

# On gravity-wave scattering by non-secular changes in depth

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The reflection of a straight-crested gravity wave by a non-secular perturbation  $h_1(x)$  in depth relative to an otherwise flat bottom of depth  $h_0$  is calculated through an expansion in  $\varepsilon \propto h_1/h_0$ . Explicit results are developed up to second order for the sinusoidal patch  $h_1 = -b \sin(m\pi x/l)$ ,  $0 < x < l$ , and reduced for Bragg resonance. Trapped modes are absent at first order but appear at second order and contribute  $O(\varepsilon^2)/O(\varepsilon^3)$  to the maximum (Bragg-resonant) reflection coefficient for odd/even  $m$ . A third-order approximation that includes the dominant contributions of the third-order components of the resonant peak of the reflection coefficient for large  $m$ , but neglects the trapped modes, predicts resonant peaks that agree with the values measured by Davies & Heathershaw (1984).

## 1. Introduction

Linear gravity waves of velocity potential  $\phi$ , free-surface displacement  $\zeta$  and frequency  $\omega$  in water of ambient depth  $h(x)$  are described by

$$[\phi(x, z, t), \zeta(x, t)] = \text{Re}\{[\Phi(x, z), i(\omega/g)\Phi(x, 0)]e^{-i\omega t}\}, \quad (1.1)$$

where the complex potential  $\Phi$  satisfies

$$\nabla^2 \Phi + \Phi_{zz} = 0 \quad (-h < z < 0), \quad (1.2)$$

$$\Phi_z = \kappa \Phi \quad (z = 0), \quad \Phi_z + \nabla h \cdot \nabla \Phi = 0 \quad (z = -h), \quad (1.3a, b)$$

$$\mathbf{x} \equiv (x, y), \quad \nabla \equiv (\partial_x, \partial_y), \quad \kappa \equiv \omega^2/g, \quad (1.4a-c)$$

and subscripts signify partial differentiation. We seek the solution of (1.2)–(1.4) for

$$h(x) = h_0 + h_1(x) \quad (0 < x < l), \quad h_1 = 0 \quad \text{in } x \leq 0 \quad \text{or } x \geq l, \quad h_1/h_0 = O(\varepsilon), \quad (1.5a-c)$$

through the expansion ( $\partial_y = 0$  throughout the subsequent development)

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \cdots, \quad \Phi_n = \Phi_n(x, z) = O(\varepsilon^n), \quad (1.6a, b)$$

where

$$\Phi_0 = e^{ik_0 x} \cosh[k_0(z + h_0)] \quad (1.7)$$

describes a straight-crested incident wave in water of uniform depth  $h_0$ . The wavenumber  $k_0$  is the positive-real root of the dispersion equation

$$k \tanh kh_0 = \kappa. \quad (1.8)$$

The solution of (1.2)–(1.7) comprises a family of trapped modes, which are

evanescent for  $|x| \gg h_0$ , and a propagated mode of wavenumber  $k_0$  with incident, reflected and transmitted components. The latter yield the asymptotic approximations

$$\Phi(x, z) \sim (e^{ik_0 x} + R e^{-ik_0 x}) \cosh[k_0(z + h_0)] \quad (x/h_0 \downarrow -\infty) \quad (1.9a)$$

$$\text{and} \quad \Phi(x, z) \sim T e^{ik_0 x} \cosh[k_0(z + h_0)] \quad (x/h_0 \uparrow \infty), \quad (1.9b)$$

wherein the reflection and transmission coefficients  $R$  and  $T$  admit the expansions

$$R = R_1 + R_2 + \dots, \quad T = 1 + T_1 + T_2 + \dots, \quad R_n, T_n = O(\varepsilon^n). \quad (1.10a-c)$$

The first-order (truncation at  $n = 1$ ) problem has been solved by Long (1973) for random  $h_1(x)$  and by Davies & Heathershaw (1984) for arbitrary  $h_1(x)$ ; see also Mei (1985) and Kirby (1986). Davies & Heathershaw (1984) compare their solution with measurements for a sinusoidal ripple bed and find that it is adequate for reflection coefficients smaller than about 0.5 but may overestimate reflection for Bragg resonance. They introduce an *ad hoc* 'correction' to represent higher-order effects, but this is superseded by more accurate calculations (Davies, Guazzelli & Belzons 1989; O'Hare & Davies 1992).

