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A linear Boltzmann equation to model wave scattering in the marginal ice zone

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Abstract

We present a linear Boltzmann equation to model wave scattering in the Marginal Ice Zone (the region of ocean which consists of broken ice floes). The equation is derived by two methods, the first based on Meylan et al. [Meylan, M.H., Squire, V.A., Fox, C., 1997. Towards realism in modeling ocean wave behavior in marginal ice zones. J. Geophys. Res. 102 (C10), 22981–22991] and second based on Masson and LeBlond [Masson, D., LeBlond, P., 1989. Spectral evolution of wind-generated surface gravity waves in a dispersed ice field. J. Fluid Mech. 202, 111–136]. This linear Boltzmann equation, we believe, is more suitable than the equation presented in Masson and LeBlond [Masson, D., LeBlond, P., 1989. Spectral evolution of wind-generated surface gravity waves in a dispersed ice field. J. Fluid Mech. 202, 111–136] because of its simpler form, because it is a differential rather than difference equation and because it does not depend on any assumptions about the ice floe geometry. However, the linear Boltzmann equation presented here is equivalent to the equation in Masson and LeBlond [Masson, D., LeBlond, P., 1989. Spectral evolution of wind-generated surface gravity waves in a dispersed ice field. J. Fluid Mech. 202, 111–136] since it is derived from their equation. Furthermore, the linear Boltzmann equation is also derived independently using the argument in Meylan et al. [Meylan, M.H., Squire, V.A., Fox, C., 1997. Towards realism in modeling ocean wave behavior in marginal ice zones. J. Geophys. Res. 102 (C10), 22981–22991]. We also present details of

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how the scattering kernel in the linear Boltzmann equation is found from the scattering by an individual ice floe and show how the linear Boltzmann equation can be solved straightforwardly in certain cases.
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1. Introduction

The marginal ice zone (MIZ) is an interfacial region of ice floes which forms at the boundary of open water and continuous ice. The major way in which the open ocean interacts with the continuous ice is through wave induced breaking, and it is this process which produces the MIZ. However, wave action does not break up the continuous ice over an infinite distance. Instead, the wave energy is dissipated by scattering from the ice floes which have formed in the MIZ. The MIZ is thus formed by wave induced breaking of the continuous ice and simultaneously shields the continuous ice from breaking. There are two aspects which need to be understood to model this process: the first is the wave induced breaking of the continuous ice, and the second is the wave scattering in the MIZ (Squire et al., 1995). This paper presents a model for the latter process.

The experimental studies of wave propagation in the MIZ reported in Wadhams et al. (1986) and Wadhams et al. (1988) have shown the following features. There is strong exponential attenuation of energy, which decreases as the wave period increases. From a narrow directional spectrum at the ice edge the wave field broadens and becomes isotropic as it evolves with increasing distance into the MIZ. The wave scattering which occurs in the MIZ is due to the scattering effects of the individual ice floes which comprise the MIZ. To understand the process of wave scattering we need to understand the scattering which any individual ice floe produces. However, the equation for the propagation of wave energy, while dependent on the scattering from individual floes, will take a form quite different to the equation of scattering from an individual floe.

Determining the scattering at the large scale from the scattering from an individual scatterer is important in many areas of physics. There are many approaches to this problem, the two most popular being the linear Boltzmann (or transport) equation, and the multiple scattering, both described in Ishimaru (1978). However, the MIZ presents a slightly more complex case than is usually encountered in scattering theories because of the random nature of the scatterers (i.e. the random geometry of ice floes) and the constant motion of the ice floes. This means that it is extremely unlikely that any kind of coherent scattering effects will be observed. However, all large scale scattering theories require as input the individual scatterers and often assume that all scatterers are identical. This can lead, in the scattering theory, to coherent scattering effects which we believe are not significant for the MIZ.

