Copyright © 1998 Elsevier Science B.V.

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1. Introduction

In this paper we consider the propagation and interaction of weakly nonlinear three-dimensional waves on the surface of a shallow fluid domain with slowly varying bottom topography. The problem of surface gravity waves on varying topography has been the subject of considerable mathematical and experimental research. The asymptotics of the diffraction of locally sinusoid waves was studied systematically in [13,19]. The diffraction of long nonlinear solitary water waves was first studied in [7,8]. There has also been considerable interest in the related study of slowly modulated solutions to integrable wave equations, and various methods have been developed for this purpose (see, for example, [1,5,14,21]. The interaction of solitary waves in a fluid domain of constant depth has been considered by many authors (see, for example, [2,6,15], and are divided into weak and strong, depending on the angle between wave fronts. Here, Section 2 and 3 are concerned with the weak interaction limit, whereas Section 4 is concerned with strong interaction.

We start from a general isotropic Boussinesq-like equation for weakly nonlinear water waves, and show, using asymptotic analysis: (1) The waves follow the characteristics of the linear problem (Section 2 and [7]). (2) Along these

[1] Tel.: (608) 262–3220; fax: (608) 263–8891; e-mail: milewski@math.wisc.edu.

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characteristics, the waves satisfy either a variable coefficient Korteweg–de Vries (Section 2 and [12]) or Kadomtsev–Petviashvili (Section 4) equation. These can be transformed to perturbed, constant coefficient equations. (3) Where waves from different ray families intersect, the waves interact yielding a nonlocal phase shifts dependent on the wave amplitudes, the angles between the rays and the local depth (Section 2). (4) For topography varying in only one direction (such as a simple beach), there is a correspondence between nearly normally incident KdV waves and normally incident KP waves (Section 5). In addition, the interaction of solitary and cnoidal waves is calculated explicitly for beach-like topography (Section 3). The solitary wave case results in a mach stem-like solution, and the cnoidal wave case results in a modulated hexagonal pattern.

2. Formulation

Waves propagating over topography are characterized by several dimensionless parameters: Nonlinearity is proportional to ϵ, the ratio of the amplitude of the wave to the characteristic depth of the fluid. Dispersion is proportional to the ratio of the depth of the fluid to the length of the waves, and is assumed, in the usual KdV scaling, to be of order $\epsilon^{1/2}$. The topographic forces are proportional to the characteristic slope of the topography, which we denote δ. Lastly, focusing and nonlinear interaction also depends on the curvature of the wavefronts. Implicitly in the formulation below, we assume that the curvature is not larger than δ in Section 2 and not larger than $\epsilon^{1/2}$ in Section 4. In all cases, the topography is varying slowly with respect to the waves, thus $\delta \ll \epsilon$.

Given these scalings, the isotropic Boussinesq equation for waves propagating in an irrotational, inviscid fluid over slowly varying topography is [16]

$$\phi_{tt} - h(\boldsymbol{\xi})\Delta\phi + \epsilon\left[-\frac{h^3}{3}\Delta^2\phi\right] + \epsilon\left[\phi_t\Delta\phi + (\nabla\phi)_t^2\right]$$
$$-\delta\left[\nabla_{\boldsymbol{\xi}} h \cdot \nabla\phi\right] + \mathrm{O}(\epsilon^2) + \mathrm{O}(\epsilon\delta) + \mathrm{O}(\delta^2) = 0. \tag{1}$$

Here, $\phi(x, y, t)$ is the velocity potential at the undisturbed free surface level $z = 0$, and h is the local depth of the fluid. The free-surface elevation η is given by $-\phi_t + \mathrm{O}(\epsilon)$, and the topography depends on the long scale $\boldsymbol{\xi} = (\xi, \eta) = \delta\mathbf{x} = \delta(x, y)$ with $\epsilon \gg \delta \gg \epsilon^2$. We define $\mu = \delta\epsilon^{-1} \ll 1$, then ϵ is a measure of dispersion and nonlinearity (which are balanced in the usual KdV limit) and μ is a measure of the local influence of the topography. We seek a solution to (1) consisting of N waves of the form

$$\phi(\mathbf{x}, t, \mu, \epsilon) = \sum_{j=1}^{N} f_j(\theta_j, \mathbf{X}, \mu) + \epsilon\phi^{(1)}(\mathbf{x}, t, \mu) + \mathrm{O}(\epsilon^2), \tag{2}$$

$$\theta_j = \frac{\Theta_j(\boldsymbol{\xi})}{\delta} - t + \epsilon\psi_j(\mathbf{x}) + \mathrm{O}(\epsilon^2), \qquad \mathbf{X} = \epsilon\mathbf{x}, \qquad \boldsymbol{\xi} = \mu\mathbf{X} = \delta\mathbf{x}, \tag{3}$$

where the various derivatives of f_j are assumed to be O(1) quantities. In order to rewrite the phase shift $\psi_j(\mathbf{x})$ as a function of $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)$, it is useful to define $\theta_j^{(0)} = \delta^{-1}\Theta_j(\boldsymbol{\xi}) - t$, and write

$$\theta_j^{(0)} = \theta_j - \epsilon\psi_j(\boldsymbol{\theta}) + \mathrm{O}(\epsilon^2). \tag{4}$$