I consider here the construction of higher-order approximations and the contributions of the trapped modes. In §2, I obtain (through Fourier transformation) a sequential solution of the boundary-value problems for the  $\Phi_n$ . In §3, I calculate the corresponding reflection and transmission coefficients to second order and then, in §4, consider the example of a sinusoidal patch,  $h = h_0 - b \sin \beta x$  ( $0 \leq x \leq l$ ),  $\beta = m\pi/l$ . The second-order approximation may be inadequate for Bragg resonance ( $2k_0 = \beta$ ) if  $m$  is large and even (in which case  $R_2$  proves to be approximately in quadrature, whereas  $R_3$  is approximately in phase, with  $R_1$ ), and in §5 I develop a third-order approximation to the Bragg-resonant reflection coefficient that is adequate for  $mb/2h_0 = O(1)$  and agrees well with Davies & Heathershaw's (1984) measured resonant peaks for  $m = 4, 8$  and  $20$  (2, 4 and 10 in their notation).

Nonlinearity would alter the linear approximation to  $R$  by  $A \equiv 1 + O(k_0^2 a^2)$ , where  $a$  is the amplitude of the incident wave. This factor must be compared with  $R_3/R_1 = 1 + O(\varepsilon^2)$  for the present (§5) approximation to the Bragg reflection coefficient, which therefore appears to require  $(k_0 a/\varepsilon)^2 \gg 1$  for its validity; however, the coefficient of  $k_0^2 a^2$  in  $A$  presumably is much smaller than that of  $\varepsilon^2$  in  $R_3/R_1$ .

The present approximation, in which the expansion parameter is a measure of the variation in depth, may be compared with the mild-slope (Mei 1983, §3.5) and modified mild-slope approximations (Chamberlain & Porter 1995; Miles & Chamberlain 1998), in which the expansion parameter is a measure of the bottom slope. See O'Hare & Davies (1992) and Suh, Lee & Park (1997) for more extensive lists of references. The present approximation is somewhat simpler, and may be more efficient, than these mild-slope approximations; however, it does not accommodate secular changes in depth, is less efficient than Mei's (1985) asymptotic approximation for a long ( $m \gg 1$ ) ripple bed, and is less powerful than some of the more sophisticated methods cited by O'Hare & Davies (1992) and Suh *et al.* (1997).

## 2. Fourier-transform solution

Substituting (1.6) into (1.2), (1.3a) and the expansion of (1.3b) about  $z = -h_0$ , we obtain the sequence (for  $n = 1, 2, \dots$ )

$$\Phi_{nxx} + \Phi_{nzz} = 0 \quad (-h_0 < z < 0), \quad (2.1)$$

$$\Phi_{nz} = \kappa \Phi_n \quad (z = 0), \quad \Phi_{nz} = -Q_{nx} \quad (z = -h_0), \quad (2.2a, b)$$

where

$$\left. \begin{aligned} Q_1 &= h_1 \Phi_{0x}, & Q_2 &= h_1 \Phi_{1x} - \frac{1}{2} h_1^2 \Phi_{0xz}, \\ Q_3 &= h_1 \Phi_{2x} - \frac{1}{2} h_1^2 \Phi_{1xz} + \frac{1}{6} h_1^3 \Phi_{0xzz}, \dots \end{aligned} \right\} \quad (z = -h_0). \quad (2.3)$$

The  $\Phi_n$  then may be determined sequentially, starting from (1.7) for  $\Phi_0$ .