While models for wave scattering in the MIZ were presented in Wadhams et al. (1986) and Wadhams et al. (1988), these models were two dimensional and therefore of limited use. The first model for wave scattering which was three dimensional and which calculated the scattering from an individual ice floe correctly was presented by Masson and LeBlond (1989) (because of the importance of this paper in our present work we will refer to this paper as M&Le). M&Le included terms for wave generation as well as scattering, and has been used in numerical wave studies (Perrie and Hu, 1996). However, the importance of M&Le is that they presented the first realistic model for wave scattering in the MIZ. The scattering theory in M&Le was based on multiple scattering theory and considered only rigid and circular ice floes. The theory was not presented in its most general form as it explicitly used circular floe geometry and derived the model for discrete time steps.

Other models for wave scattering in the MIZ have been presented. Meylan et al. (1997) developed a simple model based on the linear Boltzmann equation. The connection between the scattering at the large scale and at the individual floe scale was made by an ad hoc argument. However, the individual scattering model for an ice floe was more realistic than that used in M&Le because the floes were modelled as flexible whereas M&Le considered rigid floes. This flexure is significant for all but the smallest ice floes (as shown in the measurements of Squire, 1983; Squire and Martin, 1980). A further model was presented by Dixon and Squire (2001) which is based on Dixon and Squire (2000) in which a *Bethe–Salpeter* equation is derived for energy transport in a thin elastic plate with random material properties. However, the model in Dixon and Squire (2001) was two-dimensional (i.e. only allowed two directions of propagation) and it needs to be extended to three-dimensions to be practically useful.

In this paper, we begin by deriving a linear Boltzmann equation for wave propagation in the MIZ. In doing so we correct an error in the very similar derivation in Meylan et al. (1997). We then transform the scattering theory of M&Le by taking various limits to produce a differential rather than difference equation. From this we actually obtain a linear Boltzmann equation which is similar to the equivalent equation derived using the method of Meylan et al. (1997). We believe that this linear Boltzmann equation is a better starting point as a model for wave scattering in the MIZ because it is much simpler than the model given in M&Le, it is given as a differential equation so that it is possible to derive the most appropriate numerical scheme and the equation is independent of any assumption of circular or regularly spaced scatterers. Finally, we show how to calculate the scattering term from the equation of motion for an individual ice floe and present a straightforward method to solve the linear Boltzmann equation for some simple cases.

2. The linear Boltzmann equation for wave scattering in the MIZ

In this section we present a derivation of the linear Boltzmann equation for wave scattering in the MIZ that follows closely the derivation given in Meylan et al. (1997) but corrects an error in this earlier derivation. The linear Boltzmann equation is applicable to the propagation of wave energy through the MIZ over length and time scales large relative to the incident wavelength and wave period, respectively. Over such large scales, we assume that wave energy is incoherent. We consider the surface of an infinitely deep ocean represented in Cartesian co-ordinates by $\mathbf{r} = (x, y)$. Wave energy is propagating across this surface in all directions so that, at any point, we must consider the energy travelling in each direction. We introduce an intensity function $I(\mathbf{r}, t, \theta)$ which is the rate of flow of energy travelling in a given direction, per unit surface, per unit angle. In the absence of scatterers, we assume that the waves continue to propagate in the same direction and that the energy intensity satisfies the following equation:

$$\frac{1}{c_g}\frac{\partial}{\partial t}I(\mathbf{r}, t, \theta) + \hat{\theta}.\nabla I(\mathbf{r}, t, \theta) = 0 \tag{1}$$

(Phillips, 1977), where $\hat{\theta}$ is a unit vector in the θ direction and c_g is the speed of wave propagation (the deep water group speed). The presence of the floes will modify this expression by scattering energy, i.e. by changing the direction in which the energy is travelling.

