Surface Wave Reflection by Underwater Ridges

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ABSTRACT

The reflection of water surface waves by long undersea ridges and valleys is studied on the basis of linear longwave theory and of refraction theory. If L is the wavelength and h the local water depth, then for small $(L/h)dh/dx$ the first approximation to the reflection coefficient is established for a large class of smooth depth distributions $h(x)$.

1. Introduction

Surface wave reflection by submerged mountain ranges has been studies for tsunamis by Kajiura (1963) and Carrier (1966), among others. Kajiura emphasized that the largest reflection coefficients arise from steep steps $[(L/h)dh/dx \gg 1]$, but that is only helpful when such topographic features are actually present. Usually they are not and, in fact, $(L/h)dh/dx$ is rather small and the logical approach is in terms of a "slowly varying" seabed topography (Carrier, 1966). The physical process is then a fairly gradual modulation of the waves by the depth changes.

The natural mathematical model for this is to recognize a typical value ϵ of $(L/h)dh/dx$ as a small parameter in the linear longwave equations. However, the established type of analysis on such lines (Carrier, 1966) failed to yield an actual estimate of reflection because it runs into the WKB paradox (Mahony 1967): the established methods of wave modulation, whether classical (WKB, etc.) or modern (two-timing, averaging, etc.), aim at asymptotic expansions in powers of ϵ, but the reflection coefficient has a different asymptotic character when the seabed is smooth.

The same difficulty arises in the rigorous approach (Kreisel, 1948; Fitz-Gerald, 1977) based on the classical linear equations of surface waves (Stoker, 1957) in two dimensions free of *a priori* approximations relative to $(L/h)dh/dx$ or L/h. Exact solutions are obtainable via successive approximations but their use in estimating reflection when (L/h) dh/dx is small is impeded by the WKB paradox.

One aspect of the paradox is that estimates of reflection are readily obtained when the seabed topography is not smooth (Harband, 1977) and these indicate misleadingly that wave reflection tends to zero with increasing smoothness. Mahony (1967) has conjectured that the reflection coefficient then becomes transcendental in ϵ, and this is proved below.

This in itself is sometimes taken as a justification for neglecting reflection—but by comparison with what? That more wave energy be transmitted to one shore of the ocean, does not help in estimating the reflected energy experienced at the *opposite* shore. Moreover, Olver (1964) has emphasized that exponentially small terms need not be quantitatively smaller than first-order algebraic terms in an asymptotic expansion, even when ϵ is quite a small number. Most of all there are many circumstances when $(L/h)dh/dx$ is not very small, but it may be anticipated with confidence that the physical process is essentially a modulation of waves during propagation, and then the reflection coefficient is not particularly small; but the established methods fail to furnish any estimate for it.

This paper reverses the approach of the earlier authors. The WKB paradox is broken first and the result is used to examine questions of accuracy (Section 5).

For this, it is helpful to start with the simplest, approximate model, namely the linear longwave (or shallow-water) equation and to establish the precise, first approximation for the reflection coefficient for small $(L/h)dh/dx$ in this framework for topographies where the depth $h(x)$ is a smooth analytical function of only one horizontal coordinate x. The advantage of this model lies in the transparency and concreteness of the results. It is also the most plausible model for tsunamis.

Next, the linear longwave equation is replaced by the refraction equation, which is free of restrictions to shallow water and covers all wavelengths impartially for small $(L/h)dh/dx$. The precise, first approximation in this framework is obtained by a

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quite similar analysis in Section 4. The longwave result emerges as a special case. The advantage of this model lies in the definiteness of the result, even without restriction to two-dimensional water motion.

Its disadvantage lies in the lack of information on its accuracy and limitations, which arises from the inadequacy of present knowledge about its rigorous basis. Therefore, Section 5 adds some comments on the impact which further development of the more exact theory may be anticipated to have on the present reflection results.

2. Longwave approximation

Surface waves of sufficiently small amplitude in water of depth small compared to the wavelength are often found to satisfy the linear longwave equation (Stoker, 1957; Longuet-Higgins, 1967)

$$\nabla^2\zeta - (gh)^{-1}\partial^2\zeta/\partial t^2 + h^{-1}\nabla h\cdot\nabla\zeta = 0 \qquad (1)$$

for the surface elevation $\zeta(X,Y)$, where $h(X,Y)$ is the undisturbed water depth and g the gravitational acceleration. The water depth will here be assumed to depend on only one coordinate and to vary slowly with it, $h = h(\epsilon X)$, $\epsilon \ll 1$. Fourier analysis in Y and t then leads to a representation of ζ in terms of modes

$$\zeta = F(X)\exp i(\beta Y - \sigma t)$$

with real constants β, σ; for a direct interpretation, the real part may be taken everywhere. From (1), F must satisfy

$$d(hdF/dX)/dX + (\sigma^2/g - \beta^2 h)F = 0 \qquad (2)$$

and in terms of the topographical distance $x = \epsilon X$ and intrinsic distance

$$\xi = \int_0^x \alpha(x')dx', \quad \alpha^2 = \sigma^2/(gh) - \beta^2, \qquad (3)$$