Guided by Havelock's (1929, §5) treatment of surface-forced gravity waves, we construct the solution of the bottom-forcing problem (in which the  $Q_n(x)$  may be regarded as fluid inputs) posed by (2.1) and (2.2) through the finite Fourier-transform pair

$$\Psi(x, k) = \int_{-h_0}^0 \Phi(x, z) \cos[ik(z + h_0)] dz \quad (2.4a)$$

and

$$\Phi(x, z) = 2 \sum_k K(k) \Psi(x, k) \cos[ik(z + h_0)], \quad (2.4b)$$

where

$$K(k) = \frac{k^2 - \kappa^2}{(k^2 - \kappa^2) h_0 + \kappa}, \quad (2.5)$$

and the summation is over the positive-real root  $k_0$  and the infinite, discrete set of positive-imaginary roots,  $k = i\ell$ , of (1.8). Transforming (2.1), reducing the transform of  $\Phi_{nzz}$  through integration by parts, and invoking (2.2a, b), we obtain

$$\Psi_{nxx} + k^2 \Psi_n = -Q'_n(x), \quad (2.6)$$

the solution of which, subject to finiteness and radiation conditions for  $x \rightarrow \pm \infty$ , is given by (recall that  $Q_n = 0$  outside of  $0 < x < l$ )

$$\Psi_n(x, k) = -\frac{1}{2}(ik)^{-1} \int_0^l Q'_n(\xi) e^{ik|x-\xi|} d\xi. \quad (2.7)$$

We cast the inverse transform of  $\Psi_n$ , as determined by (2.4b), in the Green's-function form

$$\Phi_n(x, z) = \int_0^l G(x - \xi, z) Q'_n(\xi) d\xi \quad (2.8a)$$

$$= \partial_x \int_0^l G(x - \xi, z) Q_n(\xi) d\xi, \quad (2.8b)$$

where (2.8b) follows from (2.8a) through integration by parts and the invocation of  $Q_n(0) = Q_n(l) = 0$  (which follow from  $h_1(0) = h_1(l) = 0$ ),

$$G(x, z) = -\sum_k (ik)^{-1} K(k) e^{ik|x|} \cos[ik(z + h_0)] \quad (2.9a)$$

$$= -(ik_0)^{-1} K_0 e^{ik_0|x|} \cosh[k_0(z + h_0)] + \sum_{\ell} \ell^{-1} K(i\ell) e^{-\ell|x|} \cos[\ell(z + h_0)], \quad (2.9b)$$

and  $K_0 \equiv K(k_0)$ . The  $\ell$  summation comprises the trapped modes, which are evanescent for  $|x| > h_0$ . The  $k_0$  term represents the propagated mode, and the comparison of its asymptotes (for  $x \rightarrow \mp \infty$ ) with (1.9a, b) yields

$$R_n = K_0 \int_0^l Q_n(x) e^{ik_0 x} dx, \quad T_n = -K_0 \int_0^l Q_n(x) e^{-ik_0 x} dx. \quad (2.10a, b)$$

### 3. Second-order results

Proceeding through the sequence developed in §2, starting from the substitution of  $\Phi_0$  (1.7) into (2.3), and setting  $z = -h_0$ , we obtain

$$Q_1(x) = ik_0 h_1(x) e^{ik_0 x}, \quad (3.1)$$

$$R_1 = ik_0 K_0 \int_0^l h_1(x) e^{2ik_0 x} dx, \quad T_1 = -ik_0 K_0 \int_0^l h_1(x) dx, \quad (3.2a, b)$$

$$Q_2(x) = h_1(x) \partial_x \int_0^l G(x-\xi) Q_1'(\xi) d\xi, \quad (3.3)$$

$$R_2 = (ik_0)^{-1} K_0 \int_0^l Q_1(x) dx \partial_x \int_0^l G(x-\xi) Q_1'(\xi) d\xi \quad (3.4a)$$

$$= -(ik_0)^{-1} K_0 \int_0^l \int_0^l G(x-\xi) Q_1'(x) Q_1'(\xi) d\xi dx \quad (3.4b)$$