For simplicity, we take $N = 2$, but the results are straightforward to generalize. Substituting (2) into (1), yields, at leading order, the eikonal equation

$$\left(\frac{\partial\Theta_j}{\partial\xi}\right)^2 + \left(\frac{\partial\Theta_j}{\partial\eta}\right)^2 = \frac{1}{c^2(\boldsymbol{\xi})}, \tag{5}$$

where $c^2(\boldsymbol{\xi}) = h(\boldsymbol{\xi})$. Denoting the direction of the wave by the unit vector $\mathbf{k}$ and defining $\alpha_j(\boldsymbol{\xi})$, $\beta_j(\boldsymbol{\xi})$ by

$$\frac{1}{c}\mathbf{k}_j(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}}\Theta_j = (\alpha_j(\boldsymbol{\xi}), \beta_j(\boldsymbol{\xi})), \tag{6}$$

Eq. (5) can be solved by introducing characteristics (or rays) parametrized by their arclength $\boldsymbol{\xi}_j = (\xi_j(\Lambda_j), \eta_j(\Lambda_j))$ satisfying

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}\Lambda_j} = c\alpha_j, \qquad \frac{\mathrm{d}\eta_j}{\mathrm{d}\Lambda_j} = c\beta_j. \tag{7}$$

Along these characteristics, the wave vectors satisfy

$$\frac{\mathrm{d}\alpha_j}{\mathrm{d}\Lambda_j} = c\left(\frac{1}{2c^2(\boldsymbol{\xi}_j)}\right)_{\xi}, \qquad \frac{\mathrm{d}\beta_j}{\mathrm{d}\Lambda_j} = c\left(\frac{1}{2c^2(\boldsymbol{\xi}_j)}\right)_{\eta}. \tag{8}$$

The phase then satisfies

$$\frac{\mathrm{d}\Theta_j}{\mathrm{d}\Lambda_j} = \frac{1}{c(\boldsymbol{\xi}_j)}. \tag{9}$$

It is assumed here that the rays do not form caustics, although their treatment, if nonlinear effects are not too strong, would be similar to the limit described subsequently in Section 4 (see also [19] for sinusoidal waves). Recall that the curves Θ_j = constant are the wavefronts, and are everywhere perpendicular to the rays, and that, as in [7], the waves will follow the characteristics of the linear problem.

At $\mathrm{O}(\epsilon)$, we obtain

$$(\partial_t^2 - c^2\Delta)\phi^{(1)} = F^{(1)} \tag{10}$$

with the forcing term given by

$$\begin{aligned} F^{(1)} = {} & 2(\mathbf{k}_1\cdot\mathbf{k}_2 - 1)[((\partial_{\theta_2}\psi_1)(\partial_{\theta_1}f_1))_{\theta_1} + ((\partial_{\theta_1}\psi_2)(\partial_{\theta_2}f_2))_{\theta_2}] \\ & + 2c^2[\nabla_{\boldsymbol{\xi}}\Theta_1\cdot\nabla_X\partial_{\theta_1}f_1 + \nabla_{\boldsymbol{\xi}}\Theta_2\cdot\nabla_X\partial_{\theta_2}f_2] \\ & + \frac{c^2}{3}[\partial_{\theta_1}^4 f_1 + \partial_{\theta_2}^4 f_2] + \frac{3}{c^2}[(\partial_{\theta_1}f_1)(\partial_{\theta_1}^2 f_1) + (\partial_{\theta_2}f_2)(\partial_{\theta_2}^2 f_2)] \\ & + c^{-2}(1 + 2\mathbf{k}_1\cdot\mathbf{k}_2)[(\partial_{\theta_1}f_1)(\partial_{\theta_2}^2 f_2) + (\partial_{\theta_2}f_2)(\partial_{\theta_1}^2 f_1)] \\ & + \mu[c^2(\Delta_{\boldsymbol{\xi}}\Theta_1)\partial_{\theta_1}f_1 + c^2(\Delta_{\boldsymbol{\xi}}\Theta_2)\partial_{\theta_2}f_2 \\ & + (\nabla_{\boldsymbol{\xi}}h\cdot\nabla_{\boldsymbol{\xi}}\Theta_1)\partial_{\theta_1}f_1 + (\nabla_{\boldsymbol{\xi}}h\cdot\nabla_{\boldsymbol{\xi}}\Theta_2)\partial_{\theta_2}f_2]. \end{aligned} \tag{11}$$

Note that $F^{(1)}$ and thus f_j depend on $\mu \ll 1$, and that by the choice $\delta = \epsilon\mu$, $F^{(1)}$ is linear in μ.

Assuming $\phi^{(1)}$ can be written as a function of $\boldsymbol{\theta}$, (10) becomes

$$2(1 - \mathbf{k}_1\cdot\mathbf{k}_2)\partial_{\theta_1\theta_2}\phi^{(1)} = F^{(1)}. \tag{12}$$

The solvability condition that $\phi^{(1)}$ remain bounded in $\boldsymbol{\theta}$ dictates that

$$\begin{aligned} & 2c^2[\nabla_{\boldsymbol{\xi}}\Theta_j\cdot\nabla_X\partial_{\theta_j}f_j] + \frac{c^2}{3}\partial_{\theta_j}^4 f_j + 3c^{-2}(\partial_{\theta_j}f_j)(\partial_{\theta_j}^2 f_j) \\ & \quad = -\mu[c^2(\Delta_{\boldsymbol{\xi}}\Theta_j)\partial_{\theta_j}f_j + (\nabla_{\boldsymbol{\xi}}h\cdot\nabla_{\boldsymbol{\xi}}\Theta_j)\partial_{\theta_j}f_j]. \end{aligned} \tag{13}$$