We modify Eq. (1) to take into account the scattering effects of the ice floes using the general equation for the propagation of wave energy through a scattering medium,

$$\frac{1}{c_g}\frac{\partial}{\partial t}I(\mathbf{r},t,\theta)+\hat{\theta}\cdot\nabla I(\mathbf{r},t,\theta)=-\beta(\mathbf{r},\theta)I(\mathbf{r},t,\theta)+\int_0^{2\pi}S(\mathbf{r},\theta,\theta')I(\mathbf{r},t,\theta')\,\mathrm{d}\theta' \tag{2}$$

(Howells, 1960), where β is the absorption coefficient and S is the scattering function (assumed to be independent of time). Eq. (2) depends on the assumption that each floe scatters independently and that the energy from different scatterers may be added incoherently. The absorption coefficient, $\beta(\mathbf{r},\theta)$, is the fraction of energy lost by scattering and dissipative processes (assumed linear) from a pencil of radiation in direction θ, per unit path length travelled in the medium. The scattering function $S(\mathbf{r},\theta,\theta')$ specifies the angular distribution of scattered energy in such a way that,

$$S(\mathbf{r},\theta,\theta')I(\mathbf{r},t,\theta')\,\mathrm{d}\Omega\,\mathrm{d}S\,\mathrm{d}\Omega' \tag{3}$$

is the rate at which energy is scattered from a pencil of radiation of intensity $I(\mathbf{r},t,\theta')$ at an angle $\mathrm{d}\Omega'$ in direction θ', by a surface $\mathrm{d}S$ at position $\mathbf{r}$, into an angle $\mathrm{d}\Omega$ in direction θ. To apply Eq. (2) to wave scattering in the MIZ we must first estimate the scattering function $S(\mathbf{r},\theta,\theta')$ and the absorption coefficient $\beta(\mathbf{r},\theta)$.

The scattering function is determined by calculating the scattering from a single ice floe. Each ice floe scatters energy, and the energy radiated per unit angle per unit time in the θ direction for a wave incident in the θ' direction, E, is given by

$$E(\theta-\theta')=\left(\frac{H}{2}\right)^2\frac{\rho\omega g}{4k}|D(\theta-\theta')|^2, \tag{4}$$

where H is the wave height, ρ is the water density, ω and k are the radian frequency and wavenumber of the wave, respectively, and g is the acceleration due to gravity. $D(\theta-\theta')$ is the scattered amplitude for which, at a large distance, r, from the scatterer, the asymptotic amplitude of the outgoing wave in the θ direction, for an incident wave travelling in the θ' direction, is given by

$$\frac{H}{2}\frac{D(\theta-\theta')}{\sqrt{r}}. \tag{5}$$

Note that, in Eqs. (4) and (5), we have assumed that the scattering is isotropic (depends only on the difference of angle). This will not necessarily be true for a given ice floe, but we expect this to be true in the MIZ since there are no special directions in which the ice floes are oriented, and the floes are of random shape.

We must now express the scattering kernel in Eq. (2), $S(\mathbf{r},\theta,\theta')$ in terms of E, Given the definition of S (Eq. (3)), S can be found by dividing E by the rate of energy which is passing under the ice floe. The rate of energy passing under the floe is given by the product of the wave energy density ($\frac{1}{8}\rho g H^2$, since $H/2$ is the wave amplitude and we are considering only the energy in the water), the average area occupied by a floe ($A_\mathrm{f}/f_\mathrm{i}$, where A_f is the average area of the floe and

f_i is the fraction of the surface area of the ice covered ocean which is covered in ice), and the wave group speed ($\omega/2k$). This gives the following expression for S:

$$S(\theta,\theta') = \left(\frac{H}{2}\right)^2 \frac{\rho\omega g}{4k} |D(\theta-\theta')|^2 \frac{4f_i k}{\rho g H^2 A_f \omega} = \frac{f_i}{A_f} |D(\theta-\theta')|^2. \tag{6}$$

This expression is not exactly the same as the equivalent expression in Meylan et al. (1997) because, in the derivation for S in Meylan et al. (1997), the wave phase speed rather than the group speed was erroneously used.