$$= -2(ik_0)^{-1} K_0 \int_0^l Q_1'(x) dx \int_0^x G(x-\xi) Q_1'(\xi) d\xi, \quad (3.4c)$$

$$T_2 = (ik_0)^{-1} K_0 \int_0^l \bar{Q}_1(x) dx \partial_x \int_0^l G(x-\xi) Q_1'(\xi) d\xi \quad (3.5a)$$

$$= -(ik_0)^{-1} K_0 \int_0^l \int_0^l G(x-\xi) \bar{Q}_1'(x) Q_1'(\xi) d\xi dx \quad (3.5b)$$

$$= -(ik_0)^{-1} K_0 \int_0^l dx \int_0^x G(x-\xi) [\bar{Q}_1'(x) Q_1'(\xi) + \bar{Q}_1'(x) Q_1'(\xi)] d\xi, \quad (3.5c)$$

wherein  $Q_1(0) = Q_1(l) = 0$  has been invoked after the partial integrations, the reduction of (3.4b) to (3.4c) follows from the identity  $G(x-\xi) = G(\xi-x)$ , and  $\bar{Q}_1$  is the complex-conjugate of  $Q_1$ ;  $Q_3$  and  $R_3$  are given in the Appendix.

### 4. Sinusoidal patch

Consider, for example,

$$h_1 = -b \sin \beta x \quad (0 < x < l), \quad \beta l = m\pi \quad (4.1a, b)$$

( $m$  is a positive integer), the substitution of which into (3.2) and (3.4) yields

$$R_1 = i\varepsilon k_0 \beta \left( \frac{e^{2ik_0 l} \cos \beta l - 1}{\beta^2 - 4k_0^2} \right) \quad (4.2)$$

and

$$R_2 = 2i\varepsilon^2 (k_0/K_0) \sum_k K(k) H(\beta, k, k_0), \quad (4.3)$$

where

$$\varepsilon \equiv K_0 b = b/[h_0 + (2k_0)^{-1} \sinh 2k_0 h_0], \quad (4.4)$$

$$H = (ik)^{-1} \int_0^l e^{ikx} (e^{ik_0 x} \sin \beta x)' dx \int_0^x e^{-ik\xi} (e^{ik_0 \xi} \sin \beta \xi)' d\xi \quad (4.5a)$$

$$= \int_0^l e^{ikx} (e^{ik_0 x} \sin \beta x)' dx \left[ (ik)^{-1} e^{-ikx} (e^{ik_0 x} \sin \beta x) + \int_0^x e^{i(k_0-k)\xi} \sin \beta \xi d\xi \right] \quad (4.5b)$$

$$= \int_0^l e^{i(k_0+k)x} (ik_0 \sin \beta x + \beta \cos \beta x) dx \int_0^x e^{i(k_0-k)\xi} \sin \beta \xi d\xi \quad (4.5c)$$

$$= \beta^2 [\beta^2 - (k - k_0)^2]^{-1} \left\{ \frac{1}{4} (ik_0)^{-1} (1 - e^{2ik_0 l}) + ik [\beta^2 - (k + k_0)^2]^{-1} [e^{i(k+k_0)l} \cos \beta l - 1] \right\}, \quad (4.5d)$$

and we have invoked  $\sin \beta l = 0$  and  $\cos 2\beta l = 1$ . Substituting (4.5d) into (4.3), separating the propagated ( $k = k_0$ ) and trapped ( $k = i\ell$ ) terms, and invoking (4.2), we obtain

$$R_2 = 2i\varepsilon(k_0/\beta) R_1 + \frac{1}{2}\varepsilon^2(1 - e^{2ik_0 l}) + R_{2\ell}, \quad (4.6)$$

where

$$R_{2\ell} = \frac{1}{2}\varepsilon^2(\beta^2/K_0) \sum_{\ell} K(i\ell) \left\{ \frac{1 - e^{2ik_0 l}}{\beta^2 + (\ell + ik_0)^2} + \frac{4ik_0 \ell [1 - e^{-\ell l + ik_0 l} \cos \beta l]}{|\beta^2 + (\ell + ik_0)^2|^2} \right\}. \quad (4.7a)$$

