Skew Fluxes in Polarized Wave Fields

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ABSTRACT

The scalar flux due to small amplitude waves that exhibit a preferred sense of rotation or polarization is shown to consist of a component, $\mathbf{F}_S$, that is skewed, being everywhere orthogonal to the mean scalar gradient, ∇Q. The skew flux is parameterized by $\mathbf{F}_S = -\mathbf{D} \times \nabla Q$ where $\mathbf{D}$, a vector diffusivity, is a measure of particle mean angular momentum. The skew flux may affect the evolution of mean scalar since its divergence, $\nabla \cdot \mathbf{F}_S = \mathbf{U}_S \cdot \nabla Q$, may be nonzero if the velocity $\mathbf{U}_S = -\nabla \times \mathbf{D}$ is up or down the mean gradient. For statistically steady waves, $\mathbf{U}_S$ corresponds to the Stokes velocity of particle drift. Integral theorems for new skew transport and the interpretation of fixed-point measurements are discussed, and the skew flux illustrated through several examples.

1. Introduction

The determination of the horizontal fluxes of heat, carbon dioxide, nutrients and other scalar properties in the ocean is one of the primary problems in physical oceanography. It is well recognized that there are contributions to these fluxes from both the mean circulation and the time-varying current field, the latter in the form of "eddy fluxes," but it does not appear to be widely recognized in the oceanographic community that the eddy flux calculated from Eulerian measurements may have a component perpendicular to the mean scalar gradient. This component is the so-called "skew flux", discussed in the context of turbulence and magnetohydrodynamic theory by Moffatt (1983), and occurs in wave or eddy fields with a preferred sense of rotation or polarization. In fact this skew flux is in part the Eulerian signature of the scalar transport by the Stokes velocity, which if up or down gradient may affect the mean scalar concentration (Plumb 1979).

The concept of the skew flux is thus not new, although its appearance in the literature is rather recent. In the atmospheric context, Wallace (1978), Clark and Rogers (1978) and Matsuno (1980) have discussed the skew flux of chemical scalars by stratospheric planetary waves and the mean Lagrangian transport due to both the Stokes velocity and wave-induced mean Eulerian flow. As pointed out by Matsuno (1980), this mean Lagrangian transport vanishes for statistically steady, conservative (no sinks or sources) quasi-geostrophic waves on a zonal flow so that the mean scalar and zonal flow fields do not evolve in time. This connection between mean-field evolution and the Stokes drift component of the skew flux is a statement of the Charney–Drazin wave mean-flow nonacceleration theorem (e.g., Pedlosky 1979, pp. 371–378). Plumb (1979) has also pointed out this connection although with reference to the more general nonacceleration theorem of Andrews and McIntyre (1978).

Skew fluxes may also arise in the presence of turbulence that exhibits a preferred sense of rotation. Moffatt (1983) has presented a general discussion with reference to helicity and magnetohydrodynamics, although he makes no reference to the atmospheric literature.

In the oceanographic context, Haidvogel and Rhines (1983) have discussed the skew flux as a diagnostic for eddy mean-flow interaction and the transport of potential vorticity. Middleton and Garrett (1986) have presented a detailed analysis of polarized eddy motions on the Labrador shelf, although for the statistically homogeneous velocity fields they considered, the skew flux was nondivergent and hence unable to affect the evolution of the mean scalar field. More recently, Loder and Horne (1988) show that an understanding of skew fluxes is important in the interpretation of fixed mooring measurements in the presence of tidally rectified currents.

The purpose of this note is then to draw together some of the disparate formalism and examples in the literature so as to illustrate the nature and importance of skew fluxes in the ocean. For simplicity, we shall restrict our attention to small amplitude waves only and in section 2 derive a general flux formalism for the transport of a conservative scalar (no sources or sinks).

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The adoption of these assumptions will permit the scalar flux to be explicitly related to the wave velocity and mean scalar fields. The analysis here, while in part a repeat of that given by Plumb (1979), will allow the skew flux to be developed in the general context of the total scalar flux that may arise in the presence of both polarized and unsteady (growing or decaying) waves. In addition, the vector notation due to Moffatt (1983) is adopted so as to simplify and extend the formalism. The nature of the skew flux is then illustrated in section 3 through discussion of several simple cases involving surface gravity waves, tidal rectification and the effect of quasi-geostrophic waves on a mean zonal flow.

2. Scalar fluxes due to small amplitude waves

We consider the evolution of a conservative (no sources or sinks) scalar, $\tilde{q}(\mathbf{x}, t)$, in a nondivergent velocity field $\tilde{\mathbf{u}}(\mathbf{x}, t)$ so that

$$\partial\tilde{q}/\partial t + \tilde{\mathbf{u}}\cdot\nabla\tilde{q} = 0. \quad (2.1)$$

The velocity field responsible for scalar evolution is expanded in terms of a small parameter ϵ such that $\tilde{\mathbf{u}} = \mathbf{U} + \epsilon^2\mathbf{U}_I + \epsilon\mathbf{u}(\mathbf{x}, t)$. Here, $\mathbf{U}$ denotes a constant mean velocity obtained by *phase* averaging over a period of the small amplitude wave field $\epsilon\mathbf{u}(\mathbf{x}, t)$, and $\mathbf{U}_I$ a correction to $\mathbf{U}$ included so as to explicitly allow for possible wave-induced changes to the mean flow. The parameter ϵ might typically correspond to the ratio of wave-induced particle displacement to wavelength. The scalar field is chosen as $\tilde{q} = Q(\mathbf{x}, \tau) + \epsilon q(\mathbf{x}, t) + \epsilon^2 q'(\mathbf{x}, t)$ where Q the phase-averaged mean is a function of the slow time scale, $\tau = \epsilon^2 t$, with t, being the (fast) wave time scale. In general, the phase averages, denoted by an overbar, might be obtained by spatial averaging in the direction of wave propagation. Where waves are steady, in that amplitudes neither grow nor decay in time, a simple time average may be assumed.