$F(X) = f(\xi)$ must satisfy

$$d^2f/d\xi^2 + 2\phi(\xi)df/d\xi + \epsilon^{-2}f = 0, \qquad (4)$$

$$2\phi(\xi) = \alpha^{-3}(\alpha^2 - \sigma^2/2gh)h^{-1}\mathrm{dh/dx}. \qquad (5)$$

This mathematical transformation mirrors a natural, physical change of viewpoint which is most easily described in the case of normally incident waves for which $\beta = 0$. Then $\alpha = \sigma/(gh)^{1/2}$ is the familiar wavenumber of longwaves (and more generally, α is the relevant wavenumber component). Any physical measure of distance should be related to the wavelength, which varies due to modulation by the seabed, so that $d\xi = \alpha dx$ is the simplest and most familiar definition of natural distance ξ. The reference to x, rather than X, reflects the expectation that significant changes occur only over many wavelengths, so that wave distance ξ is more conveniently related to hundreds of wavelengths, so to speak; the individual-wave distance is ξ/ϵ. Physically, there is no good reason to refer to map distance x from here on. The endless mathematical difficulties that arise from needless reference to map distance x are illustrated by Chapman and Mahony (1978).

The physically natural form (4) of the longwave equation shows the influence of the seabed on wave development to be entirely described by the modulation function ϕ. For normally incident waves ($\beta = 0$), it is simply

$$\phi = h^{-1/4}d(h^{1/4})/d\xi,$$

i.e., the relative rate of change of the depth function $h^{1/4}$ that describes amplitude enhancement due to energy conservation on the most familiar, crude longwave approximation. Physically, depth is more naturally measured by the wave depth function G introduced in Section 4 but for longwaves the distinction is slight. The decisive role of ϕ in governing wave modulation is therefore plausible but it is a local measure of depth variation and the task at hand is to discover those overall properties of ϕ which determine wave reflection.

For the sake of a clear-cut, classical definition of reflection, $h(x)$ will be assumed to tend to positive limits h_+, h_- as $x \to \pm\infty$ such that the corresponding wavenumber limits α_+, α_- are also positive and $dh/dx \to 0$. The problem of transmission and reflection is then to find the numbers τ, r such that

$$e^{-i\xi/\epsilon}f \to \tau \quad \text{as} \quad \xi \to \infty$$
$$\text{but} \quad e^{i\xi/\epsilon}(f - e^{i\xi/\epsilon}) \to r \quad \text{as} \quad \xi \to -\infty. \qquad (6)$$

If σ be chosen >0, this radiation condition specifies an incident wave $f \sim \exp(i\xi/\epsilon)$ of unit amplitude together with a reflected wave $r\exp(-i\xi/\epsilon)$ of (unknown) amplitude r at $\xi = -\infty$, but only a purely outgoing wave $f \sim \tau\exp(i\xi/\epsilon)$ at $\xi = +\infty$. The sense of wave travel with respect to x depends on the choice of sign for α. The amplitudes τ, r are actually complex numbers, but the phase of r is rarely of interest and the aim here will be to determine $|r|$, which is independent of the signs of σ and α, for the following class of seabed topographies:

(A) $h(x)$ and $h\alpha^2 = \sigma^2/g - \beta^2 h$ have positive lower bounds (and of course, h is bounded above). This excludes consideration of wave trapping (Ursell, 1952; Longuet-Higgins, 1967; Shen et al., 1968; Meyer, 1970) for which questions other than reflection tend to be of primary interest.

(B) $h'(x)$ is assumed smooth and absolutely integrable because the seabed is normally of sedimentary origin.

Further technical qualifications will be introduced in Section 3.