$$= \varepsilon^2(\beta^2/K_0) e^{i(k_0 l - \pi/2)} \operatorname{Im} \sum_{\ell} K(i\ell) \left[ \frac{e^{ik_0 l} - e^{-\ell l} \cos \beta l}{\beta^2 + (\ell + ik_0)^2} \right], \quad (4.7b)$$

and (4.7b) follows from (4.7a) through the identity

$$\frac{4ik_0 \ell}{|\beta^2 + (\ell + ik_0)^2|^2} = \frac{1}{\beta^2 + (\ell - ik_0)^2} - \frac{1}{\beta^2 + (\ell + ik_0)^2} \quad (4.8)$$

and the reality of

$$K(i\ell) = (\ell^2 + \kappa^2)/[(\ell^2 + \kappa^2)h_0 - \kappa]. \quad (4.9)$$

The series in (4.7b) may be summed by invoking the rigid-lid approximations  $\ell \simeq s\pi/h_0$  ( $s = 1, 2, \dots$ ) and  $K(i\ell) \simeq 1/h_0$  for the trapped modes and neglecting the exponentially small terms to obtain

$$R_{2\ell} = \varepsilon^2(\beta^2/K_0) e^{i(k_0 l - \pi/2)} \operatorname{Im} (e^{ik_0 l} S), \quad (4.10a)$$

$$S \equiv \sum_{\ell} \frac{K(i\ell)}{\beta^2 + (\ell + ik_0)^2} = \frac{1}{2i\pi\beta} \sum_{s=1}^{\infty} \left[ \frac{1}{s + i(h_0/\pi)(k_0 - \beta)} - \frac{1}{s + i(h_0/\pi)(k_0 + \beta)} \right] \\ = (2i\pi\beta)^{-1} \{ \psi[1 + i(h_0/\pi)(k_0 + \beta)] - \psi[1 + i(h_0/\pi)(k_0 - \beta)] \}, \quad (4.10b)$$

where  $\psi$  is the digamma function (Abramowitz & Stegun 1964, hereinafter referred to as AS, §6.3).

## 5. Bragg resonance

Bragg resonance occurs for  $k_0 l = \frac{1}{2}\beta l = \frac{1}{2}m\pi$ , for which (4.2) and (4.6) reduce to

$$R_1 = \frac{1}{4}m\pi\varepsilon, \quad R_2 = (\sin^2 \frac{1}{2}m\pi + \frac{1}{4}im\pi)\varepsilon^2 + R_{2\ell}, \quad (5.1a, b)$$

and the approximation (4.10) yields

$$R_{2\ell} = \frac{1}{2}i^{m-1} m\varepsilon^2 (K_0 l)^{-1} \operatorname{Im} \left\{ i^{m-1} \left[ \psi \left( 1 + i(m + \frac{1}{2}) \frac{h_0}{l} \right) - \psi \left( 1 - i(m - \frac{1}{2}) \frac{h_0}{l} \right) \right] \right\} \quad (5.2a)$$

$$= i^{m-1} \varepsilon^2 (\mu_0/\pi) \operatorname{Im} \{ i^{m-1} [\psi(1 + i\mu_+) - \psi(1 - i\mu_-)] \}, \quad (5.2b)$$

where  $\mu_0 \equiv k_0/K_0 = k_0 h_0 + \frac{1}{2} \sinh 2k_0 h_0$ ,  $\mu_{\pm} \equiv \left( \frac{2m \pm 1}{m\pi} \right) k_0 h_0$ . (5.3a, b)

Letting  $m$  be either odd or even and invoking AS, §6.3 (13) and (17), we obtain

$$R_{2\ell} = \frac{1}{2}\varepsilon^2 \mu_0 [\coth(\pi\mu_+) + \coth(\pi\mu_-) - (\pi\mu_+)^{-1} - (\pi\mu_-)^{-1}] \quad (m \text{ odd}) \quad (5.4a)$$