With the above, (2.1) may be transformed to the frame moving with the mean flow $\mathbf{U}$ and expanded in powers of ϵ so that

$$O(\epsilon)\partial q/\partial t + \mathbf{u}\cdot\nabla Q = 0 \quad (2.2)$$

$$O(\epsilon^2)\partial Q/\partial\tau + \nabla\cdot(\overline{\mathbf{u}q}) + \mathbf{U}_I\cdot\nabla Q = 0 \quad (2.3)$$

where (2.3) is obtained using the further transformation $\partial/\partial t \rightarrow \partial/\partial t + \epsilon^2\partial/\partial\tau$ and by averaging over the fast scale t. Equation (2.3) shows that the mean scalar may evolve due to advection by the wave-induced mean velocity and through divergence of the flux

$$\mathbf{F} = \overline{\mathbf{u}q}. \quad (2.4)$$

The flux may also affect the growth in scalar variance, $\overline{q^2}$, or streakiness, since from (2.2) we have

$$O(\epsilon^2)\partial\left(\tfrac{1}{2}\overline{q^2}\right)\Big/\partial t + \mathbf{F}\cdot\nabla Q = 0. \quad (2.5)$$

To determine the nature of the flux, $\mathbf{F}$, we define $\mathbf{X} = \int \mathbf{u}\,dt$ to be the wave-induced particle displacement (correct to order ϵ), so that (2.2) may be integrated to yield $q = -\mathbf{X}\cdot\nabla Q$ and the ith Cartesian component of (2.4) written as

$$F_i = -\overline{u_iX_j}\,\partial Q/\partial x_j \quad (2.6)$$

with implied summation over j. The product $\overline{u_iX_j}$ clearly plays the role of a diffusivity and following Plumb (1979), we define the symmetric and skew symmetric tensor components

$$K_{ij} = \frac{1}{2}(\overline{u_iX_j} + \overline{u_jX_i}) = \frac{1}{2}\partial(\overline{X_iX_j})/\partial t \quad (2.7)$$

$$S_{ij} = \frac{1}{2}(\overline{u_iX_j} - \overline{u_jX_i}) \quad (2.8)$$

with flux contributions

$$F_{K_i} = -K_{ij}\partial Q/\partial x_j \quad (2.9)$$

$$F_{S_i} = -S_{ij}\partial Q/\partial x_j \quad (2.10)$$

so that $\mathbf{F} = \mathbf{F}_K + \mathbf{F}_S$.

The above relations for the total flux $\mathbf{F}$ depend on the assumption that $\tilde{q}$ is a conservative scalar. For example, replacing the right hand side of (2.2) by the source term $\partial s/\partial t$ results in $q = s - \mathbf{X}\cdot\nabla Q$ so that the total flux now contains an additional term: $\mathbf{F} = \overline{\mathbf{u}s} + \mathbf{F}_K + \mathbf{F}_S$. Consideration of nonconservative terms such as $\overline{\mathbf{u}s}$ has been given by Plumb (1979). Here however, we will focus on the nature of the symmetric and skew flux components that are most directly related to the particle displacement field.

From (2.7), the symmetric diffusivity K_{ij} is clearly associated with nonsteady waves and gives rise to a flux component $\mathbf{F}_K$ that may be up or down the mean gradient ∇Q. The symmetric flux $\mathbf{F}_K$ may thus affect the scalar streakiness through (2.5) and if also divergent, the mean scalar concentration through (2.3). It should also be noted that the symmetric flux will in general have a component that is orthogonal to ∇Q since the product $\mathbf{F}_K \times \nabla Q$ need not vanish. This skew component of $\mathbf{F}_K$ is in fact related to the anisotropic nature of the wave field. For example, if the off-diagonal terms of K_{ij} are zero, then components such as $(\mathbf{F}_K \times \nabla Q)_1 = [(K_{33} - K_{22})\partial Q/\partial x_2]\partial Q/\partial x_3$ will only vanish if particle displacements grow equally in all directions.

a. *The skew flux*

The flux of most interest here however is the component $\mathbf{F}_S$ that is associated with the skew diffusivity which will be shown below to be a measure of the preferred sense of particle rotation. The skew flux is everywhere orthogonal to ∇Q since $F_{S_i}\partial Q/\partial x_i = -S_{ij}\partial Q/\partial x_j\,\partial Q/\partial x_i = S_{ji}\partial Q/\partial x_j\,\partial Q/\partial x_i$ must vanish, and therefore cannot affect the streakiness of the scalar field (to order ϵ^2), although it may still affect the evolution of Q if divergent.

Wave fields that give rise to the skew flux are characterized by a preferred sense of rotation or polarization since S_{ij} can only be nonzero when the velocity component u_i is correlated with the orthogonal displacement X_j. An example illustrating the origin of the skew flux is presented in Fig. 1 for the case of a spatially uniform, anticlockwise rotating wave field. The wave field is assumed steady, so that $K_{ij} = 0$, and at $t = 0$ we take the velocity field to be directed up the mean gradient, ∇Q, with a scalar value of Q_0 at the origin. One quarter period later, the velocity field has rotated so as to displace the mean scalar field in the y direction. The negative u velocities at this time are thus associated with negative scalar perturbations, $-\Delta Q$, indicating a positive flux in the x-direction and perpendicular to ∇Q. This flux is reinforced on the remainder of the wave cycle, since the subsequent rotation leads to positive perturbations of both u and Q.

To proceed further we adopt the vector notation due to Moffatt (1983), and note that since $\mathbf{F}_S \cdot \nabla Q = 0$ we may write

$$\mathbf{F}_S = -\mathbf{D} \times \nabla Q, \tag{2.11}$$

where after some manipulation we obtain

$$\mathbf{D} = \frac{1}{2}\overline{\mathbf{X} \times \mathbf{u}}, \tag{2.12}$$

so that $\mathbf{D} = (S_{32}, S_{13}, S_{21})$ includes all three independent components of the diffusivity S_{ij}. In the form (2.11), the skew flux $\mathbf{F}_S$ is clearly orthogonal to ∇Q and, from (2.12) is determined by a vector skew diffusivity $\mathbf{D}$ that is perpendicular to the plane of polarized motions (see Fig. 2). The diffusivity $\mathbf{D}$ is also a measure of the mean angular velocity of particle motion with a sense given by the right-hand rule: for the example shown in Fig. 1, $\mathbf{D} = (0, 0, \overline{u_2 X_1})$ is directed out of the page. Note that $\mathbf{D}$ is in fact a pseudovector in that its product with some vector changes sign under a transformation from a left to right handed coordinate system. For waves with no preferred sense of rotation $\mathbf{D}$ is thus zero.