Direct verification shows

$$v(\xi) = n^{-1/2}g(\xi)[1 - a(\xi)] \exp(-i\xi/\epsilon) \quad (7)$$

to satisfy (4) if

$$n^{-1}dn/d\xi = 2\phi(\xi), \quad (8)$$

$$g(\xi) = \exp \int_{\xi}^{\infty} \phi(s)a(s)ds, \quad (9)$$

$$da/d\xi = 2ia/\epsilon + (a^2 - 1)\phi, \quad a(-\infty) = 0. \quad (10)$$

In the case of normally incident waves ($\beta = 0$), $n^{-1/2} = h^{-1/4}$, so that the factor $n^{-1/2}$ in (7) is seen to represent the amplitude enhancement expected from energy conservation already on the crudest long-wave model. The factor $\exp(-i\xi/\epsilon)$ in (7) represents a wave appropriate to the local depth and $(1 - a)g$ is the unknown function; g is used merely to simplify the differential equation for a. Another approach to (10) is found, for example, in Kajiura (1963). If a bar denotes complex conjugation, (10) yields

$$d(a\bar{a})/d\xi = (a\bar{a} - 1)(a + \bar{a})\phi$$

because $\phi(\xi)$ is real, and since $a \to 0$ as $\xi \to -\infty$, it follows that $a\bar{a} = |a|^2 < 1$ for all ξ. Direct integration of (10) therefore shows

$$a(\xi)e^{-2i\xi/\epsilon} = \int_{-\infty}^{\xi} \{[a(s)]^2 - 1\}\phi(s)e^{-2is/\epsilon}ds \quad (11)$$

$$\to \int_{-\infty}^{\infty} \{a^2 - 1\}\phi e^{-2is/\epsilon}ds = a_+ \quad \text{as} \quad \xi \to \infty \quad (12)$$

because hypothesis (B) implies the absolute integrability of $\phi(\xi)$. The conjugate, $\bar{v}$, is another solution, found independent of v because $|a| < 1$, and f must therefore be a linear combination of v and $\bar{v}$. A straightforward computation now leads from (6) to the following representation of the transmission and reflection coefficients as functionals of $a(\xi)$:

$$\tau = (n_-/n_+)^{1/2}(1 - |a_+|^2)/\bar{g}(-\infty),$$

$$r = \bar{a}_+g(-\infty)/\bar{g}(-\infty).$$

Thus $|r|$ is represented as a near-Fourier transform

$$|r| = |a_+| = \left|\int_{-\infty}^{\infty} (a^2 - 1)\phi e^{-2i\xi/\epsilon}d\xi\right|. \quad (13)$$

It may be helpful to interrupt the analysis here for two comments. The first concerns the meaning of $a(\xi)$. In many ways, it is best to regard it merely as an auxiliary function of technical significance; there are other formulations of the same analysis (Chapman and Mahony, 1978) which do not involve this function. On the other hand, the connection between the reflection coefficient r and the total change $a_+ = a(\infty) - a(-\infty)$ of a tempts one to interpret $a_+ - a(\xi)$ as a kind of local reflection coefficient (Ogawa and Yoshida, 1959; Kajiura, 1963). That leads to a temptation to compute reflection via approximations to the local values of $a(\xi)$, which are easy to find. Both the differential equation (10) and integral equation (11) for $a(\xi)$ indicate that $|a(\xi)| \ll 1$ [in fact, $|a| = O(\epsilon)$], and if a^2 be therefore neglected against unity in (10) or (11), the going is easy. . . .

The second comment concerns the reason why that road is liable to lead to quite wrong results, and it goes to the heart of the matter. The integral in (13) has the awkward property that the integrand is not small but fluctuates so rapidly that almost all of its contributions to the integral cancel one another. Indeed, if we write the integral in its two parts,

$$\left|\int_{-\infty}^{\infty} \phi e^{-2i\xi/\epsilon}d\xi - \int_{-\infty}^{\infty} a^2\phi e^{-2i\xi/\epsilon}d\xi\right|,$$

then the mutual cancellations in the first integral are so severe that the whole integral is much smaller than even $|a|^2$ (i.e., is $\ll\epsilon^2$). Even the second integral is afflicted by the cancellation sickness (sometimes referred to as a WKB-paradox), it is also $\ll\epsilon^2$, and in the end, both integrals turn out to be of comparable magnitude. The smallness of $|a(\xi)|$ is deceptive. To avoid wrong reflection results (to which many literature references could be given), we should therefore aim to break the WKB-paradox by reformulating the integral in (13).

Indeed, the embarrassment of cancellations can be turned to advantage by the method of complex embedding familiar in Fourier integral theory (Titchmarsh, 1937). It starts with thinking of the wave distance ξ as a complex variable. The smoothness of sedimentary seabeds then suggests looking at modulation functions $\phi(\xi)$ which are analytic functions. The original path of integration in (13) along the real axis of ξ may then be deformed into the lower half-plane where the magnitude of the offending factor $\exp(-2i\xi/\epsilon)$ decreases rapidly with ϵ. For maximum advantage, the path should be shifted to a line parallel to the real ξ axis so that a constant factor $\exp(2 \text{ Im } \xi/\epsilon)$ can be extracted from the integral and the lowest possible value $-m$, say, of Im ξ should be used to get the smallest possible such factor; the cancellation sickness of (13) must then necessarily be cured.