Introducing the fast variable λ_j along the characteristic with the scaling $\Lambda_j = \mu\lambda_j$, the first term in (13) is the derivative along the ray. Thus (13) becomes

$$\partial_{\theta_j\lambda_j} f_j + \tfrac{3}{2}c^{-3}(\partial_{\theta_j} f_j)(\partial_{\theta_j}^2 f_j) + \tfrac{1}{6}c\partial_{\theta_j}^4 f_j = -\mu\mathcal{H}\partial_{\theta_j} f_j, \tag{14}$$

where

$$\mathcal{H}(\Lambda_j) = \frac{1}{2}c\Delta_{\xi}\Theta_j + c^{-1}\nabla_{\xi}c\cdot\mathbf{k}_j = \frac{1}{2}\left[\nabla_{\xi}\cdot\mathbf{k} + \frac{\mathbf{k}}{c}\cdot\nabla_{\xi}c\right]. \tag{15}$$

Defining $u_j = \partial_{\theta_j} f_j$, each wave satisfies a forced, slowly varying coefficient, KdV equation along the rays:

$$\partial_{\lambda_j}u_j + \frac{3}{2c^3}u_j\partial_{\theta_j}u_j + \frac{c}{6}\partial_{\theta_j}^3 u_j = -\mu\mathcal{H}u_j. \tag{16}$$

Note that u_j is proportional to the horizontal velocity component of the fluid in the direction of the ray, and to the the free surface displacement. One can transform (16) into a constant coefficient equation by

$$u_j = c^4 v_j(\theta_j, \widehat{\lambda}_j), \qquad \frac{\mathrm{d}\widehat{\lambda}_j}{\mathrm{d}\lambda_j} = c, \tag{17}$$

whence,

$$\partial_{\widehat{\lambda}_j}v_j + \frac{3}{2}v_j\partial_{\theta_j}v_j + \frac{1}{6}\partial_{\theta_j}^3 v_j = -\mu\left[\frac{1}{c}\mathcal{H} + 4\frac{\mathbf{k}_j}{c^2}\cdot\nabla_{\xi}c\right]v_j. \tag{18}$$

We now calculate the phase shifts ψ_j due to interactions between these waves. Eq. (10) now reads

$$\begin{aligned}2(1-\mathbf{k}_1\cdot\mathbf{k}_2)\partial_{\theta_1\theta_2}\phi^{(1)} = {} & 2(\mathbf{k}_1\cdot\mathbf{k}_2 - 1)[((\partial_{\theta_2}\psi_1)(\partial_{\theta_1}f_1))_{\theta_1} + ((\partial_{\theta_1}\psi_2)(\partial_{\theta_2}f_2))_{\theta_2}] \\ & + c^{-2}(1+2\mathbf{k}_1\cdot\mathbf{k}_2)[(\partial_{\theta_1}f_1)(\partial_{\theta_2}^2 f_2) + (\partial_{\theta_2}f_2)(\partial_{\theta_1}^2 f_1)].\end{aligned} \tag{19}$$

Assuming all the coefficients are O(1) and integrating this equation one obtains

$$\phi^{(1)} = -[\psi_1\partial_{\theta_1}f_1 + \psi_2\partial_{\theta_2}f_2] + \frac{1+2\mathbf{k}_1\cdot\mathbf{k}_2}{2c^2(1-\mathbf{k}_1\cdot\mathbf{k}_2)}[f_1\partial_{\theta_2}f_2 + f_2\partial_{\theta_1}f_1]. \tag{20}$$

The homogeneous solution has been neglected since it is the sum of functions of one phase, and thus can be incorporated into f_j by a redefinition of the leading order solution. Although one could take $\psi_1 = \psi_2 = 0$ consistently at this order (see [2]), this choice would lead to secular terms at the next order in most cases, including solitary and periodic waves. We use the method of [6,20] to extract the phase shifts at this order of the perturbation. Fixing θ_2 in (20) with $\psi_1 = \psi_2 = 0$, we note that $\phi^{(1)}$ does not tend to zero at infinity and it is easy to see that this leads to secular terms at the next order in $\phi^{(2)}$. Anticipating this, we remove these terms at this order by imposing the stricter solvability condition that $\phi^{(1)} \to 0$ as $\boldsymbol{\theta} \to \infty$ yielding

$$\psi_1 = \frac{1+2\mathbf{k}_1\cdot\mathbf{k}_2}{2c^2(1-\mathbf{k}_1\cdot\mathbf{k}_2)}f_2, \tag{21}$$

$$\psi_2 = \frac{1+2\mathbf{k}_1\cdot\mathbf{k}_2}{2c^2(1-\mathbf{k}_1\cdot\mathbf{k}_2)}f_1. \tag{22}$$

These phase shifts are nonlocal in the sense that they depend on the integral of the free surface displacement of the other wave. If the waves are solitary, then the phase shift in θ_1 of u_1 is felt where u_2 is largest. The nature of this interaction is discussed in more detail in Section 3.