We can determine β from the absorption cross section, σ_a, and the ice cover fraction f_i (M&Le, p. 58). The absorption cross section, σ_a, may be estimated from the total scattering or from experimental measurements. The expression for β is the following:

$$\beta = \int_0^{2\pi} \frac{f_i}{A_f} |D(\theta-\theta')|^2 \, d\theta' + \sigma_a \frac{f_i}{A_f}. \tag{7}$$

Combining Eqs. (2), (6) and (7), the following linear Boltzmann equation for wave scattering in the MIZ is obtained:

$$\frac{1}{c_g}\frac{\partial I}{\partial t} + \hat{\theta}.\nabla I = \int_0^{2\pi} \frac{f_i}{A_f} |D(\theta-\theta')|^2 I(\theta') \, d\theta' - \left(\int_0^{2\pi} \frac{f_i}{A_f} |D(\theta-\theta')|^2 \, d\theta' + \sigma_a \frac{f_i}{A_f}\right) I(\theta). \tag{8}$$

3. The multiple scattering theory of M&Le

The scattering theory of M&Le was the first model which properly accounted for the three dimensional scattering which occurs in the MIZ. The model was derived using multiple scattering and was presented in terms of a time step discretisation and only for ice floes with a circular geometry. Their scattering theory included the effects of wind generation, nonlinear coupling in frequency and wave breaking. However, what was original in their work was their equation for the scattering of wave energy by ice floes. M&Le began with the following equation for the evolution of wave scattering:

$$\frac{\partial I}{\partial t} + c_g \hat{\theta}.\nabla I = (S_{in} + S_{ds})(1 - f_i) + S_{nl} + S_{ice}, \tag{9}$$

where S_{in} is the input of wave energy due to wind forcing, S_{ds} is the dissipation of wave energy due to wave breaking, S_{nl} is the non-linear transfer of wave energy and S_{ice} is the wave scattering. Similarly, the terms S_{in}, S_{ds}, and S_{nl} could be added to Eq. (2). However, the purpose of this paper is to derive a consistent equation for S_{ice}. M&Le solved Eq. (9) in the isotropic (no spatial dependence) case. Furthermore, they did not actually determine S_{ice} but derived a time stepping procedure to solve the isotropic solution using multiple scattering. We will derive S_{ice} from their time stepping equation.

M&Le derived the following difference equation as a discrete analogue of Eq. (9)

$$I(f_n,\theta;t+\Delta t) = [\mathbf{T}]_{f_n}[I(f_n,\theta;t) + ((S_{in} + S_{ds})(1 - f_i) + s_{nl})\Delta t], \tag{10}$$

where f_n is the wave frequency (M&Le, Eq. (51)). It is important to realise that $[\mathbf{T}]_{f_n}$ is a function of Δt in the above equation. We are interested only in the wave scattering term so we will set the terms due to wind input (S_{in}), wave breaking (S_{ds}) and non-linear coupling (S_{nl}) to zero. These terms can be readily included in any model if required. M&Le discretised the angle θ into n evenly spaced angles θ_i between $-\pi$ and π. $[\mathbf{T}]f_n$ is then given by

$$(T_{ij})_{f_n} = A^2\{\hat{\beta}|D(\theta_{ij})|^2\Delta\theta + \delta(\theta_{ij})(1 + |\alpha_c D(0)|^2) + \delta(\pi - \theta_{ij})|\alpha_c D(\pi)|^2\}, \tag{11}$$

where $\theta_{ij} = |(\theta_i - \theta_j)|$ (M&Le, Eq. (42)). In Eq. (11), $\hat{\beta}$ (this notation is chosen to follow from M&Le who used β and to avoid confusion with the expression for β in Eq. (2) and which is used in Howells (1960) and Meylan et al. (1997)) is a function of Δt given by

