Gravity Wave Reflection at a Discontinuity in Bottom Slope

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ABSTRACT

The one-dimensional reflection of a gravity wave at a discontinuity in bottom slope is calculated from a Green–Liouville (WKB) solution of the mild-slope equation.

1. Introduction

We consider here the one-dimensional reflection of the gravity wave

$$\zeta(x, t) = \operatorname{Re}\{Z(x)e^{-i\omega t}\} \tag{1}$$

of complex amplitude $Z(x)$ and frequency ω in water of gradually varying depth $h(x)$ on the assumptions that h is continuous, $h' \equiv dh/dx$ may be discontinuous, and

$$k|Z| \ll 1, \quad \epsilon \equiv \max|h'/kh| \ll 1, \tag{2a,b}$$

where $k(x)$ is the wavenumber determined by the dispersion relation

$$k \tanh kh = \omega^2/g \equiv \kappa. \tag{3}$$

We also assume that h is constant in $x < x_0$ and $x > x_1$, that the wave is incident with the assigned amplitude A_0 from $x < x_0$, and that there is no reflection from $x > x_1$, and posit

$$Z = A_0[e^{ik_0(x-x_0)} + R_0 e^{-ik_0(x-x_0)}] \quad (x < x_0) \tag{4a}$$

and

$$Z = A_1 e^{ik_1(x-x_1)} \quad (x > x_1), \tag{4b}$$

where $k_{0,1} \equiv k(x_{0,1})$, and the reflection coefficient R_0 and the amplitude A_1 are to be determined. A complementary solution for a wave incident from $x > x_1$ may be constructed by analogy with, and superimposed on, (4). Moreover, $x_0 \downarrow -\infty$ or $x_1 \uparrow \infty$ or both are admissible.

Our model is, to be sure, idealized, but it serves to illustrate that the reflection induced by discontinuities in slope is quite small, and hence that the geometrical-optics approximation (Keller 1958) for wave propagation over bottom topography has a wider range of validity than otherwise might have been expected.

2. Mild-slope equation

The assumptions (1) and (2), together with the neglect of dissipation, lead to the mild-slope equation (Mei 1983, section 3.5), which we cast in the form

$$\frac{d}{d\theta}\left(C\frac{dZ}{d\theta}\right) + CZ = 0, \tag{5}$$

where

$$\theta = \int_{x_0}^{x} k(x)dx, \quad C \equiv \frac{d\omega}{dk} = \frac{1}{2}\frac{\omega}{k}\left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{6a,b}$$

is the group velocity, k is determined by (3), and an error factor of $1 + O(\epsilon^2)$ is implicit. The Green–Liouville (or WKB) solution of (5), subject to (4), yields [cf. Mei's (1983, section 4.5) treatment of the corresponding shallow-water problem]

$$Z = A_0(C_0/C)^{1/2}[e^{i\theta} + R(x)e^{-i\theta} + O(\epsilon^2)],$$

$$R(x) = \frac{1}{2}\int_{x_1}^{x} e^{2i\theta(\xi)}\frac{C'(\xi)}{C(\xi)}d\xi, \tag{7a,b}$$

$$A_1 = A_0(C_0/C_1)^{1/2}e^{i\theta_1}, \quad R_0 = R(x_0). \tag{8a,b}$$

Note that $R = O(\epsilon)$, $R = R_0$ in $x < x_0$, and $R = 0$ in $x > x_1$ (by virtue of $C' = 0$ in $x < x_0$ or $x > x_1$). It remains true, as in the geometrical-optics approximation (Keller 1958), that the mean energy flux is constant within $1 + O(\epsilon^2)$:

$$\frac{\partial}{\partial x}\left(\frac{1}{2\pi}\int_0^{2\pi} CEd\theta\right) = 0, \quad E \equiv \frac{1}{2}g|Z|^2. \tag{9a,b}$$

Oblique incidence may be accommodated by adding $ik_y y$ to the exponent in (1) and replacing k by $k_x \equiv (k^2 - k_y^2)^{1/2}$ in (6a); C by $C_x \equiv k_x C/k$ in (5), (7), and (8); and $k_{0,1}$ by $(k_x)_{0,1}$ in (4).