3. Interaction of waves on a gentle beach

We now specialize the results of Section 2 for the case where $h = h(\xi)$ and periodic and solitary wave solutions to (18) are sought. This case can be explicitly computed by the methods introduced by [9,12,14] for perturbations of the KdV equation. Eqs. (7)–(9) read

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}\Lambda_j} = c\alpha_j, \quad \frac{\mathrm{d}\eta_j}{\mathrm{d}\Lambda_j} = c\beta_j, \quad \frac{\mathrm{d}\alpha_j}{\mathrm{d}\Lambda_j} = c\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{2c^2}\right), \quad \frac{\mathrm{d}\beta_j}{\mathrm{d}\Lambda_j} = 0, \quad \frac{\mathrm{d}\Theta_j}{\mathrm{d}\Lambda_j} = \frac{1}{c}. \tag{23}$$

It is preferable here to write the solution as a function of ξ instead of Λ:

$$\beta_j(\xi) = \beta_j^0, \qquad \alpha_j^2(\xi)c^2(\xi) = 1 - (\beta_j^0)^2 c^2(\xi), \tag{24}$$

$$\eta_j(\xi) - \eta_j^0 = \int_{\xi^0}^{\xi} \frac{\beta_j^0}{\alpha_j}\mathrm{d}\xi, \qquad \Theta_j(\xi) - \Theta_j^0 = \int_{\xi^0}^{\xi} \frac{1}{\alpha_j c^2}\mathrm{d}\xi, \tag{25}$$

where superscripts indicate the value at $\Lambda_j = 0$ (or $\xi = \xi^0$). With this solution, (18) becomes

$$\partial_{\widehat{\lambda}_j} v_j + \tfrac{3}{2} v_j \partial_{\theta_j} v_j + \tfrac{1}{6}\partial_{\theta_j}^3 v_j = -\mu[\tfrac{1}{2}\alpha_j' + 5c^{-1}c'\alpha_j]v_j. \tag{26}$$

We seek a solution to (26) of the form

$$v_j = v_j^{(0)} + \mu v_j^{(1)}, \tag{27}$$

where

$$v_j^{(0)} = a_j(\Lambda_j)\,\mathrm{sech}^2\left(\omega_j(\Lambda_j)\left(\theta_j - \frac{1}{\mu}\int_0^{\mu\widehat{\lambda}_j} \gamma_j(s)\,\mathrm{d}s\right)\right), \quad \gamma_j = \frac{\omega_j^2}{6}, \quad a_j = \frac{\omega_j^2}{3}. \tag{28}$$

The analysis here is very similar to [9] and the solvability condition for $v_j^{(1)}$ yields the adiabatic variation of the solitary wave amplitude. We do not repeat that calculation here, although it is equivalent to conservation of energy for (26). The result, regarding a_j as a function of ξ is

$$\frac{1}{a_j}\frac{\mathrm{d}a_j}{\mathrm{d}\xi} = -\frac{4}{3}\left[\frac{1}{2}\frac{\alpha_j'}{\alpha_j} + 5\frac{c'}{c}\right], \qquad \frac{a_j(\xi)}{a_j^0} = c^{-20/3}\alpha_j^{-2/3}. \tag{29}$$

Returning to the physical quantity, with $u_j = c^4 v_j$, we have

$$u_j = A_j(\xi)\,\mathrm{sech}^2\left(\omega_j(\xi)\left(\theta_j - \frac{1}{\mu}\int_{\xi^0}^{\xi} \frac{\gamma_j(s)}{\alpha_j(s)}\,\mathrm{d}s\right)\right), \quad \gamma_j = \frac{\omega_j^2}{6}, \quad A_j = \frac{\omega_j^2 c^4}{3}, \tag{30}$$

where

$$\frac{A_j(\xi)}{A_j^0} = c^{-8/3}\alpha_j^{-2/3} \sim c^{-2} \quad \text{as } c \to 0. \tag{31}$$

In addition, by solving for $v_j^{(1)}$, one finds that there is a small amplitude trailing "tail" behind the solitary wave (see [9,12,14]). We shall only be interested in the leading order calculation here.