$$\hat{\beta} = \int_0^{c_g\Delta t} \rho_e(r)\,\mathrm{d}r \tag{12}$$

(M&Le, p. 68). The function $\rho_e(r)$ gives the "effective" number of floes per unit area effectively radiating waves under the single scattering approximation which is to assume that the amplitude of a wave scattered more than once is negligible. It is given by

$$\rho_e(r) = \frac{2}{\sqrt{3}D_{\text{av}}^2\left(1 - \frac{8a^2}{\sqrt{3}D_{\text{av}}^2}\right)^{1/2}}\left(1 - \frac{8a^2}{\sqrt{3}D_{\text{av}}^2}\right)^{r/2a} \tag{13}$$

(M&Le, Eq. (29), although there is a typographical error in their equation which we have corrected) where D_{av} is the average floe spacing and a is the floe radius (remembering that M&Le considered circular floes). The energy factor A is given by,

$$A = \left(1 + |\alpha_c D(0)|^2 + |\alpha_c D(\pi)|^2 + \hat{\beta}\int_0^{2\pi} |D(\theta)|^2\mathrm{d}\theta + f_d\right)^{\frac{1}{2}} \tag{14}$$

(M&Le, Eq. (52)), where the term f_d represents dissipation and is given by

$$f_d = \mathrm{e}^{\frac{f_i}{A_f}\sigma_a c_g \Delta t} - 1$$

(M&Le, Eq. (53)) and α_c, the "coherent" scattering coefficient, is given by

$$\alpha_c = \left(\frac{2\pi}{k}\right)^{1/2}\exp\left(\frac{\mathrm{i}\pi}{4}\right)\frac{2}{\sqrt{3}D_{\text{av}}^2\left(1 - \frac{8a^2}{\sqrt{3}D_{\text{av}}^2}\right)^{1/2}}\int_0^{c_g\Delta t}\exp(\mathrm{i}kx_s)\left(1 - \frac{8a^2}{\sqrt{3}D_{\text{av}}^2}\right)^{x_s/2a}\mathrm{d}x_s. \tag{15}$$

It should be noted that the upper limit of integration for α_c was given as infinity in M&Le. This is appropriate in the steady case only; it should have been changed to $c_g\Delta t$ in the time dependent case. However, this corrrection leads to only negligible quantitative changes to the results.

We will transform the M&Le scattering operator $\mathbf{T}$ by taking the limit as the number of angles used to discretise θ tends to infinity. On taking this limit, the operator $\mathbf{T}(\Delta T)$ becomes

$$\mathbf{T}(\Delta T)I(\theta) = A^2\left\{\hat{\beta}\int_0^{2\pi} |D(\theta - \theta')|^2 I(\theta')\,\mathrm{d}\theta' + I(\theta)\right\}.$$

The scattering theory of M&Le depends on the values of the time step Δt and the correct solution is found for small time steps. We will now find the equation in the limit of small time steps by taking the limit as Δt tends to zero. As we shall see, when this limit is taken, there is a considerable simplification in the form of the equation. Since

$$I(t+\Delta t) = \mathbf{T}(\Delta t)I(t),$$

we obtain the following expression for the time derivative of I,

$$\frac{\partial I}{\partial t} = \lim_{\Delta t\to 0}\left(\frac{\mathbf{T}(\Delta t)I(t) - I(t)}{\Delta t}\right).$$