Now, the path of integration can be deformed until, and only until, it approaches a singular point of the integrand. That requires only consideration of the modulation function $\phi(\xi)$ given by (5) because (10) or (11) can be used (Coddington and Levinson, 1955) to show $a(\xi)$ to be analytic where $\phi(\xi)$ is. The best number m that could be hoped for is therefore the *distance between the real ξ axis and the singular point of $\phi(\xi)$ nearest to that axis.* We are thus led to suspect a fundamental connection between wave reflection and the breakdown of analyticity of the modulation function with increasing distance from the real ξ axis.

3. Analysis

It is desirable to state clearly to what extent these conjectures have been proved. For simplicity, attention will be restricted now to water depth functions $h(x)$ which are analytic functions. [More general classes of smooth functions can be treated as limits of analytic functions (Meyer and Guay, 1974; Stengle, 1977).] By assumption A and Eqs. (3), and (5), $\phi(\xi)$ is then also analytic, and since it is real for real ξ, its domain of analyticity is symmetrical with respect to the real ξ axis. The forms of breakdown of this analyticity covered by available mathematical theory may be described by the following hypothesis:

(C) The modulation function $\phi(\xi)$ is analytic for $|\mathrm{Im}\xi| \leq m$, except for a set (without limit point) of singular points with $|\mathrm{Im}\xi| = m > 0$ arising from "transition points" x_0 of the wavenumber α or depth h of the type

$$\left.\begin{aligned} \alpha(x) &= (x - x_0)^{\nu/2}\alpha_1(x), && \nu \text{ real} \\ h(x) &= (x - x_0)^{\lambda}h_1(x), && \lambda \leq 1 \end{aligned}\right\}, \tag{14}$$

with functions $\alpha_1(x)$, $h_1(x)$ analytic and nonzero at x_0. Thus $2m$ denotes the width of the analytic domain of $\phi(\xi)$.

The new viewpoint of complex embedding also requires an extended interpretation of the radiation condition (6) which defines reflection, and an adequate one is as follows:

(D) $\phi(\xi) \to 0$ as $|\mathrm{re}\xi| \to \infty$ uniformly in $|\mathrm{Im}\xi| < m$, and for $\mathrm{Im}\xi = -m$, $\phi(\xi)$ is absolutely integrable with respect to $\mathrm{Re}\xi$ over $(-\infty,-M)$ and (M,∞) for some M independent of ϵ. In terms of the raw depth $h(x)$, this is assured by the analogous hypothesis on $h'(x)$, i.e, replace $\phi(\xi)$ by $h'(x)$, ξ by x, and m by $m/\alpha_{+,-}$ in (D).

It may be noted that the hypotheses (A) to (D) cover a large class of smooth seabed topographies. A simple example is $h(x) = 1 + k_1 \tanh x$, a natural model of smooth transition in ocean depth, with $0 < k_1 < \max[1,(\sigma^2/g\beta^2) - 1]$ for (A). Another is $h(x) = 2 - \exp(-x^2)$, a model of a symmetrical ridge, with $\sigma^2 > 2g\beta^2$ for (A). In some special cases the solution of (4)–(6) is known (Epstein, 1930) in terms of classical functions and illustrates (Kajiura, 1963) the general, asymptotic reflection results to be derived now.

The hypothesis (C), (D) imply that the line $\mathrm{Im}\xi = -m$ is a path of integration equivalent to the real axis for the integral in (13), provided the singular points of $\phi(\xi)$ are avoided by indentations of the path. For definiteness, envisage first the case where the only transition point x_0 is a zero [$\nu > 0$ in (14)] of $\alpha^2 = \sigma^2/gh - \beta^2$ and $\beta \neq 0$. The modulation function $\phi(\xi)$ then has a branch point at $\xi(x_0) = \xi_0$ such that

$$(\xi - \xi_0)\phi(\xi) \to \mu\nu/2 \quad \text{as} \quad \xi \to \xi_0, \qquad \mu = 1/(2 + \nu) \tag{15}$$

by (3) and (5). Away from it, the Riemann-Lebesgue lemma and a contracting map argument can be used to solve the nonlinear integral equation (11) in principle to show that

$$|a(\xi)| \to 0 \quad \text{as} \quad \epsilon \to 0 \quad \text{for} \quad \mathrm{Im}\xi \geq \mathrm{Im}\xi_0 = -m, \qquad \mathrm{Re}\xi \leq \gamma_1, \tag{16}$$