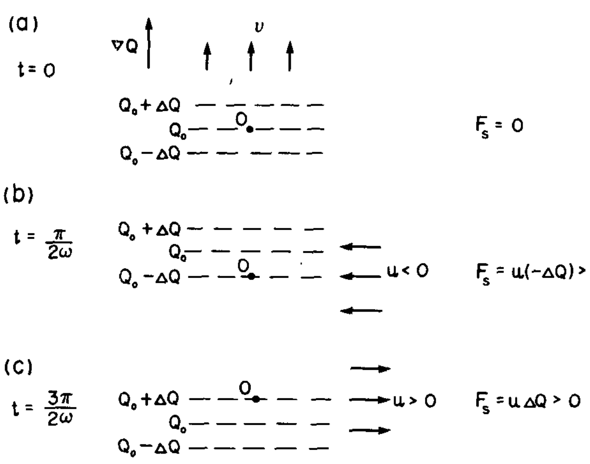

FIG. 1. The origin of the skew flux for a spatially uniform anticlockwise rotating velocity field, with ∇Q in the y direction. At $t = 0$ the scalar value at the origin equals its mean value, Q_0, so $\mathbf{F}_S$ is zero. One quarter period later, (b) the scalar field is displaced such that perturbation scalar at the origin is $-\Delta Q$ and $F_S = (-u)(-\Delta Q)$ is positive. The positive flux is reinforced over the remainder of the cycle, (c).

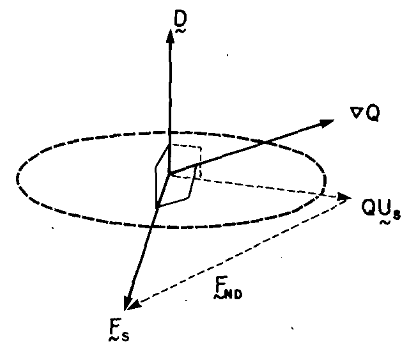

FIG. 2. A vector schematic of the skew flux $\mathbf{F}_S$, its advective, $Q\mathbf{U}_S$, and nondivergent $\mathbf{F}_{ND}$, constituents and the relation to the skew diffusivity $\mathbf{D}$ and mean gradient ∇Q. For clarity motion is assumed to be confined to the plane of mean polarized motion, the dashed circle, so that $\mathbf{U}_S$ is perpendicular to $\mathbf{D}$. Note, that in general $\mathbf{U}_S$ may contain a component that is up or down the mean gradient ∇Q, and that $\mathbf{F}_S$ is perpendicular to ∇Q. Where waves are steady, the total scalar flux is given by $\mathbf{F}_S$ and the nondivergent velocity is equal to the Stokes velocity $\mathbf{U}_{ST}$.

The formalism may be extended by noting that as an identity, the skew flux may be written as

$$\mathbf{F}_S = -Q\nabla \times \mathbf{D} + \nabla \times (Q\mathbf{D}) \tag{2.13}$$

where the second term, denoted $\mathbf{F}_{ND} = \nabla \times (Q\mathbf{D})$, is nondivergent and hence cannot affect the evolution of Q. The first term is of an advective nature with

$$\mathbf{U}_S \equiv -\nabla \times \mathbf{D} \tag{2.14}$$

playing the role of a nondivergent velocity and related to the Stokes velocity of particle drift, $U_{ST_i} = \overline{\mathbf{X} \cdot \nabla u_i} = \partial(S_{ij} + K_{ij})/\partial x_j$, through

$$\mathbf{U}_{ST} = \mathbf{U}_S + \mathbf{U}_K \tag{2.15}$$

since $(\nabla \times \mathbf{D})_i = -\partial S_{ij}/\partial x_j$ and where $U_{K_i} \equiv \partial K_{ij}/\partial x_j$. The component $U_{K_i} = \frac{1}{2}\partial^2\overline{X_i X_j}/\partial x_j \partial t$, is divergent and arises only in the presence of nonsteady waves and where wave amplitudes, and thus mean square displacements $\overline{X_i X_j}$, might grow or decay in space. The nondivergent velocity $\mathbf{U}_S$ arises due to the orbital motion of fluid particles and thus their preferred sampling of velocity crests (rather than troughs) as they are advected in the direction of increasing wave phase. This is of course the conventional explanation for the Stokes drift.

With the above definitions, the skew flux may be written as

$$\mathbf{F}_S = \mathbf{U}_S Q + \mathbf{F}_{ND},$$

so that with $\nabla \cdot \mathbf{F}_S = \mathbf{U}_S \cdot \nabla Q$, the evolution equation for Q is

$$\partial Q/\partial \tau + (\mathbf{U}_S + \mathbf{U}_I)\cdot \nabla Q = \mathbf{U}_K \cdot \nabla Q + K_{ij}\partial^2 Q/\partial x_i \partial x_j \quad (2.16)$$

where the right hand side results from the divergence of the symmetric flux. Note, that the symmetric contribution is in part advective and in part diffusive. The skew flux on the other hand is purely advective since it can only affect the evolution of Q if there exists a component of $\mathbf{U}_S$ that is up or down the mean gradient, ∇Q. The relations between $\mathbf{F}_S$, $\mathbf{F}_{ND}$ and $\mathbf{U}_S$ are sketched in Fig. 2 where for clarity we have assumed that particle motions are always confined to the plane of mean polarized motion so that $\mathbf{U}_S \cdot \mathbf{D} = 0$.