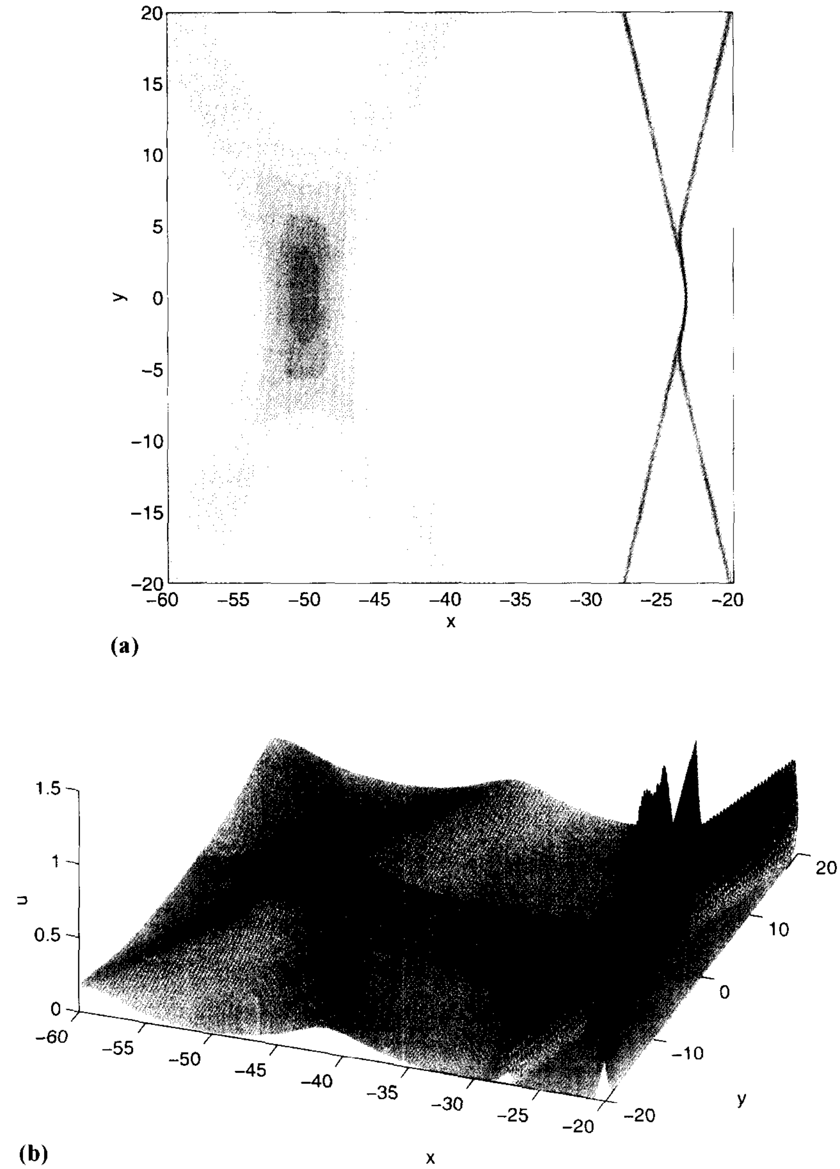

Fig. 1. Solitary waves interacting on a sloping beach $h(x) = -(0.1)x$. The waves propagate from left to right, and the solution is shown at two different times. Initially, $\mathbf{k}_1 \approx (0.9, 0.45)$, $\mathbf{k}_2 \approx (0.9, -0.45)$ (the fronts are at 45° to each other) and the amplitude of each wave is 0.1725. At the later time the interaction is clear at the intersection of the two waves. (a) Grayscale plot. (b) Surface plot. Note: The solutions shown are a local approximation around the interaction point of the solutions of Section 3. In this approximation, the wavefronts are straight, whereas those of Section 3 are curved.

The phase shift due to the interaction of two waves is given by

$$\psi_1 = \frac{1 + 2\mathbf{k}_1 \cdot \mathbf{k}_2}{2c^2(1 - \mathbf{k}_1 \cdot \mathbf{k}_2)} \int u_2 \, \mathrm{d}\theta_2, \tag{32}$$

with a similar expression for ψ_2. The shift is most pronounced at the intersection of the two wavefronts, and the total phase shift is given by

$$\psi_1^{\max} = \frac{1 + 2\mathbf{k}_1 \cdot \mathbf{k}_2}{2c^2(1 - \mathbf{k}_1 \cdot \mathbf{k}_2)} \left(2\frac{A_2(\xi)}{k_2(\xi)} \right) \sim \frac{2\sqrt{3A_2^0}}{(\beta_1^0 - \beta_2^0)^2} c^{-2} \quad \text{as } c \to 0. \tag{33}$$

This, together with (30) and (31) describes the leading order wave field. The sign of ψ_1 corresponds to a retardation of wave 1 as it intersects wave 2. Similarly wave 2 is retarded when it intersects wave 1. Thus, the wavefronts forms a "mach-stem", where both waves superpose for a finite distance. Figs. 1(a) and (b) show, locally, such a solution at two different times. At the later time the "mach-stem" with a large amplitude is clearly visible.

For periodic waves, we similarly can construct the wave field using (23) and (25) and seeking solutions to (26) as in (27) with $v_j^{(0)}$ having mean zero, of the form

$$v_j^{(0)} = \frac{4}{3}\left(\frac{K}{\pi}\right)^2 \omega_j^2 \left[\kappa_j^2 \operatorname{cn}^2 \left(\frac{K}{\pi}\omega_j \left(\theta_j - \frac{1}{\mu}\int_0^{\mu\widehat{\lambda}_j} \gamma_j(s)\,\mathrm{d}s \right), \kappa_j \right) + 1 - \kappa_j^2 - \frac{E}{K} \right], \tag{34}$$

where $\operatorname{cn}(z, \kappa)$ is the elliptic function of modulus κ and $E(\kappa)$, $K(\kappa)$ are the complete elliptic integrals of the first and second kinds, respectively. K is the quarter-period of cn and thus, (34) has period $2\pi/\omega$ in θ. The solution has the dispersion relation

$$\gamma_j = -\frac{2}{3}\left(\frac{K}{\pi}\right)^2 \omega_j^2 \left[\kappa_j^2 + 3\frac{E}{K} - 2 \right]. \tag{35}$$

If we consider waves at a given frequency ω_j^0 generated at some position ξ^0, then, by conservation of waves, we must fix $\omega_j = \omega_j^0$. At ξ^0 the waves have wavelength $2\pi c^0/\omega_j^0$. Then γ_j and the modulus κ_j, which parametrizes the amplitude, depend on $\widehat{\Lambda_j} = \mu\widehat{\lambda_j}$, and thus on ξ. Their dependence is obtained by a solvability condition as follows: The problem for $v_j^{(1)}$ is

$$-\gamma_j \partial_{\theta_j} v_j^{(1)} + \tfrac{3}{2}\partial_{\theta_j}(v_j^{(1)} v_j^{(0)}) + \tfrac{1}{6}\partial_{\theta_j}^3 v_j^{(1)} = -[\tfrac{1}{2}\alpha_j' + 5c^{-1}c'\alpha_j] v_j^{(0)} - \partial_{\widehat{\Lambda}_j} v_j^{(0)}. \tag{36}$$