We can calculate this limit as follows:

$$\begin{aligned}\lim_{\Delta t\to 0}\left(\frac{\mathbf{T}(\Delta t)I(t) - I(t)}{\Delta t}\right) &= \lim_{\Delta t\to 0}\left(\frac{A^2\left\{\hat{\beta}\int_0^{2\pi}|D(\theta-\theta')|^2 I(\theta')\,\mathrm{d}\theta' + I(\theta)\right\} - I(\theta)}{\Delta t}\right)\\ &= c_g\rho_{\mathrm{e}}(0)\int_0^{2\pi}|D(\theta-\theta')|^2 I(\theta')\,\mathrm{d}\theta' - c_g\rho_{\mathrm{e}}(0)\int_0^{2\pi}|D(\theta-\theta')|^2\,\mathrm{d}\theta'\\ &\quad + \frac{f_{\mathrm{i}}}{A_{\mathrm{f}}}\sigma_{\mathrm{a}}c_g I(\theta).\end{aligned} \tag{16}$$

We can simplify Eq. (16) by using Eq. (13). The value of $\rho_{\mathrm{e}}(0)$ is given by

$$\rho_{\mathrm{e}}(0) = \frac{2}{\sqrt{3}D_{\mathrm{av}}^2\left(1-\frac{8a^2}{\sqrt{3}D_{\mathrm{av}}^2}\right)^{1/2}} = \frac{f_{\mathrm{i}}}{A_{\mathrm{f}}\sqrt{1-4f_{\mathrm{i}}/\pi}}, \tag{17}$$

where we have used the fact that $f_{\mathrm{i}} = 2\pi a^2/\sqrt{3}D_{\mathrm{av}}^2$ and $A_{\mathrm{f}} = \pi a^2$.

If we substitute our expressions for $\rho_{\mathrm{e}}(0)$ in Eq. (16) and include the spatial term (which was not in M&Le since they assumed isotropy) and divide by C_g, we obtain the following linear Boltzmann equation:

$$\begin{aligned}\frac{1}{c_g}\frac{\partial I}{\partial t} + \hat{\theta}.\nabla I &= \frac{1}{\sqrt{1-4f_{\mathrm{i}}/\pi}}\int_0^{2\pi}\frac{f_{\mathrm{i}}}{A_{\mathrm{f}}}|D(\theta-\theta')|^2 I(\theta')\,\mathrm{d}\theta'\\ &\quad - \left(\frac{1}{\sqrt{1-4f_{\mathrm{i}}/\pi}}\int_0^{2\pi}\frac{f_{\mathrm{i}}}{A_{\mathrm{f}}}|D(\theta-\theta')|^2\,\mathrm{d}\theta' + \sigma_{\mathrm{a}}\frac{f_{\mathrm{i}}}{A_{\mathrm{f}}}\right)I(\theta).\end{aligned} \tag{18}$$

If we compare Eqs. (8) and (18) we see that they are identical except for the factor $1/\sqrt{1-4f_{\mathrm{i}}/\pi}$ in the two components resulting from the scattering. This difference comes from the fact that, in M&Le, multiple scattering is neglected by using an effective density, ρ_{e}, in lieu of the number density ρ_0. As shown in Eq. (17), the effective density is related to the number density as $\rho_{\mathrm{e}}(0) = \rho_0/\sqrt{1-4f_{\mathrm{i}}/\pi}$. In summary, we have shown that, by taking the limit as the number of angles tend to infinity and as the time step Δt tends to zero in the scattering equation of M&Le, we obtain a linear Boltzmann equation equivalent to the equation given in Meylan et al. (1997) (once the error in this earlier work has been corrected).

4. Determining the scattering amplitude

The central difficulty in applying Eqs. (8) and (18) is the determination of the scattering amplitude $D(\theta - \theta')$. The scattering amplitude is found by solving the boundary value problem which arises when an isolated floe is subject to linear wave forcing. The exact equations depend on the equations chosen to model the movement of the ice floe. The solution to the equations of motion depends on the model used to describe an ice floe. The principal difference between the ice floe models used by Meylan et al. (1997) and M&Le is the following. Meylan et al. (1997) assumed that the ice floes had negligible submergence but could flex, while M&Le assumed the floes were rigid but allowed for submergence. Both models have different ranges of validity (although typical ice floes tend to be relatively thin). Of course, either ice floe model could have been used in the large scale scattering models derived by M&Le and Meylan et al. (1997). Here we simply present the equation for $D(\theta)$ independent of the equation used to model the ice floe.