For steady, polarized wave fields, the symmetric flux and diffusivity vanish so that the nondivergent and Stokes velocities become equal

$$\mathbf{U}_{ST} = -\nabla \times \mathbf{D} \quad (2.17)$$

and $\mathbf{D} = (\overline{u_3 X_2}, \overline{u_1 X_3}, \overline{u_2 X_1})$. The total scalar flux is then equal to the skew component alone

$$\mathbf{F} = \mathbf{U}_{ST} Q + \mathbf{F}_{ND} \quad (2.18)$$

and Q evolves through the mean Lagrangian advection of scalar:

$$\partial Q/\partial \tau + (\mathbf{U}_{ST} + \mathbf{U}_I)\cdot \nabla Q = 0. \quad (2.19)$$

The Stokes component of Lagrangian velocity is a kinematic quantity obtainable from the specified wave field. The wave-induced change to the mean flow $\mathbf{U}_I$ on the other hand is determined by the dynamics of the wave field through Reynolds stress divergence (e.g., Andrews 1980). While a detailed discussion of wave induced mean Eulerian flow is beyond the scope of this paper, several simple examples are given in the next section.

b. Interpretation of point observations: steady waves

Many regions of the ocean are characterized by steady waves, tides, or waves which may be regarded as steady over several wave periods (e.g., internal waves). A condition for wave steadiness is derived in the following section for the case of growing surface gravity waves. However, where waves are steady, conservative scalar transport is characterized by the skew flux alone and $\mathbf{F} = \mathbf{U}_{ST} Q + \mathbf{F}_{ND}$. The problem then arises of how point estimates of scalar transport may be correctly interpreted since the nondivergent component $\mathbf{F}_{ND}$ may be relatively large and estimates of $\mathbf{F}$ in no way indicative of the Stokes flux component $\mathbf{U}_{ST} Q$, the true scalar transport.

To elucidate the differences between point and spatially integrated estimates of $\mathbf{F}$ and $\mathbf{U}_{ST} Q$ we shall consider the net flux through the cross-shelf vertical plane depicted in Fig. 3. In addition we assume for simplicity that both $\mathbf{U}$ and $\mathbf{U}_I$ are zero. In this case, Stokes theorem implies that the net Stokes or Lagrangian flux may be written as

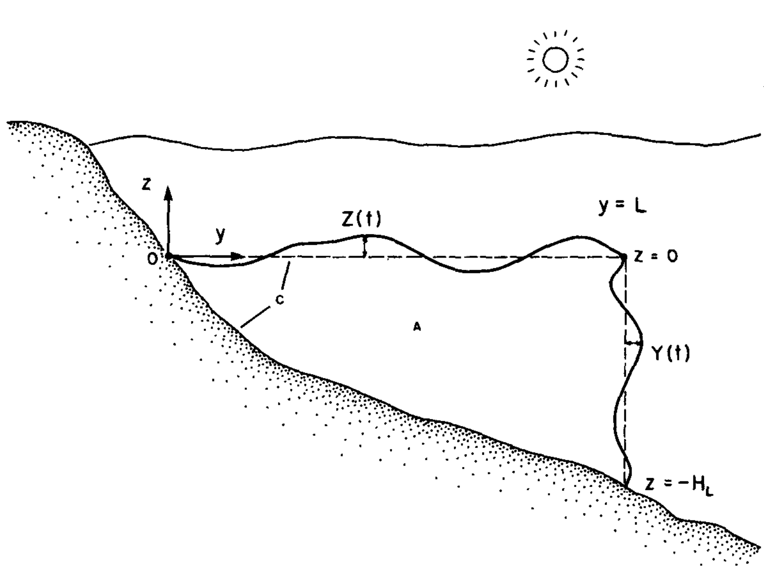

FIG. 3. The plane of integration for (2.20) through which the net Stokes and skew fluxes are evaluated for steady waves. The area A is bounded by a fixed contour C, the dashed curve and the wave-induced particle displacements are indicated by $Z(t)$ and $Y(t)$.

$$\iint_A Q\mathbf{U}_{ST}\cdot d\mathbf{A} = \iint_A \mathbf{F}\cdot d\mathbf{A} - \int_C Q\mathbf{D}\cdot d\mathbf{r} \quad (2.20)$$

where the contour C bounds the fixed area A under consideration. Evaluation of the contour integral, which arises from the nondivergent flux, is simplified since the plane of polarized motions must always be parallel to rigid boundaries, at such boundaries, so that $\mathbf{D}d\mathbf{r} = 0$. For the example in Fig. 3, the net Stokes flux may then be written as

$$\iint_A Q U_{ST} d\mathbf{A} = \underset{\text{(a)}}{\iint_A F dA} + \underset{\text{(b)}}{\int_0^L [QD_2]_0 dy} - \underset{\text{(c)}}{\int_{-H_L}^0 [QD_3]_L dz} \quad (2.21)$$

where $D_3 = -\overline{uY}$ and $D_2 = \overline{uZ}$ are evaluated along the fixed domain boundary, and which in general will not vanish. Thus, even in a net sense, the Stokes flux may not be equal to the net (skew) flux that might be directly estimated from fixed-point observations. The origin of this difference lies in the nondivergent terms (b) and (c) in (2.21). These terms while evaluated along the fixed domain boundary, in fact arise from the fluid parcel displacements across the boundary as suggested by the schematic, Fig. 3.

To show that this is so, (2.21) can be rederived by time averaging the net instantaneous flux due to all particles

$$I = \overline{\int_0^{\tilde{Y}} \int_{-H(y)}^{\tilde{Z}} \tilde{q}\tilde{u} dz dy} \quad (2.22)$$

over the time-varying domain shown in Fig. 3, where $\tilde{Y} = L + \epsilon Y(t)$ and $\tilde{Z} = \epsilon Z(t)$. Working to $O(\epsilon^2)$, the expression (2.22) for the Lagrangian or Stokes scalar flux may be approximated by

$$I = \epsilon \int_0^L \left\{ \underset{\text{(a)}}{\epsilon \int_{-H}^0 \overline{qu} dz} + \underset{\text{(b)}}{\overline{\int_0^{\tilde{Z}} Qu dz}} \right\} dy + \underset{\text{(c)}}{\epsilon \int_{-H_L}^0 \overline{\int_L^{\tilde{Y}} Qu dy} dz} \quad (2.23)$$

since the intervals, $(L, \tilde{Y})$ and $(0, \tilde{Z})$ are of order ϵ. The integrals (b) and (c) take account of the time varying extent of the fluid domain and may be further approximated by $\epsilon Q\overline{uZ}$, at $z = 0$, and $\epsilon Q\overline{uY}$ at $y = L$, since again $\tilde{Z}$ and $\tilde{Y} - L$ are small.