The linear operator on the left-hand side of (36) has, as its adjoint, the operator $-\gamma\partial_{\theta_j} + \frac{3}{2}v_j^{(0)}\partial_{\theta_j} + \frac{1}{6}\partial_{\theta_j}^3$. This operator has the linearly independent homogeneous periodic solutions (see [10]) 1, $v_j^{(0)}$. The right-hand side of (36) is orthogonal to the constant function since $v_j^{(0)}$ has mean zero, but imposing orthogonality with respect to $v_j^{(0)}$ yields the solvability condition

$$\partial_\xi \int_0^{2\pi/\omega_j} \frac{1}{2}(v_j^{(0)})^2\,\mathrm{d}\theta = -\left[\frac{1}{2}\frac{\alpha_j'}{\alpha_j} + 5\frac{c'}{c} \right] \int_0^{2\pi/\omega_j} (v_j^{(0)})^2\,\mathrm{d}\theta. \tag{37}$$

Again, (37) is the statement of conservation of energy for (26) to this order. Thus, defining $\mathcal{E}_j = \frac{1}{2}\int_0^{2\pi/\omega_j}(v_j^{(0)})^2\,\mathrm{d}\theta$, we have, in terms of κ_j,

$$\mathcal{E}_j = \frac{8\pi K}{\kappa_j \omega_j^0}\left(\frac{2}{3}\right)^2 \left(\frac{K}{\pi}\right)^4 (\omega_j^0)^4 \left[\frac{1}{3}\left((2 - 3\kappa_j^2)(1 - \kappa_j^2) + (2\kappa_j^2 + 1)\frac{E}{K} \right) - \left(1 - \kappa_j^2 - \frac{E}{K} \right)^2 \right]. \tag{38}$$

The solution to (37) is then

$$\frac{\mathcal{E}_j(\xi)}{\mathcal{E}_j^0} = \alpha_j^{-1} c^{-10}, \tag{39}$$

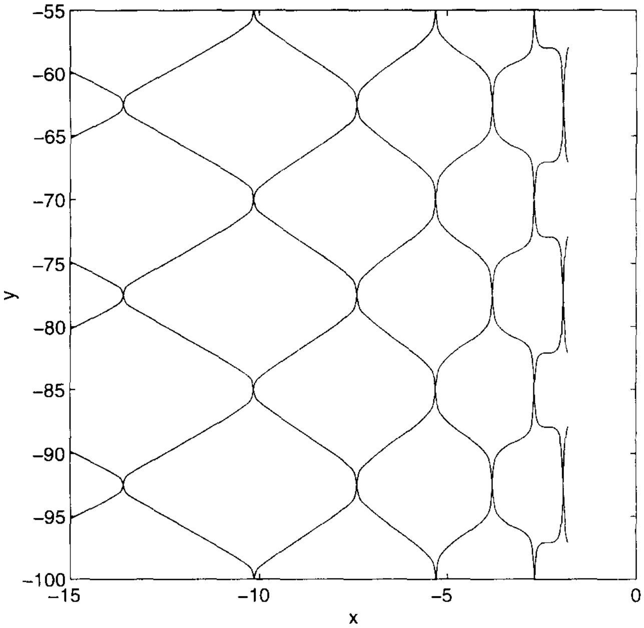

Fig. 2. Figure indicating the crests of periodic waves interacting on a sloping beach $h(x) = -(0.1)x$. The waves propagate from left to right, and at each intersection of the crests there is a phase shift. This figure was obtained by considering a periodic train of solitons instead of cnoidal waves. The near shore pattern is hexagonal.

and the phase shifts are given by (32). The phase shift has mean zero, but, since the waves have sharp crests and flat troughs, the interaction is most pronounced at the intersection of the crests. Near the beach since, as before, $\psi_j \sim c^{-2}$. The solutions here appear like periodic tilings of the wavefronts shown in Fig. 1, as shown in Fig. 2. The result is a modulated array of hexagons whose aspect ratio and wave segments parallel to the beach grow as the waves approach the beach. Similar patterns have been observed in the Ocean [17].

4. Strong interaction and the KP equation

In the previous sections we assumed that the waves were locally one-dimensional. If the waves are slowly modulated in the direction parallel to the wavefronts on a length scale shorter than the topographic scales, they may be described by a modified KP equation. To illustrate the introduction of another scale, consider the initial value problem for one wave in (1). Suppose $\phi(\mathbf{x}, 0, \epsilon) = g(\mathbf{x}, \epsilon)$. Then, taking the gradient of (2)

$$\nabla g = (\nabla_{\boldsymbol{\xi}}\Theta)(\partial_\theta f) + \mathrm{O}(\epsilon). \tag{40}$$

This means that the form (2) of the solution implies that the initial conditions can have O(1) variations only along rays. All other variation must be $\mathrm{O}(\epsilon)$. Another distinguished limit of the asymptotic expansion of the solution of (1) is the case when the initial data is allowed to vary on a length scale of $\mathrm{O}(\epsilon^{1/2})$ perpendicular to the rays (i.e. a modulation along the wavefronts).

Alternatively, when two waves interact strongly, the free surface evolution is also described by a modified KP equation. From (22) one can see that if $1 - \mathbf{k}_1 \cdot \mathbf{k}_2$ becomes small then the original asymptotic expansion is no longer valid. If $\nu(\boldsymbol{\xi})$ is the angle between the waves, $1 - \cos\nu = 1 - \mathbf{k}_1 \cdot \mathbf{k}_2$. Thus, when $1 - \cos\nu(\boldsymbol{\xi}) = \mathrm{O}(\epsilon)$, (i.e. the waves are almost parallel since $\nu(\boldsymbol{\xi}) = \mathrm{O}(\epsilon^{1/2})$), one must introduce a new expansion. The two waves are combined into one wave with $O(\epsilon^{1/2})$ modulation along the wavefront. We note that in the case of Section 3, this procedure is more involved since not only $1-\cos\nu(\boldsymbol{\xi}) = O(\epsilon)$, but $A = O(\epsilon^{-1})$ and thus the waves are no longer in the weakly nonlinear regime.