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3. Discontinuous bottom slope

We now suppose that h' is discontinuous at the end points x_0 and x_1, integrate (7b) by parts, and neglect $O(\epsilon^2)$ to obtain

$$R_0 = -\left.\frac{e^{2i\theta}C'}{4ikC}\right|_{x_0+}^{x_1-} = ih'F(\kappa h)e^{2i\theta}|_{x_0+}^{x_1-},$$

$$F(\kappa h) = \frac{(1-\kappa h)\sinh 2kh}{(2kh+\sinh 2kh)^2}; \qquad (10a,b)$$

$F(\kappa h)$, as determined from (3) and (10b), is plotted in Fig. 1. It has a minimum of -0.06 at $\kappa h = 2.0$ and vanishes exponentially as $\kappa h \uparrow \infty$. It follows that reflection is quite small for $\kappa h > 1$ and is likely to be significant only in the shallow water domain, in which

$$F \approx \frac{1}{8}(\kappa h)^{-1/2}\left(\frac{1-\kappa h}{1+\frac{1}{6}\kappa h}\right),$$

$$\theta \approx \int_{x_0}^{x}(\kappa/h)^{1/2}\left(1-\frac{1}{6}\kappa h\right)dx \quad (\kappa h \ll 1). \qquad (11a,b)$$

The particular case of constant slope, $h' = \sigma$ in (x_0, x_1), for which (10a) reduces to

$$R_0 = i\sigma\left[F(\kappa h_0) - F(\kappa h_1)\exp\left(i\int_0^l kdx\right)\right],$$

$$l \equiv \frac{h_1 - h_0}{\sigma}, \qquad (12a,b)$$

has been solved by Booij (1983) through numerical integration of (5) for $\kappa h_0 = 0.2$ and $\kappa h_1 = 0.6$. Our approximation to $|R_0|$, which is plotted in Fig. 2, agrees with his result within the resolution of his numerical plot for $\sigma < 0.4$ and is closer to the result calculated from a numerical integration of the full (linear) equations of motion for $0.4 < \sigma < 1$. But, as is evident from (12a), our result diverges for $\sigma \uparrow \infty$, whereas Green's approximation for shallow-water reflection at a discontinuity in depth implies the asymptote $|R_0| \sim (\sqrt{3}-1)/(\sqrt{3}+1) = 0.27$.

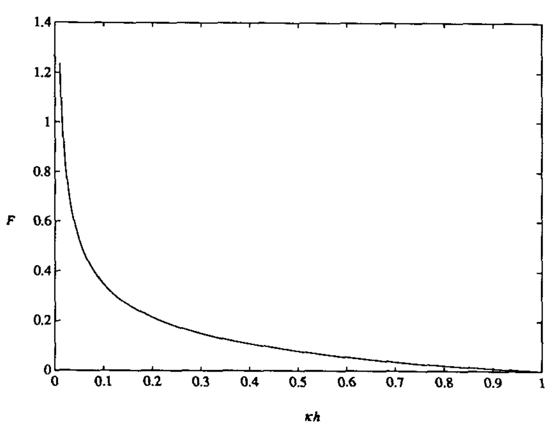

FIG. 1. Plot of $F(\kappa h)$, as determined from (3) and (10b): $F \sim \frac{1}{8}(\kappa h)^{-1/2}$ as $\kappa h \downarrow 0$, and $-0.06 < F < 0$ in $\kappa h > 1$.

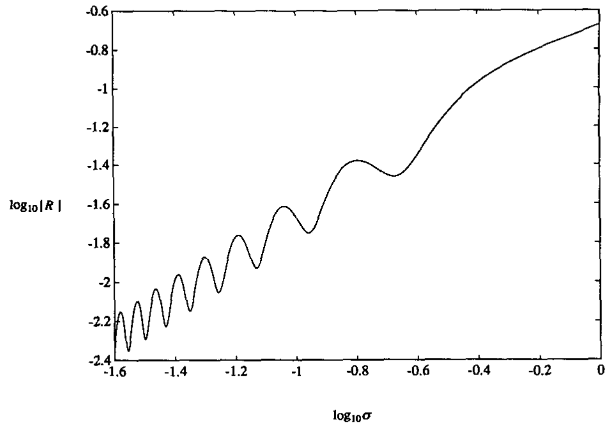

FIG. 2. Plot of $|R_0|$ for a ramp with uniform slope $\sigma \equiv \tan\alpha$ between an upstream depth $h_0 = 0.2/\kappa$ and a downstream depth of $h_1 = 0.6/\kappa$. Note that $|R_0|^2$ is oscillatory with a frequency inversely proportional to σ, but this oscillation is distorted by the logarithmic scale.

We remark that $Z'(x)$, as well as $Z(x)$, is continuous in the present approximation even though h' is discontinuous. But if the $O(\epsilon)$ reflected wave is neglected, (7) reduces to the geometrical-optics approximation, and Z' then exhibits $O(\epsilon)$ discontinuities at the corresponding discontinuities in h' (cf. Lighthill 1948). If h' is continuous at x_0 and x_1, $R_0 = O(\epsilon^2)$ (cf. Mei 1983, p. 140), and the mild-slope equation is inadequate for its determination. However, the geometrical-optics approximation then is adequate for typical applications.

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