Now since $\mathbf{F} = \overline{qu}$, term (a) in (2.23) simply represents the skew flux in (2.21) integrated over the fixed domain. With the identification, $D_2 = \overline{uZ}$ and $D_3 = -\overline{uY}$, terms (b) and (c) in (2.23) are thus also identical to their counterparts in (2.21) and arise from the time-varying extent of the fluid domain. These terms, and hence $\mathbf{F}_{ND}$, represent a Stokes *correction* to the net skew flux, which is Eulerian in character as it is obtained by first time averaging $\tilde{q}\tilde{u}$ at a point. An analogy with the Stokes correction to Eulerian *mass* transport may be immediately drawn by putting $\tilde{q}$ equal to unity in (2.20). Since $\mathbf{U} = 0$, the net mean Lagrangian transport is simply given by the Stokes correction:

$$\iint_A \mathbf{U}_{ST}\cdot d\mathbf{A} = -\int_C \mathbf{D}\cdot d\mathbf{r}.$$

Returning to (2.21) it may now be seen that the net Stokes component of the skew flux may, in principle, be determined from fixed-point observations if allowance is made for the nondivergent flux terms (b) and (c). In principal these terms and the net skew flux could be estimated directly from fixed-point observations although the required density of such measurements might be prohibitively large. Measurements elucidating the kinematics of the wave field responsible for scalar transport would be of great help in this regard since the skew diffusivity $\frac{1}{2}\overline{\mathbf{X}\times\mathbf{u}}$ and Stokes velocity $\overline{\mathbf{X}\cdot\nabla\mathbf{u}}$ might then be directly determined.

3. Illustration of the skew flux

To illustrate the skew flux formalism we consider the scalar transport that may arise in the presence of surface gravity, tidal and quasi-geostrophic waves. For *steady* surface gravity waves, it will be shown that scalar transport is due solely to advection by the Stokes drift. The case of unsteady, evanescent gravity waves is also examined so as to illustrate the relative importance of skew and symmetric contributions to scalar transport. The wave-induced Eulerian mean transport will not be determined since the example is rather contrived and the analysis lengthy. However, in the cases of tidal and quasi-geostrophic wave propagation, stress divergence will be simply shown to lead to an additional drift of scalar through the induction of Eulerian mean velocities. For quasi-geostrophic waves, the possibility of evolution of the mean flow arises since it is assumed to be in thermal wind balance with the mean scalar field.

a. Steady surface gravity waves

We consider first the example of a shallow water, surface gravity wave propagating in the x-direction. A flat bottom at depth $z = -H$ is assumed, and the velocity field is prescribed by the real parts of

$$u = A\exp(i\theta) \quad (3.1)$$

$$w = -iAm(z + H)\exp(i\theta) \quad (3.2)$$

where A is a real constant, $\theta = mx - \omega t$ and $\omega^2 = m^2gH$. With the small amplitude assumption the vertical component of particle displacement is equal to

$$Z = A\omega^{-1}m(z + H)\exp(i\theta) \tag{3.3}$$

and zero at the sea floor, $z = -H$.

The wave field given by (3.1)–(3.3) implies that fluid parcels rotate in a clockwise sense, in the x-z plane, since u and Z are positively correlated. The polarized nature of the wave field is simply represented by the skew diffusivity (2.12) where here

$$\mathbf{D} = (0, \overline{uZ}, 0) = \left(0, \frac{1}{2}A^2\omega^{-1}m(z + H), 0\right) \tag{3.4}$$

is everywhere normal to the x-z plane of polarization, and increases in magnitude as distance, $z + H$, from the sea floor increases. The Stokes velocity associated with the wave field is from (2.17) given by

$$\mathbf{U}_{ST} = U_{ST}(1, 0, 0) \tag{3.5}$$

where $U_{ST} = \frac{1}{2}A^2 m/\omega$, and is both spatially uniform and in the direction of wave propagation.

In the presence of a vertically stratified scalar $Q = Q_0 + \gamma(z + H)$, the wave motion results in fluctuations in scalar concentration that from (2.2) are equal to

$$q = -Am(z + H)\gamma\omega^{-1}\exp(i\theta).$$

The resultant skew flux

$$\mathbf{F}_S = -U_{ST}[\gamma(z + H), 0, 0] \tag{3.6}$$

that would be measured at a point, is along isolines of $Q(z)$ but opposite in direction to the actual (Stokes) flux

$$Q\mathbf{U}_{ST} = U_{ST}[Q_0 + \gamma(z + H), 0, 0] \tag{3.7}$$

due to the nonzero contribution from the nondivergent flux

$$\mathbf{F}_{ND} = -U_{ST}[Q_0 + 2\gamma(z + H), 0, 0]. \tag{3.8}$$

Thus, if the nondivergent flux is not properly accounted for, Eulerian observations of $\mathbf{F}_S$ would yield a completely misleading picture of scalar transport. We note that the skew flux (3.6) is nondivergent, so that the mean scalar field will remain unchanged. Alternatively, there does not exist a component of $\mathbf{U}_{ST}$ that lies up or down ∇Q so that $\mathbf{U}_{ST}\cdot\nabla Q = \nabla\cdot\mathbf{F}_S = 0$. In this regard note that the Eulerian velocity $\mathbf{U}_I$ is identically zero since it may be shown that the only divergent Reynolds stress term $\partial\overline{w^2}/\partial z$ is balanced by an $O(\epsilon^2)$ mean vertical pressure gradient.