With this as motivation, the appropriate expansion in this case is

$$\phi(\mathbf{x}, t, \mu, \epsilon) = \sum_{j=1}^{N} f_j(\theta_j, \theta_j^{\perp}, \mathbf{X}, \mu) + \epsilon\phi^{(1)}(\mathbf{x}, t, \mu) + O(\epsilon^2), \tag{41}$$

where

$$\theta_j = \frac{\Theta_j(\boldsymbol{\xi})}{\delta} - t + \epsilon\psi_j(\mathbf{x}) + O(\epsilon^2), \quad \theta_j^{\perp} = \epsilon^{1/2}\frac{\Theta_j^{\perp}(\boldsymbol{\xi})}{\delta} + O(\epsilon^{3/2}). \tag{42}$$

For such an asymptotic expansion to be valid, the constraint on δ is that $\epsilon \gg \delta \gg \epsilon^{3/2}$. Imposing that the $\theta^{\perp}$ variation is perpendicular to the rays with $\nabla_{\boldsymbol{\xi}}\Theta_j \cdot \nabla_{\boldsymbol{\xi}}\Theta_j^{\perp} = 0$, the dependence of the solution on $\theta^{\perp}$ introduces into (11) the additional term

$$F^{(1)\perp} = c^2[(\partial_\xi\Theta_j^{\perp})^2 + (\partial_\eta\Theta_j^{\perp})^2]\partial^2_{\theta_j^{\perp}} f_j. \tag{43}$$

Here, we may choose, by an appropriate parametrization of the wavefronts

$$(\partial_\xi\Theta_j^{\perp})^2 + (\partial_\eta\Theta_j^{\perp})^2 = 1. \tag{44}$$

Thus, Eq. (14) becomes

$$\partial_{\theta_j\lambda_j} f_j + \tfrac{3}{2}c^{-3}(\partial_{\theta_j} f_j)(\partial^2_{\theta_j} f_j) + \tfrac{1}{6}c\partial^4_{\theta_j} f_j + \tfrac{1}{2}c\partial^2_{\theta_j^{\perp}} f_j = -\mu\mathcal{H}(\Lambda_j)\partial_{\theta_j} f_j. \tag{45}$$

With the introduction of the free surface displacement $u_j = \partial_{\theta_j} f_j$ and differentiating (45) one obtains the KP equation

$$\partial_{\theta_j}[\partial_{\lambda_j} u_j + \tfrac{3}{2}c^{-3}u_j(\partial_{\theta_j} u_j) + \tfrac{1}{6}c\partial^3_{\theta_j} u_j] + \tfrac{1}{2}c\partial^2_{\theta_j^{\perp}} u_j = -\mu\mathcal{H}(\Lambda_j)\partial_{\theta_j} u_j. \tag{46}$$

A transformation similar to (17) (with the $\theta_j^{\perp}$ dependence unchanged) gives

$$\partial_{\theta_j}\left[\partial_{\widehat{\lambda}_j} v_j + \frac{3}{2}v_j(\partial_{\theta_j} v_j) + \frac{3}{2}\partial^3_{\theta_j} v_j\right] + \frac{1}{2}\partial^2_{\theta_j^{\perp}} v_j = -\mu\left[\frac{1}{c}\mathcal{H} + 4\frac{\mathbf{k}_j}{c^2}\cdot\nabla_{\boldsymbol{\xi}} c\right]\partial_{\theta_j} v_j. \tag{47}$$

5. KP waves on normally incident rays

We now study more carefully the case of a single family of normally incident rays in the KP regime, that is, taking $h(\xi)$, and $\beta = 0$. We emphasize that this does not correspond to normally incident waves, since the KP equation has two spatial dimensions and, in particular, allows for oblique KdV-like solutions. The goal is to show that oblique traveling wave solutions to KP are, to the order considered here, equivalent to almost-normally incident traveling KdV waves.

For normally incident rays, from (23), and (25) we have $\lambda = X$. Defining $\widehat{X}$ according by $\mathrm{d}\widehat{X}/\mathrm{d}X = c$, $\widehat{\xi} = \mu\widehat{X}$, and $\widehat{Y} = \theta^{\perp} = \epsilon^{1/2}y$, 46 for $v(\theta, \widehat{X}, \widehat{Y}) = c^{-4}u(\theta, X, \widehat{Y})$ becomes

$$\partial_\theta\left[\partial_{\widehat{X}} v + \frac{3}{2}v\partial_\theta v + \frac{1}{6}\partial^3_\theta v\right] + \frac{1}{2}\partial^2_{\widehat{Y}} v = -\mu\frac{9}{2c}\frac{\mathrm{d}c}{\mathrm{d}\widehat{\xi}}\partial_\theta v. \tag{48}$$

The KP equation and similarly (48) has a rotation-like symmetry since it is invariant under the transformation

$$\widehat{X} \to \widehat{X}, \qquad \widehat{Y} \to \widehat{Y} + \sigma \widehat{X}, \qquad \theta \to \widehat{\theta} - \sigma \widehat{Y} - \tfrac{1}{2}\sigma^2 \widehat{X}. \tag{49}$$