for any $\gamma_1 < \mathrm{Re}\xi_0$. [The details of this and the following arguments are found in Meyer (1975).] Near the transition point, on the other hand, with $f(\xi) = y(z)$ and $d\xi/dz = n(\xi)$ defined by (8), Eq. (4) becomes

$$\frac{d^2y}{dz^2} + \left(\frac{n}{\epsilon}\right)^2 y = 0,$$

$$n = n_0(z - z_0)^{\nu/2}[1 + \sum_1^\infty n_j(z - z_0)^j], \tag{17}$$

which meets the hypotheses of Langer (1932). His analysis shows $y(z)$ to be locally approximable in terms of Hankel functions, but their asymptotic expansions show only those of the first kind to be compatible with (16). Apart from a constant factor, Langer's approximation thus becomes

$$y \sim (\rho/n)^{1/2}H_\mu^{(1)}(\rho), \quad \rho = \eta e^{-i\pi}, \quad \eta = (\xi - \xi_0)/\epsilon,$$

as $\epsilon \to 0$ in the η-plane cut along its negative imaginary axis, which is found to imply, by (7) to (9),

$$a(\xi) \sim [\gamma H_{1-\mu}^{(1)}(\rho) - H_\mu^{(1)}(\rho)]/ [\gamma H_{1-\mu}^{(1)}(\rho) + H_\mu^{(1)}(\rho)], \tag{18}$$

with $\gamma = \exp(\mu\nu\pi i/2)$, and the power series and asymptotic expansions of the Hankel functions (Watson, 1944) now show $(a^2 - 1)\phi$ to be absolutely integrable with respect to $\mathrm{Re}\xi$ even at ξ_0, despite (15). A further contracting map argument for (11) shows $a(\xi)$ to remain bounded at $\mathrm{Im}\xi = -m$ for $\mathrm{Re}\xi > \mathrm{Re}\xi_0$.

This information on $a(\xi)$ suffices to show that (13) may be evaluated by the principle of stationary phase (Jones, 1966) with dominant contribution only from the critical point ξ_0. This gives by (11) and the asymptotic expansion of (18) (Watson, (1944)

$$|r|e^{2m/\epsilon} \to 2\left|\cos\frac{\pi}{2 + \nu}\right| \quad \text{as} \quad \epsilon \to 0. \tag{19}$$

The left-hand side of (19) gives the order of magnitude $\exp(-2m/\epsilon)$ of the reflection coefficient for naturally gentle seabed topography ($\epsilon \ll 1$). Reflection therefore depends most of all on the analytic width $2m$ of the modulation function $\phi(\xi)$. This is not, perhaps, a topographical scale which physical intuition might have suggested as characteristic of wave reflection. It is not related directly to either the total depth change or the maximal seabed slope $|dh/dx|$ nor even the maximum of $|d\phi/d\xi|$. Rather, m is the radius of convergence of the Taylor series of $\phi(\xi)$ at that real value of ξ where this radius is smallest. Perhaps, m may best be called the wave distance characterizing the maximal, local *intensity of modulation*. Its ratio to the wavelength scale ϵ is the parameter of most influence on wave reflection.

For a quantitative evaluation of reflection, the right-hand side of (19) is also important, and it is seen to depend on the exponent in (14), i.e., on the qualitative character of the transition point x_0 that limits the analytic width of modulation. As an example, consider the smooth shelf slope represented by

$$h = D[1 + \tfrac{1}{2}\tanh(x/l)]$$

with constants D, l over which the depth increases by a factor 3, for oblique waves with transverse wavenumber, say, $\beta = \sigma/(2gD)^{1/2}$. Then α^2 has a sequence of roots where $\tanh x' = 2$, $x' = x/l$ but h also has a sequence of poles on the imaginary x' axis. For $\mathrm{Re}x' = 0$, $\tanh x' = iw$ with real w increasing monotonely with $\mathrm{Im}x'$. By (3) $2(l\sigma)^{-1}(6gD)^{1/2}\xi = \log f(\tanh x') - \log f(0)$ with

$$f(v) = (1 + v)^3(1 - v)^{-1}[4 - v + |12 - 3v^2|^{1/2}] \times [4 + v + |12 - 3v^2|^{1/2}]^{-3}.$$