The net Stokes flux, integrated from $z = -H$ to $z = 0$, is readily calculated from (3.7) as

$$\int_{-H}^{0} QU_{ST}dz = Q(-H/2)U_{ST}H \tag{3.9}$$

and differs from that based solely on the skew flux

$$\int_{-H}^{0} F_S dz = [Q(-H) - Q(-H/2)]U_{ST}H \tag{3.10}$$

due to the net contribution from $\mathbf{F}_{ND}$. The net nondivergent contribution, which arises from fluid particle displacements across $z = 0$, may be determined from the integral theorem (2.20) through evaluation of the skew diffusivity along the surface $z = 0$. For the example in hand the net nondivergent contribution is equal to

$$\int_{-H}^{0} F_{ND}dz = -Q(0)D_2(0) = -Q(0)U_{ST}H \tag{3.11}$$

which may be verified to be the difference between the net skew and Stokes fluxes, (3.10) and (3.9) above. The example also illustrates the analogous origins of the net nondivergent flux $\mathbf{F}_{ND}$ and Stokes mass transport: setting Q equal to unity in (3.9)–(3.11), the net Stokes transport (3.9) is recovered as a correction (3.11) to the mean Eulerian transport (3.10) (here zero), and thus due also to the time varying extent of the fluid domain.

b. *Unsteady surface gravity waves*

A simple model for unsteady gravity waves may be obtained by allowing frequency and wavenumber in (3.1)–(3.2) to assume the complex values $\omega = \sigma + i\nu$ and $m = k + il$, where $\sigma/k = \nu/l = (gH)^{1/2}$. In this case, the evanescent wave field and particle displacements grow in time (and space), leading to the symmetric diffusivities:

$$\left.\begin{aligned} K_{11} &= \frac{1}{2}A^2\nu\alpha \\ K_{13} &= \frac{1}{2}A^2\nu l(z + H)\alpha \\ K_{33} &= \frac{1}{2}A^2\nu|m|^2(z + H)^2\alpha \end{aligned}\right\} \tag{3.12}$$

where $\alpha = \exp(2[\nu t - lx])/|\omega|^2$. The skew diffusivity is given by

$$S_{13} = \frac{1}{2}A^2\sigma k(z + H)\alpha \tag{3.13}$$

and reduces to the steady analog (3.4) if $\nu = l = 0$, while those above, (3.12), vanish. Indeed, for slowly growing waves where ν/σ is small, the skew diffusivity is much greater than the symmetric components, (3.12).

A more pertinent estimate of the relative importance of the skew and symmetric fluxes may be obtained from their contributions to (2.16), the evolution equation for mean scalar:

$$\partial Q/\partial\tau + (\mathbf{U}_S - \mathbf{U}_K + \mathbf{U}_I)\cdot\nabla Q = K_{ij}\partial^2 Q/\partial x_i\partial x_j. \tag{3.14}$$

While the wave-induced mean $\mathbf{U}_I$ may be nonzero for the unsteady waves considered, we focus rather on those means which result from flux divergence and obtain from (3.12)–(3.13)

$$U_K = -\nu^2 U_S/\sigma^2 \tag{3.15}$$

$$W_K = W_S \tag{3.16}$$

where $U_S = \frac{1}{2}A^2k\sigma\alpha$ and $W_S = A^2\nu k^2(z + H)\alpha$. The vertical velocity components that arise are equal although each has an opposing sign in the evolution equation and so cannot affect the mean scalar field. In the x-direction the skew velocity U_S will dominate the velocity U_K due to the growth in particle displacements provided the wave field grows slowly in time. Note also that U_S is also of order σ/ν larger than the vertical components, (3.16).

Finally, we also estimate the ratio of skew advection $U_S\partial Q/\partial x$, to the three symmetric diffusion terms in the right hand side of (3.14). Assuming Q to vary horizontally and vertically with the scales L and H, the ratios of skew advection to the diffusive terms involving K_{11}, K_{13} and K_{33} are of order, $kL\sigma/\nu$, σ^2/ν^2 and $\sigma/(\nu kL)$. Typically we might expect the horizontal scalar scale L to greatly exceed the wavelength so that kL is large and only the vertical diffusive term may be important for slowly growing waves. Where $\sigma/(\nu kL)$ is of order one however, both vertical diffusion and skew advection will be of equal importance in determining the local evolution of Q.

c. *Topographic rectification of tidal currents*

The third example we consider provides a model for the flux of heat due to steady polarized tidal currents on the side of a submarine bank. Specifically, we consider barotropic tidal motion on the side of a long bank (Fig. 4) under the assumptions of a rigid-lid sea surface, uniformity in the along-isobath (x-) direction, weak friction (parameterized by a linear bottom stress law with a constant coefficient λ), and weak nonlinearity (tidal excursions $\ll$ topographic length scale). With the horizontal tidal current at a deep-water position $y = L$ specified to be $(u, v) = (0, A_L \cos\omega t)$ where subscript L indicates evaluation at that position, the tidal velocity at other cross-isobath (y-) positions is (see Huthnance 1973 or Loder 1980)

$$u(y, t) = \frac{f}{\omega}A(1 - H/H_L)\cos(\omega t + \theta)$$

$$v(y, t) = A\cos\omega t$$

$$w(y, z, t) = AzH^{-1}H_y\cos\omega t$$

where $A(y) = A_LH_L/H$, the water depth H is a function of y only, and $\theta = \tan^{-1}(-\omega H/\lambda)$. It can again be easily shown that the skew diffusivity vector is

$$\mathbf{D} = D_a[0, -zH_y/H, 1] \tag{3.17}$$

where

$$D_a(y) = -\frac{1}{2}f\omega^{-2}A^2(1 - H/H_L),$$

and the Stokes velocity is

$$\mathbf{U}_{ST} = \frac{1}{2}f\omega^{-2}A^2H^{-1}H_y[-1, 0, 0]. \tag{3.18}$$

Note in the final determination of (3.17) and (3.18), and (3.19) below, the limit of weak friction $\omega H/\lambda \gg 1$ or $\theta = -90$ deg is taken. In this limit, the plane of wave rotation, normal to the diffusivity (3.17), varies with both the cross-isobath and vertical position (Fig. 4) and the Stokes velocity (3.18) is directed along isobaths.

In the present example there is also a nonzero mean Eulerian velocity $\mathbf{U}_I$ associated with the tidal current interaction (Huthnance 1973; Loder 1980). Under the assumption of negligible cross-isobath mean Eulerian current, the $O(\epsilon^2)$ mean momentum equation in the x-direction reduces to a balance between stress divergence and bottom friction

$$(H\overline{vu})_y = -\lambda U_I \tag{3.19}$$

and results in the mean velocity

$$\mathbf{U}_I = |\mathbf{U}_{ST}|(2 - H/H_L)[1, 0, 0].$$

Thus, over the side of the bank, the tidal velocity is rectified resulting in an along-isobath mean Eulerian current directed with shallow water to the right, and an oppositely-directed Stokes velocity, Fig. 4, so that the mean Lagrangian current, $(\mathbf{U}_{ST} + \mathbf{U}_I)$, is significantly less than the Eulerian.

