On Damped Resonant Interactions

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ABSTRACT

It is shown that the evolution equations for a triad of weakly damped, resonantly interacting waves are isomorphic to the corresponding equations for undamped waves (and therefore may be integrated in terms of elliptic functions) if the damping coefficient is the same for each member of the triad. This condition is satisfied for topographic Rossby waves for which dissipation is through a turbulent Ekman layer and the wavelengths are small compared with the Rossby radius of deformation

1. Introduction

The resonant interaction of a triad of waves of the form

$$\psi_n = A_n(\tau)\cos(\omega_n t - \mathbf{k}_n\cdot\mathbf{x}),\quad n = 1, 2, 3, \tag{1}$$

is governed by the evolution equations

$$\dot{A}_n + \alpha_n A_n + \gamma_n A_{n+1}A_{n-1} = 0,\quad n = 1, 2, 3, \tag{2}$$

where A_n is a slowly (on the scale of $1/\omega_n$) varying amplitude; τ is a slow time; $\dot{A}_n \equiv dA_n/d\tau$; the indices are cyclic, such that $A_4 \equiv A_1$ and $A_0 \equiv A_3$; the frequencies and wave numbers satisfy the resonance conditions

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0,\quad \omega_1 + \omega_2 + \omega_3 = 0 \tag{3a,b}$$

and the dispersion relation

$$\omega_n = \omega(\mathbf{k}_n); \tag{4}$$

the interaction coefficients satisfy

$$\gamma_1 + \gamma_2 + \gamma_3 = 0 \tag{5}$$

for appropriately scaled A_n; α_n is a measure of weak (by hypothesis) damping. If $A_n = O(\epsilon)$, *slow* and *weak* are defined by $\dot{A}_n = O(\epsilon\omega_n A_n)$ and $\alpha_n = O(\epsilon\omega_n)$ as $\epsilon \to 0$.

The α_n are unequal in the general case, but there is an important category of problems for which

$$\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha. \tag{6}$$

For example, $\alpha_n = -2\nu|\mathbf{k}_n|^2$ for capillary–gravity waves on a clean, free surface (McGoldrick, 1965), but (Pedlosky, 1982, after converting to dimensional notation) $\alpha = (A_V f/2D^2)^{1/2}$ is independent of n for quasi-geostrophic planetary (Rossby) waves in a fluid layer of thickness D that rotates about the vertical with the angular velocity $\frac{1}{2}f$ and is subjected to turbulent friction through an Ekman layer (at the bottom) in which the vertical eddy viscosity is A_V, provided that laminar dissipation and turbulent dissipation outside of the Ekman layer are negligible and the wavelength is small compared with the Rossby radius of deformation ($|\mathbf{k}_n|^2 \gg f^2/gD$, which is equivalent to $K_n^2 \gg F$ in Pedlosky's dimensionless notation). I proceed to show that (2) are isomorphic to the corresponding evolution equations for undamped waves if (6) holds.

2. Reduction

I first note that, for the scaling adopted here, an appropriate measure of the energy in the triad is

$$E = \frac{1}{2}(A_1^2 + A_2^2 + A_3^2). \tag{7}$$

Multiplying (2) through by A_n, summing over n, and invoking (5)–(7), we obtain

$$\dot{E} + 2\alpha E = 0, \tag{8}$$

from which it follows that E decays like $\exp(-2\alpha\tau)$. This suggests the change of variable

$$A_n = \hat{A}_n(\hat{\tau})e^{-\alpha\tau},\quad \hat{\tau} = \alpha^{-1}(1 - e^{-\alpha\tau}), \tag{9a,b}$$

which reduces (2) to

$$\frac{d\hat{A}_n}{d\hat{\tau}} + \gamma_n\hat{A}_{n+1}\hat{A}_{n-1} = 0,\quad n = 1, 2, 3. \tag{10}$$

It follows from (10) that the evolution equations (2), subject to (6), are isomorphic to the corresponding equations for undamped waves. The latter may be integrated in terms of elliptic functions (McGoldrick, 1965); accordingly, so also may (2).

3. Arbitrary phases

If (1) is generalized to

$$\psi_n = A_n(\tau)\cos[\omega_n t - \mathbf{k}_n\cdot\mathbf{x} + \phi_n(\tau)], \tag{11}$$

where ϕ_n is, by hypothesis, a slowly varying phase angle, (2) are generalized to

$$\dot{A}_n + \alpha_n A_n + \gamma_n A_{n+1} A_{n-1} \cos\phi = 0, \qquad (12a)$$

$$A_n \dot{\phi}_n - \gamma_n A_{n+1} A_{n-1} \sin\phi = 0, \qquad (12b)$$

where

$$\phi \equiv \phi_1 + \phi_2 + \phi_3. \qquad (13)$$

The simplest solution of (12b) is given by

$$\phi_n = \text{constant}, \quad \sin\phi = 0, \qquad (14a,b)$$

which reduces (12a) to (2), either directly if $\phi = 0$ or after changing the sign of any one of the A_n if $\phi = \pi$.

I have not proved that the solution (14) is stable or that (12) does not admit other solutions; however, independently of the solution for ϕ_n but subject to (6), the transformation (9) reduces (12) to

$$\frac{d\hat{A}_n}{d\hat{\tau}} + \dot{\gamma}_n \hat{A}_{n+1} \hat{A}_{n-1} \cos\phi = 0, \qquad (15a)$$

$$\hat{A}_n \frac{d\phi_n}{d\hat{\tau}} - \gamma_n \hat{A}_{n+1} \hat{A}_{n-1} \sin\phi = 0, \qquad (15b)$$

thereby generalizing the preceding isomorphism.

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REFERENCES

McGoldrick, L. F., 1965: Resonant interactions among capillary–gravity waves. *J. Fluid Mech.*, **21**, 305–331.

Pedlosky, Joseph, 1982: *Geophysical Fluid Dynamics*, Springer-Verlag, 3.26 and 4.12.