On the imaginary x' axis, therefore,

$$f(\tanh x') = F(iw)/\overline{F(iw)}^3,$$

$$F(iw) = 1 + [3 + |12 + 3w^2|^{1/2}](1 + iw)/(1 + w^2),$$

and $\arg F(iw)$ grows monotonely from 0 to $\pi/3$ as w grows from 0 to ∞, so that $\arg f(\tanh x')$ grows from 0 to $4\pi/3$ as we follow the imaginary axis from the origin to the first pole of h. At $x' = \tfrac{1}{2}\log 3$, on the other hand, $\tanh x' = \tfrac{1}{2}$ and as $\mathrm{Im}x'$ increases by $i\pi/2$, $\tanh x'$ traces a path to 2 through positive arguments and $\arg f(\tanh x')$ increases along it only from 0 to π. Hence, the root of α^2 at $\tanh x' = 2$, $f(\tanh x') = (\tfrac{1}{4}) \exp i\pi$ limits the analytic width of $\phi(\xi)$ and

$$m = \pi l\sigma/(24gD)^{1/2}.$$

Since the roots of α^2 are simple, $\nu = 1$ in (14) and $2\cos[\pi/(\nu + 2)] = 1$ and from (19)

$$|r| = \exp[-\pi l\sigma/(6gD\epsilon^2)^{1/2}].$$

For normally incident waves ($\beta = 0$) on the same shelf slope, by contrast, the analytic width of $\phi(\xi)$ is similarly found to be limited by the pole of h at $x' = -i\pi/2$, with

$$\mathrm{Im}\xi_0 = m = \pi l\sigma/(6gD)^{1/2}$$

and $\lambda = -1$ in (14). This also gives rise to a simple root of α^2, however, and

$$|r| = \exp[-2\pi l\sigma/(6gD\epsilon^2)^{1/2}].$$

The changeover of transition point suggests existence of an intermediate incidence near which two transition points make comparable contributions to reflection.

Comparison of these reflection coefficients for waves incident at different angles on the same topography illustrates the marked increase of reflection with obliqueness of incidence. This is expected since increase of β in (3) ultimately brings a root of α^2 close to the real ξ axis [which is excluded by (A) for the present account] and for still greater obliqueness, refraction theory predicts almost complete reflection (Shen *et al.*, 1968).

For waves normally incident ($\beta = 0$) on a ridge represented by

$$H(x) = D[2 - \exp(-x^2/l^2)]$$

the analytic width of the modulation function is found from (3) to be

$$2m = \sigma l(gD)^{-1/2} \int_1^2 u^{-1}[(2 - u)\log u]^{-1/2}du$$
$$= 2.70\sigma l(gD)^{-1/2}.$$

It arises from the simple roots of h at $x = \pm i(\log 2)^{1/2}$ so that $\lambda = 1$ in (14), and the solution f of (4) may then be approximated locally by Hankel functions of zero order and (18), (19) hold with $\mu = (2 + \nu)^{-1} = 0$. The reflection coefficient is therefore

$$|r| = 2\exp(-2m/\epsilon)$$

in this case.

Other cases may arise, e.g., singular transition points of α with $\nu < 0$ in (14); for $\nu \leqslant -2$ such a point is mapped on no ξ. For $-2 < \nu < 0$, a modification (Meyer, 1974) of Langer's (1932) analysis can be employed to repeat the calculations; the details are different (Meyer, 1975), but the result is still (19). The same result is obtained for a singular transition point of h (or G in Section 4) with $\lambda < 0$ in (14) and $\beta \neq 0$ in (3), which corresponds to $\nu = 2\lambda/(1 - \lambda)$. A regular transition point of h (or G) with $0 < \lambda < 1$ in (14) corresponds to $\nu = \lambda/(1 - \lambda)$ in (17), and (19) still gives $|r|$.

Finally, several transition points could share the minimum of $|\mathrm{Im}\xi|$; if those with $\mathrm{Im}\xi = -m < 0$ be ordered from left to right, in the sense of increasing $\mathrm{Re}\xi$, then the analysis so far sketched covers the part of the line $\mathrm{Im}\xi = -m$ to the left of the second transition point. But the turning point and contraction arguments can then be applied afresh to the second and further turning points, generally at the expense of local approximation by Hankel functions of both kinds (Meyer, 1975). The limit of $|r| \exp 2m/\epsilon$, moreover, is the sum of individual terms of the form (19) for each of these critical points, with the appropriate approximation to $a(\xi)$ for the respective points.

The examples illustrate the dimensional dependence of reflection on

$$\sigma(gD)^{-1/2}l/\epsilon = 2\pi l/L$$

(Kajiura, 1963), where σ is the frequency, D the depth scale, l the horizontal, topographic scale and L, the wavelength which, for normal incidence in particular, is $L = 2\pi(gh)^{1/2}\epsilon/\sigma$ in x units, by (3) and (6). The dependence of reflection on this parameter is very strong, $|r|$ is a power of $\exp(-2\pi l/L)$, and the precise power depends on the modulation intensity measured by the analytic width $2m$ of $\phi(\xi)$. It is noteworthy that, if a modulation function ϕ be the sum of two analytic functions of significantly different analytic width, then that of smaller width controls reflection, the other has only a relatively negligible influence. This does not preclude that the latter could account for most of the total depth change or for most of the steepness of the seabed.