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| $m$                       | 1               | 4            | 8            | 20           |
|---------------------------|-----------------|--------------|--------------|--------------|
| $b/h_0$                   | $\frac{1}{3}$   | 0.320        | 0.320        | 0.160        |
| $k_0 h_0$                 | $\frac{1}{2}$   | 0.491        | 0.491        | 0.982        |
| $\varepsilon$             | 0.153           | 0.148        | 0.148        | 0.058        |
| $R_1$ (5.1 <i>a</i> )     | 0.120           | 0.464        | 0.927        | 0.904        |
| $R_{2\ell}$ (5.4)         | 0.0077          | $O(10^{-4})$ | $O(10^{-4})$ | $O(10^{-4})$ |
| $R_2$ (5.1 <i>b</i> )     | $0.031 + i.018$ | 0.069i       | 0.138i       | 0.052i       |
| $ R_1 + R_2 $             | 0.152           | 0.469        | 0.937        | 0.906        |
| $ R_1 + R_2 + R_3 $ (5.9) | 0.155           | 0.446        | 0.746        | 0.670        |
| $\tanh R_1$               | 0.119           | 0.433        | 0.729        | 0.718        |
| $ R _{DH}$                | —               | 0.45         | 0.72         | 0.68         |
| $(k_0 a)_{DH}$            | —               | 0.027        | 0.027        | 0.054        |

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TABLE 1. Peak reflection coefficient for the sinusoidal patch (4.1), as calculated in §5 and measured by Davies & Heathershaw (1984) (DH).

or 
$$R_{2\ell} = i\varepsilon^2(\mu_0/\pi)(\mu_+^2 - \mu_-^2) \sum_{n=1}^{\infty} \frac{n}{(n^2 + \mu_+^2)(n^2 + \mu_-^2)} \quad (\mu \text{ even}). \quad (5.4b)$$

It follows from (5.1) and (5.4) that if  $m$  is even  $R_2$  is in quadrature with  $R_1$  and therefore contributes only  $O(\varepsilon^3)$  to  $|R|$ . But if  $m$  is odd  $R_2$  has an in-phase (with  $R_1$ ) component and contributes  $O(\varepsilon^2)$  to  $|R|$ .

The second-order approximation described by (5.1)–(5.4) is adequate for  $m\varepsilon \ll 1$ . The simplest case is a half-wave bump, for which  $m = 1$ ,

$$R_1 = \frac{1}{4}\pi\varepsilon, \quad R_2 = (1 + \frac{1}{4}i\pi)\varepsilon^2 + R_{2\ell}, \quad (5.5a, b)$$

and 
$$R_{2\ell} = \frac{1}{2}\varepsilon^2\mu_0[\coth(3k_0 h_0) + \coth(k_0 h_0) - \frac{4}{3}(k_0 h_0)^{-1}]. \quad (5.6)$$

Adding (5.5 *a*) and (5.5 *b*), we place the resulting second-order approximation in the form

$$R = \frac{1}{4}\pi\varepsilon + \varepsilon^2\{1 + \frac{1}{4}i\pi + \frac{1}{2}[1 + (2k_0 h_0)^{-1} \sinh 2k_0 h_0] \times [k_0 h_0 (\coth 3k_0 h_0 + \coth k_0 h_0) - \frac{4}{3}]\}. \quad (5.7)$$

But if  $m\varepsilon = O(1)$  the second-order approximation is inadequate for the calculation of the peak (Bragg-resonant) reflection coefficient, and it is necessary to include the third-order contribution of the propagated mode. This calculation is sketched in the Appendix and yields

$$R_3 = \varepsilon^3[-\frac{1}{3}(\frac{1}{4}m\pi)^3 + \frac{13}{8}i(\frac{1}{4}m\pi)^2 + O(\frac{1}{4}m\pi)], \quad (5.8)$$

in which the neglected terms are of the same order as the trapped-mode component  $R_{3\ell}$ . Adding the dominant parts of (5.1 *a*), (5.1 *b*) for  $m$  large and even, and (5.8), we obtain

$$R = R_1 - \frac{1}{3}R_1^3 + i\varepsilon(R_1 + \frac{13}{8}R_1^2). \quad (5.9)$$