For small angles (σ small) the transformation 49 approaches a rotation. Also, $\widehat{Y}$ independent, zero mean solutions of KP are solutions of the KdV equation. Thus, we may generate an oblique traveling solution of 48 by seeking a solution to KP of the form

$$v = F\left(\theta - \frac{1}{\mu}\int_0^{\mu\widehat{X}} \gamma \,\mathrm{d}s\right), \tag{50}$$

$$\theta = \frac{1}{\delta}(\Theta - \Theta_0) = \int_{\xi^0}^{\xi} \frac{1}{c}\,\mathrm{d}\xi, \tag{51}$$

where $F(\theta, \widehat{X})$ is a solution of KdV, and rotating it with (49).

We wish to compare (50) to solutions of (26), with $\beta = \mathrm{O}(\epsilon^{1/2})$, of the form

$$v = F\left(\theta - \frac{1}{\mu}\int_{\xi^0}^{\xi} \frac{\gamma}{\alpha}\,\mathrm{d}s\right), \tag{52}$$

$$\theta = \frac{1}{\delta}(\Theta - \Theta_0) = \int_{\xi^0}^{\xi} \frac{1}{c}\,\mathrm{d}\xi + \frac{1}{2}\beta^2 \int_{\xi^0}^{\xi} c\,\mathrm{d}\xi + \mathrm{O}(\beta^4), \tag{53}$$

$$\frac{1}{\mu}\int_{\xi^0}^{\xi} \frac{\gamma}{\alpha}\,\mathrm{d}s = \frac{1}{\mu}\int_{\xi^0}^{\xi} \gamma c\,\mathrm{d}s + \mathrm{O}(\beta^2), \tag{54}$$

where the ray is at

$$y = \frac{1}{\delta}(\eta - \eta_0) = \beta \int_{\xi^0}^{\xi} c\,\mathrm{d}\xi + \mathrm{O}(\beta^3). \tag{55}$$

Choosing $\sigma = -\epsilon^{-1/2}\beta$ in (49) and substituting into (50) yields

$$v = F\left(\theta + \beta y - \frac{1}{2}\beta^2 \frac{1}{\delta}\int_{\xi^0}^{\xi} c\,\mathrm{d}\xi - \frac{1}{\mu}\int_{\xi^0}^{\xi} \gamma c\,\mathrm{d}s\right). \tag{56}$$

Upon inserting (53) and (54) into (52) and inserting (55) and (51) into (56), the equivalence of the two to $\mathrm{O}(\beta^2)$ is shown. We note that this is shown with $\sigma = -\epsilon^{-1/2}\beta$ fixed (not a function of ξ). This result indicates that for almost normally incident waves, one should study the simpler case of KP waves on normally incident rays. A further application of this result is in the modulation of multi-phase solutions of KP (see Section 6). In that case, if the original waves are nearly normally incident, one should study the normally incident problem.

6. Conclusions

The diffraction and interaction of weakly nonlinear waves on variable topography has been studied, and there are some questions that warrant further research. One would like to extend the modulation results of Section 3 to multi-phase solutions of the KP equations. In the simplest case, one can study the adiabatic variation of normally incident two phase solutions of KP. Two phase solutions of KP are usually written in terms of θ functions and correspond, in most cases, to two-dimensional periodic traveling waves [4,17]. These solutions have compared well with experimental measurements of waves in water of constant depth [11], and comparisons have been attempted locally in water of non-constant depth [3].

For (48) with $\mu = 0$, two phase solutions have the form $v = f(\phi_1, \phi_2)$ where f is periodic with period 2π in both phases and $\phi_j = \omega_j(\theta + \sigma_j \widehat{Y} - \gamma_j \widehat{X})$ There are three relations between the nine dynamically significant parameters: $\omega_1, \omega_2, \sigma_1, \sigma_2, \gamma_1, \gamma_2$ and three "amplitude" parameters (related to the three elements of the symmetric Riemann matrix in [18]). For the problem where waves of given frequencies are generated offshore we should fix ω_1, ω_2, while Section 5 indicates that σ_1, σ_2 should be fixed. This leaves five free parameters, with three equations relating them. Thus, for the problem (48), with $\mu \neq 0$ and $v = f(\phi_1, \phi_2, \widehat{\xi})$, where $\phi_j = \omega_j(\theta + \sigma_j \widehat{Y} - \int^{\widehat{\xi}} \gamma_j \mathrm{d}s)$ one needs two solvability conditions to completely constrain the evolution of the waves. Similarly to the periodic problem in Section 3 one can easily obtain the condition

$$\partial_{\widehat{\xi}} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{2} f^2 \,\mathrm{d}\phi_1 \,\mathrm{d}\phi_2 = -\frac{9c'}{2c} \int_0^{2\pi} \int_0^{2\pi} f^2 \,\mathrm{d}\phi_1 \,\mathrm{d}\phi_2. \tag{57}$$

At present the method to obtain the second solvability condition is unclear (using, for example, [1]), although physically it seems that there should be a statement of conservation of energy for each phase.

An interesting observation of ([3]) is that even though the data was gathered for waves that had broken far offshore, their evolution seemed remarkably similar to multi-phase KP solutions. Obviously, the various asymptotic equations discussed here do not admit breaking waves, although some other criteria may be applied to the waves to model breaking. Nevertheless, it seems that the qualitative results may apply past breaking.

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