A little paradox arises therefrom when an observed seabed topography is to be modeled mathematically. Should the scatter in the data be smoothed out? At first sight, it may appear that it should not, for if a little analytic wrinkle $\psi(\xi)$ be added to a modulation function $\phi(\xi)$, then ψ is likely to have much the smaller analytic width and must therefore determine reflection. Modulation theory, however, is based on the premise $l/L \gg 1$, and for its application, the data must be smoothed to the extent that the length scale l of topographic variation is rather large compared with the wavelength L. If that procedure leaves a bump in the data, on the other hand, then this bump is indeed likely to dominate wave reflection.

4. Refraction approximation

When the surface wavelength is not long compared with the water depth, the motion decays rapidly with increasing depth below the surface and the local influence of the seabed on the motion is reduced accordingly. Over many wavelengths, nonetheless, gradual changes in water depth may have an important effect on the development of the surface wave motion. Refraction theory aims to estimate this slow, cumulative effect without restriction to "shallow water."

A mathematical model for this has been proposed by Berkhoff (1973), Jonsson *et al.* (1976) and Lozano and Meyer (1976). It is motivated by the simplifications arising from small wave amplitude and slowness of the variation in water depth; the first suggests linearization of the surface condition and the second, a vertical structure of the motion dependent to a first approximation on the local depth, but not on its gradient. It follows that the velocity potential has, to this approximation, the form

$$\Phi = \Psi(x,y)e^{-i\sigma t}\cosh[k(z+h)]/\cosh(kh) \quad (20)$$

with dispersion relation

$$k\tanh(kh) = \sigma^2/g \quad (21)$$

relating the local wavenumber $k(x,y)$ to the depth $h(x,y)$ and frequency σ. This leads (Lozano and Meyer, 1976) to the equation

$$\left.\begin{aligned} &\nabla(G\nabla\Psi) + k^2G\Psi = 0 \\ &G = [\sinh(2kh) + 2kh]/[4k\cosh^2(kh)] \end{aligned}\right\} \quad (22)$$

for the surface potential $\Psi(x,y)$; the wave-depth function G here represents the attenuated measure of water depth that effectively modulates the wave when the water is not shallow; the surface elevation is $\zeta = i\sigma\Psi(x,y)$.

A rigorous derivation of this model from the classical linear theory under very restricted circumstances has been given by Harband (1977). Moreover, Lozano and Meyer (1976) have confirmed that (22) yields the appropriate approximation to the known exact solutions of the classical, linear water wave equations and contains the heuristic models in most common use, e.g., the ray theory of water wave refraction.

If the water depth $h(x)$ depends only on one Cartesian coordinate $x = \epsilon X$, as in the preceding sections, then also $k = k(x)$, $G = G(x)$, by Eqs. (21) and (22) and Fourier analysis of (22) leads again to modes $\Psi(X,Y) = F_1(X)\exp i(\beta Y)$ which are now governed by

$$d(GdF_1/dX)/dX + (k^2 - \beta^2)GF_1 = 0. \quad (23)$$

This is of the same form as (2), and in the longwave limit $kh \to 0$, (21) and (22) show $k^2 \sim \sigma^2/(gh)$ and $G \sim h$, so that (2) is the longwave limit of (23). Conversely, the analysis of Sections 2 and 3 applies equally to (23) if

$$\left.\begin{aligned} &\alpha^2(x) = k^2 - \beta^2, \quad \xi = \int_0^x [k^2(x') - \beta^2]^{1/2}dx' \\ &2\phi(\xi) = (\alpha^2G)^{-1}d(\alpha G)/dx \end{aligned}\right\} \quad (24)$$

in the place of Eqs. (3) and (5). The reflection coefficient is therefore again given by (19),

$$|r| \sim 2e^{-2m/\epsilon}\left|\cos\frac{\pi}{\nu+2}\right| \quad \text{as} \quad \epsilon \to 0, \quad (25)$$

with $2m$ now denoting the analytic width of the modulation function defined by (24) and ν, the order of the transition point that limits this width, as defined by (14) with G in place of h and α defined by (24). If several transition points share the same m, then they contribute additively to $|r|$.

5. Classical linear reflection

The theory of reflection outlined in the preceding sections estimates a small effect on the basis of an approximate theory. This raises the question whether the approximate theory constitutes a framework within which real water wave reflection can be studied at all?