For a background scalar gradient $\nabla Q = (\alpha, \beta, \gamma)$, the skew flux in this example is

$$\mathbf{F}_S = -D_a[-(\beta + \gamma zH_y/H), \alpha, \alpha zH_y/H]$$

with nonzero components in all three directions and a significant contribution from $\mathbf{F}_{ND}$, since the only nonzero component of the Stokes flux $Q\mathbf{U}_{ST}$ is in the x-direction. The skew flux may also affect the slow evolution of Q since it is divergent in the y-z plane. This divergence is simply an Eulerian signature of the Stokes flux in the x-direction, since from $\nabla\cdot\mathbf{F}_S = \mathbf{U}_{ST}\cdot\nabla Q$ it follows that

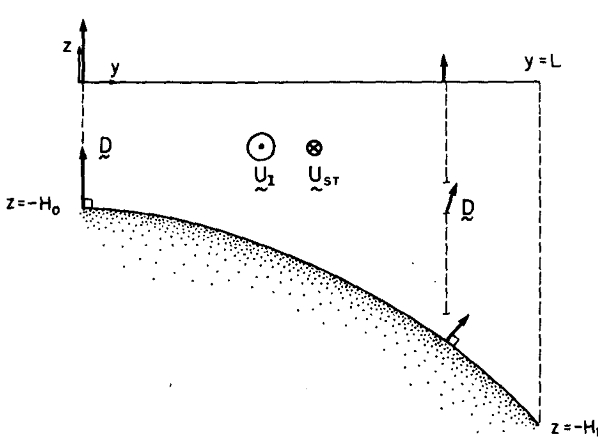

FIG. 4. Schematic for the example of tidal rectification on the edge of a long bank that is uniform in the along-isobath (x-)direction. The along-isobath Stokes and wave-induced mean Eulerian velocities are indicated. (Note: the directions of the skew diffusivity are in error by 180°.)

$$\alpha U_{ST} = \partial(\overline{vq})/\partial y + \partial(\overline{wq})/\partial z. \quad (3.20)$$

The Stokes flux divergence (3.20) together with the mean Eulerian contribution αU_I determine the evolution of Q through (2.19).

To illustrate the significance of this example we choose a topographic profile, $H = H_0 \exp(by^2)$ with $H_0 = 50$ m, $L = 20$ km, $H_L = 200$ m, $A_L = 0.2$ m s^{-1}, $f = 10^{-4}$ s^{-1} and $\omega = 1.4 \times 10^{-4}$ s^{-1} so as to crudely model the semidiurnal tide on the northern side of Georges Bank (Loder 1980). With these parameters, the mean Lagrangian current, $\mathbf{U}_{ST} + \mathbf{U}_I$, takes the form of an along-isobath jet, with a maximum speed of 0.04 m s^{-1} at $y = 8$ km (where $U_{ST} = -0.06$ m s^{-1}).

Taking the scalar to be temperature, with a mean gradient of $(2 \times 10^{-5}, -3 \times 10^{-4}, 0.15)$°C m^{-1} and representative of the northern side of Georges Bank, the skew temperature flux at mid-depth in the jet maximum is $(0.4, 0.01, 3 \times 10^{-5})$°C m s^{-1}. As discussed by Loder and Horne (1988), the along-isobath component of this flux is in reasonable agreement with that observed at the semidiurnal period in a cross-spectral analysis of current meter velocity and temperature data. Indeed, the flux at this period was found to dominate contributions from other frequency bands.

The *cross*-isobath skew fluxes predicted here for barotropic tidal rectification are smaller than those observed on Georges Bank. Loder and Horne (1988) show however, that the inclusion of the influences of stratification and internal friction on the tidal current interaction results in predicted skew fluxes that are of comparable magnitude to those observed. The predicted fluxes are the Eulerian signature of cross-bank transport by Stokes velocities, suggesting that nonlinear tidal current interactions play a major role in the cross-bank exchange of nutrients and other scalars on Georges Bank.

d. Quasi-geostrophic waves

Here we consider the effect of quasi-geostrophic waves on a zonal mean flow, $\mathbf{U} = (U(z), 0, 0)$ that is in thermal wind balance with the mean scalar (density) field, $Q = Q(y, z)$. Wave-induced vertical particle displacements are assumed small compared with the shear scale U/U_z so that, locally, U is effectively constant and the formalism of section 2 again applies. In addition, the quasi-geostrophic waves are assumed to be steady and to propagate in the zonal (x-) direction so that fluxes of density are of the skewed form $\mathbf{F}_S$ and $\mathbf{D}$ is independent of x.

As shown in section 2, the mean field Q and, through thermal wind balance, $\mathbf{U}$ may only evolve if there exists a component of the Stokes velocity directed up or down the mean gradient, ∇Q. Here $Q = Q(y, z)$ so that the relevant components of $\mathbf{U}_{ST}$ are

$$V_{ST} = -\partial D_1/\partial z, \quad W_{ST} = \partial D_1/\partial y \quad (3.21)$$

where, from (2.11), we note that $D_1 = \overline{vq}(\partial Q/\partial z)^{-1} = -\overline{wq}(\partial Q/\partial y)^{-1}$ since $\partial Q/\partial x = 0$. Quasi-geostrophic waves may also induce changes in the mean Eulerian flow through the generation of divergent fluxes of momentum and density. For the steady, conservative waves considered, Charney and Drazin (1961) have shown that these wave-induced mean velocities are given by

$$V_I = \partial \chi/\partial z, \quad W_I = -\partial \chi/\partial y \quad (3.22)$$

where $\chi = \overline{vq}(\partial Q/\partial z)^{-1}$ (see Pedlosky 1979, for a simple account.) However, since $D_1 = \chi$, the wave-induced Eulerian mean and Stokes velocities cancel exactly so that by (2.19), $\partial Q/\partial \tau = 0$ and the mean scalar (and velocity) field is unaffected by the wave motion. The result is in essence the Charney–Drazin wave mean-flow nonacceleration theorem although their derivation showed that the induced mean velocities (3.22) were cancelled by the divergence of the fluxes of momentum and density in the equations of motion. That the Charney–Drazin theorem represents a cancellation of wave-induced mean and Stokes velocities is not new and was given in the generalized Lagrangian mean theory derived by Andrews and McIntyre (1978). These authors extended the theorem to cover finite amplitude wave motions in a stratified, Boussinesq fluid. In addition, Plumb (1979) and Matsuno (1980) have noted the connection with the skew diffusivity, the latter in the context of planetary wave propagation in the stratosphere.