The real part of (5.9) comprises the first two terms in the  $R_1$  expansion of Mei's (1985) asymptotic ( $\beta l \uparrow \infty$  with  $R_1$  fixed) approximation ( $x = 0$  in Mei's (3.24))

$$R \sim \tanh R_1, \quad (5.10)$$

but the imaginary part differs significantly from, Mei's  $M$ .

Numerical values of the above approximations for the half-wave bump and the experimental configurations of Davies & Heathershaw are given in table 1. The

experimental values of  $(k_0 a/\varepsilon)^2$  are small, as appears to be required for the neglect of nonlinearity (see last paragraph in §1), for  $m = 4$  and 8, but not for  $m = 20$ ; however, the coefficient of  $k_0^2 a^2$  in the correction for nonlinearity typically is small, and Davies & Heathershaw's results do not appear to be amplitude dependent.

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### Appendix. Reduction of $R_3$

The reduction of  $Q_3$  and  $R_3$  follows that of  $Q_2$  and  $R_2$  in §3 and yields

$$Q_3(x) = h_1(x) \partial_x \int_0^l G(x-\xi) Q_2'(\xi) d\xi + \frac{1}{6} i k_0^3 h_1^3(x) e^{i k_0 x} \quad (\text{A } 1)$$

and

$$R_3 = -(i k_0)^{-1} K_0 \int_0^l dx \int_0^x G(x-\xi) [Q_1'(x) Q_2'(\xi) + Q_1'(\xi) Q_2'(x)] d\xi \\ + \frac{1}{6} i k_0^3 K_0 \int_0^l h_1^3(x) e^{2i k_0 x} dx. \quad (\text{A } 2)$$

We restrict further consideration to the Bragg-resonant sinusoidal patch, for which  $h_1$  is given by (4.1) and  $k_0 l = \frac{1}{2} \beta l = \frac{1}{2} m \pi$ , and neglect the third-order contributions of the trapped modes. Substituting the  $k_0$  component of  $G$  from (2.9b) into (3.3) and invoking  $k_0 = \frac{1}{2} \beta$ , we obtain

$$Q_2(x) = \frac{1}{2} i b^2 K_0 \sin \beta x E'(x), \quad (\text{A } 3)$$

$$\text{where} \quad E(x) = \frac{1}{2} i \beta (l-x) e^{-i k_0 x} - (1 - \cos \beta x - \frac{1}{2} i \sin \beta x) e^{i k_0 x}. \quad (\text{A } 4)$$

The corresponding approximation to  $R_3$ , obtained by integrating the terms in  $Q_1'$  and  $Q_2'$  by parts and substituting  $Q_1$  and  $Q_2$  from (3.1) and (A 3), is

$$R_3 = \frac{1}{16} (k_0 b)^3 K_0 l - 2(i k_0)^{-1} K_0^2 \int_0^l Q_1(x) Q_2(x) dx \\ - K_0^2 \int_0^l e^{i k_0 x} dx \int_0^x e^{-i k_0 \xi} [Q_1(x) Q_2(\xi) + Q_1(\xi) Q_2(x)] d\xi. \quad (\text{A } 5a)$$

$$= \varepsilon^3 \left[ -\frac{1}{3} \left( \frac{1}{4} m \pi \right)^3 + \left( \frac{77}{64} + \frac{1}{8} \mu_0^2 \right) \left( \frac{1}{4} m \pi \right) + \frac{13}{8} i \left( \frac{1}{4} m \pi \right)^2 + \frac{1}{3} i \sin^2 \left( \frac{1}{2} m \pi \right) \right]. \quad (\text{A } 5b)$$

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