The water is assumed irrotational and inviscid and the wave amplitude is assumed sufficiently small that we can linearise all the equations. The water motion is represented by a velocity potential which is denoted by ϕ. The coordinate system is the standard Cartesian coordinate system with the z-axis pointing vertically up. The water occupies the region $-\infty < z < 0$. We denote the free surface by Γ_s (located at $z = 0$) and the wetted surface of the ice floe by Γ_w.

The linearised boundary value problem for the fluid velocity potential $\phi(\mathbf{r}, z)$ subject to an incoming wave of frequency ω is

$$\left.\begin{aligned} &\nabla^2\phi = 0, && -\infty < z < 0, \\ &\frac{\partial\phi}{\partial z} = 0, && z \to -\infty, \\ &\frac{\partial\phi}{\partial z} = k\phi, && z \in \Gamma_s, \\ &\frac{\partial\phi}{\partial z} = L\phi, && z \in \Gamma_w \end{aligned}\right\} \tag{19}$$

(e.g. Meylan and Squire, 1996; Peter et al., 2004 or M&Le). The Laplace's equation comes from the fact that the water is irrotational and inviscid. The vanishing of the normal derivative at $z = -\infty$ is the no-flow through the boundary condition at the bottom of the infinitely deep ocean. The condition at the free surface is the standard linear free surface condition. At the wetted surface of the ice floe the exact equation of motion depends on the way in which the floe is modelled (for example whether it is flexible or rigid) and we represent this by the operator L. To actually solve Eq. (19) requires us to choose a specific model for the ice floe and hence to determine L. Meylan et al. (1997) assumed that the ice floes had negligible submergence but could flex. The solution to Eq. (19) for this case is described in Meylan and Squire (1996); Meylan (2002); Peter et al. (2004). M&Le assumed the floes were rigid but allowed for submergence and the solution to Eq. (19) in this case is described in Masson and LeBlond (1989) and Sarpkaya and Isaacson (1981).

Eq. (19) requires boundary conditions as x or y tend to infinity which are found from the incident or driving wave, denoted ϕ^{In}. We assume that ϕ^{In} is a plane wave travelling in the x direction,

$$\phi^{\mathrm{In}}(\mathbf{r}, z) = \frac{\omega H}{2k} \mathrm{e}^{\mathrm{i}kx} \mathrm{e}^{kz}. \tag{20}$$

The condition as $|\mathbf{r}| \to \infty$ is the standard Sommerfeld radiation condition (e.g. Wehausen and Laitone, 1960)

$$\sqrt{|\mathbf{r}|}\left(\frac{\partial}{\partial|\mathbf{r}|} - \mathrm{i}k\right)(\phi - \phi^{\mathrm{In}}) = 0, \quad \text{as } |\mathbf{r}| \to \infty. \tag{21}$$

Once we have found the solution to Eq. (19), we obtain the absolute value of the scattering amplitude D as

$$|D(\theta)| = \frac{1}{H/2}\left(\frac{k}{2\pi}\right)^{1/2}\left(\frac{\omega}{g}\right)|\mathbf{H}(\pi + \theta)|,$$

where the Kochin function $\mathbf{H}(\tau)$ is

$$\mathbf{H}(\tau) = \iint_{\Gamma_s}\left(-\frac{\delta\phi}{\delta n} + \phi\frac{\delta}{\delta n}\right)\mathrm{e}^{kz}\mathrm{e}^{\mathrm{i}k(x\cos\tau + y\sin\tau)}\,\mathrm{d}S, \tag{22}$$

where $\delta/\delta n$ is the inward normal derivative.