What is established here, and by the above authors, is that the skew diffusivity and the associated Stokes velocities are of direct importance in understanding wave–mean flow interactions. When waves are no longer steady, or sources/sinks of momentum or density exist, the Stokes and wave-induced mean Eulerian velocities may not cancel, so that the mean scalar field may evolve in time.

4. Summary and discussion

Our purpose was to provide an introduction to the concept of skew diffusion, and to illustrate its importance to scalar transport in the ocean. Drawing upon the analysis of Plumb (1979) and Moffatt (1983), the scalar flux $\overline{\mathbf{u}q}$, due to small amplitude waves, was shown to decouple into a symmetric flux $\mathbf{F}_K$, with a component in the direction of the mean gradient ∇Q, and a skew component $\mathbf{F}_S$, perpendicular to ∇Q. The symmetric flux is of the form $F_{K_i} = -K_{ij}\partial Q/\partial x_j$ and for a conservative scalar, arises only in the presence of nonsteady (growing or decaying) waves.

In contrast, the skew flux $\mathbf{F}_S$, was shown to result from a preferred sense of rotation of the wave field, and to be parameterized in terms of the mean scalar concentration, Q, and a vector skew diffusivity $\mathbf{D}$, through $\mathbf{F}_S = -\mathbf{D} \times \nabla Q$: the skew diffusivity is a measure of mean angular velocity through $\mathbf{D} = \frac{1}{2}\overline{\mathbf{X} \times \mathbf{u}}$, and perpendicular to the plane of polarized motions. A second representation for the skew flux was given by $\mathbf{F}_S = \mathbf{U}_S Q + \mathbf{F}_{ND}$, where $\mathbf{U}_S$ and $\mathbf{F}_{ND}$ are nondivergent velocities and fluxes. This form emphasizes the advective nature of the skew flux since its divergence is simply, $\nabla \cdot \mathbf{F}_S = \mathbf{U}_S \cdot \nabla Q$.

Indeed, to second order in wave amplitude, the evolution of Q over a slow time scale τ was shown to be given by

$$\partial Q/\partial \tau + (\mathbf{U}_S + \mathbf{U}_I)\cdot\nabla Q = \mathbf{U}_K\cdot\nabla Q + K_{ij}\partial^2 Q/\partial x_i \partial x_j \quad (2.16)$$

where the right hand side results from the divergence of $\mathbf{F}_K$ and $U_{K_i} = \partial K_{ij}/\partial x_j$. The velocity $\mathbf{U}_I$ denotes a second order wave-induced correction to the mean Eulerian flow that will in general arise from Reynolds stress divergence. No kinematic formalism for $\mathbf{U}_I$ may be given however since it must be determined from the equations of motion in hand.

For steady waves the symmetric flux and right hand side of (2.16) vanish and the total scalar flux $\overline{uq}$ is given by $\mathbf{F}_S$. The nondivergent velocity $\mathbf{U}_S$ in this case is equal to the Stokes velocity of particle drift illustrating the scalar transport to be real and of a Lagrangian nature. Point or integral estimates of $\mathbf{F}_S$ obtained from fixed mooring data may not however be indicative of the true Stokes transport. The nondivergent component $\mathbf{F}_{ND}$ at a point or integrated over a plane will in general be nonzero and in fact represents a Stokes correction to the net skew flux by allowing for the time varying extent of the fluid domain. Estimates of the nondivergent, Stokes and skew fluxes might be obtained however from a knowledge of the wave-field kinematics since these determine the skew diffusivity,

$$\mathbf{D} = \frac{1}{2}\overline{\mathbf{X}\times\mathbf{u}}.$$

The analogous origins of the Stokes velocity and nondivergent flux were illustrated in the example of steady surface gravity wave propagation in the presence of a vertically stratified scalar. The skew flux here was also shown to be determined by horizontally directed Stokes and nondivergent fluxes, so that Q is unaffected, that are of similar magnitude but opposite sign, a result illustrating the misleading nature of isolated measurements of scalar flux. The relative importance of skew and symmetric diffusion was also determined for the case of *unsteady*, evanescent gravity waves and the arbitrary mean scalar field, $Q(\mathbf{x}, \tau)$. For waves with a growth rate ν that is small compared with wave frequency σ, the horizontal component of skew advection was shown to dominate the advective symmetric components in (2.16). In addition, skew advection will be as important as pure symmetric diffusion provided that σ/ν is of order kL, where k denotes wavelength and L the horizontal scale of mean scalar variability.

Further illustration of the skew formalism was provided by the examples of steady tidal and quasi-geostrophic wave propagation where the transport of scalar is due both to Stokes and wave-induced mean Eulerian velocities. In the barotropic tidal model for Georges Bank, a simple balance between bottom friction and stress divergence results in a second order along-isobath mean Eulerian velocity. With the Stokes velocity, the resultant Lagrangian current takes the form of an along-isobath jet that is up-gradient so that the mean scalar may indeed evolve. However, while the predicted along-isobath skew flux of heat is in agreement with observations, a more elaborate model is required to represent the cross-isobath skew and Stokes fluxes (Loder and Horne 1988).

In the case of steady quasi-geostrophic waves, the Stokes and wave-induced mean velocities were shown to cancel exactly so that the skew flux of heat in this case cannot affect the mean density field and through thermal wind, the mean flow. This result is otherwise known as the Charney–Drazin wave mean-flow nonacceleration theorem. These examples are by no means exhaustive. Indeed, simple calculations suggest that the skew flux may be important in regions of internal Kelvin wave or short shelf wave activity, where Stokes velocities may be comparable to advection by other processes.

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