5. Numerical solution of the transport equation

We present here a simple method to solve the linear Boltzmann equation. It involves simplifying assumptions of the spatial or temporal independence of the solution as well as a discretisation of the equation in angle. We begin by assuming that the solution is only a function of the x spatial co-ordinate and time, i.e. there is no y dependence of the solution. We also consider a uniform MIZ so that the scattering function, 5, is a function only of θ and θ' and β is a constant. The only variation we allow spatially is that the MIZ occupies the region $x > b$, i.e. the ice edge is at $x = b$. This will allow us to consider a wave spectrum which enters the MIZ from the open ocean. Under these assumptions Eqs. (8) and (18) become,

$$\frac{1}{c_g}\frac{\partial I}{\partial t} + \cos\theta\frac{\partial I}{\partial x} = \begin{cases} -\beta I + \int_0^{2\pi} S(\theta - \theta')I(\theta')\,\mathrm{d}\theta', & x > b, \\ 0, & x < b. \end{cases} \tag{23}$$

To solve Eq. (23) we convert the problem to a matrix equation by introducing a discretisation in angle. We use a discrete ordinate method (Case and Zweifel, 1967) and represent the angular coordinate by a discrete set of n angles evenly spaced between 0 and $2\pi\left(\theta_j = \frac{2\pi j}{n}, 0 \leqslant j \leqslant n-1,\right)$. This approximation converts Eq. (23) to the following equation:

$$\frac{1}{c_g}\frac{\partial \vec{I}}{\partial t} - \frac{\partial}{\partial x}\mathbf{D}\vec{I} = \begin{cases} -\beta\vec{I} + \mathbf{S}\vec{I}, & x > b, \\ 0, & x < b. \end{cases} \tag{24}$$

In Eq. (24) the intensity $\vec{I}$ is now a vector of functions of x and t for each angle θ_j and the elements of the matrices $\mathbf{D}$ and $\mathbf{S}$ are given by,

$$d_{ij} = \begin{cases} -\cos(\theta_i), & i = j, \\ 0, & i \neq j, \end{cases} \tag{25}$$

and

$$s_{ij} \begin{cases} -\beta + S(\theta_i - \theta_i)\frac{2\pi}{n}, & i = j, \\ S(\theta_i - \theta_j)\frac{2\pi}{n}, & i \neq j, \end{cases} \tag{26}$$

respectively. Eq. (24) can be easily solved in the stationary (no time dependence), or isotropic (no spatial dependence) case. Meylan et al. (1997) solved the stationary problem and M&Le solved the isotopic problem (with wind forcing etc.). In the stationary case Eq. (24) reduces to, setting the ice edge to $b = 0$,

$$-\frac{\partial}{\partial x}\mathbf{D}\vec{I} = -\beta\vec{I} + \mathbf{S}\vec{I}, \quad x > 0. \tag{27}$$

For the isotropic case, Eq. (24) reduces to, setting the ice edge to $b = -\infty$,

$$\frac{1}{c_g}\frac{\partial}{\partial t}\vec{I} = -\beta\vec{I} + \mathbf{S}\vec{I}, \quad t > 0. \tag{28}$$

Eqs. (27) and (28) can be solved by straightforward matrix methods (e.g. Ishimaru, 1978). Eq. (27) requires boundary conditions (the wave spectrum at the ice edge $x = 0$ and a condition as $x \to \infty$) and Eq. (28) requires an initial condition (the wave spectrum at $t = 0$).

6. Summary

We have shown that the scattering theory of M&Le can be reduced to the linear Boltzmann equation if the discrete equation is converted to a differential equation by taking the appropriate limits. We also showed that this linear Boltzmann equation is equivalent to the linear Boltzmann equation presented in Meylan et al. (1997) with an error corrected. The difference between the two theories is a term which comes from the fact that M&Le explicitly neglected multiple scattering. Finally, we have shown how the scattering term is calculated from the equation of motion for an individual ice floe and how the linear Boltzmann equation can be solved in certain situations.

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