Part of the answer may be that for tsunamis, e.g., the linear longwave equation (1) appears as a thoroughly plausible framework and within it, (19) follows rigorously. In any case, the objection does not apply to a study of underwater reflection in the framework of the classical, linear theory (Stoker, 1957) based on linearization of the surface condition for small surface wave amplitude (unless circumstances be envisaged, e.g., on beaches, under which nonlinear effects might make the dominant contribution to reflection). It may therefore be appropriate to reexamine the present results from the viewpoint of the classical, linear theory.

A famous study of reflection on that theory for two-dimensional motion is due to Kreisel (1948), but his approach aims primarily at reflection from engineering works of scale comparable to the wavelength and does not therefore result in close bounds for reflection by topographical features. A new mathematical study was undertaken by Fitz-Gerald (1977) also for the two-dimensional case, to develop a rigorous solution procedure generating successive approximations for deep water. Both studies envisage that the depth tends to constant values as $x \to \pm\infty$, and the asymptotic form of the potential as $x \to \pm\infty$ is therefore modeled exactly by refraction theory and the definition of the reflection coefficient implied in Section 4 agrees precisely with the exact definition. The exact reflection coefficient is an integral (Fitz-Gerald, 1977)

$$|r| = \left| \int_{-\infty}^{\infty} P(\xi) e^{2i\xi/\epsilon} d\xi \right| . \qquad (26)$$

closely analogous to (13). An approach to reflection via refraction theory is therefore justified in principle.

It follows from the analogy between (26) and (13) that the magnitude of (26) again depends primarily on the distance between the real ξ axis and the nearest singular point of $P(\xi)$. That distance, however, remains to be estimated, even to a first approximation, and the result (25) remains accordingly subject to revision. For instance, if that distance should be found to differ from the half-width m of $\phi(\xi)$ by $O(\epsilon)$, then the factor of $\exp(-2m/\epsilon)$ in (25) would be substantially inaccurate. The longwave approximation, on the other hand, involves both the limits $\epsilon \to 0$ and $kh \to 0$, and the same factor in (19) might be more accurate, if kh is sufficiently small.

The exact solution of the classical linear equations for a special shelf slope has been obtained by Roseau (1952) and its reflection properties have been discussed by Kajiura (1963). The approach through a special velocity potential differs much from the direct approach to reflection in this article and a detailed comparison has not yet been made but the results appear to support refraction theory when the waves are slowly modulated. Unless the water be deep throughout, that requires a gentle slope in this solution, and the reflection coefficient is then $|r| = \exp(-2m/\epsilon)$ where $\epsilon \ll 1$ is a measure of the slope, D is the water depth over the shelf and m, the positive root of $m \tanh m = \sigma^2 D/g$. Thus $m = kD$ from (21), where k is the wavenumber over the shelf. One conclusion is that, except for deep water, reflection by slow modulation is here dominated by the top of the shelf slope where the modulation function has its smallest analytic width. The toe of the slope has little influence on reflection.

An interesting facet of the special solution is that it permits us to discuss separately the limits of zero shelf slope ϵ and zero water depth D over the shelf. As D decreases so does m, and for waves that are long over the shelf, $\sigma(D/g)^{1/2} \to 0$ and $m \sim \sigma(D/g)^{1/2}$. As $D \to 0$ for fixed slope, $|r| \to 1$ for all nonzero slopes from the exact solution. As $\epsilon \to 0$ for fixed $D > 0$, by contrast, $|r| \to 0$. Indeed the double limit $D \to 0$, $\epsilon \to 0$ of this special solution displays the features of the beach reflection paradox (Meyer and Taylor, 1972) and thereby suggests the intriguing possibility that waves of small enough amplitude on a sloping beach are completely reflected before they reach the shoreline. That process, however, is outside the scope of refraction theory, which applies only when $\sigma(D/g)^{1/2}/\epsilon \gg 1$.

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REFERENCES

Berkhoff, J. C. W., 1973: Computation of combined refraction-diffraction. *Proc. 13th Conf. Coastal Eng.* J. W. Johnson, Ed., Amer. Soc. Civ. Engrs., 471–490.

Carrier, G. F., 1966: Gravity waves on water of variable depth. *J. Fluid Mech.*, **24**, 641–659.

Chapman, P. B., and J. J. Mahony, 1978: Reflection of waves in a slowly varying medium. *SIAM J. Appl. Math.*, **34**, 303–319.

Coddington, E. A., and N. Levinson, 1955: *Theory of Ordinary Differential Equations*. McGraw-Hill, 429 pp.

Epstein, P. S., 1930: Reflection of waves in an inhomogeneous absorbing medium. *Proc. Nat. Acad. Sci., U. S.*, **16,** 627–637.

Fitz-Gerald, G. F., 1977: The reflexion of plane gravity waves travelling in water of variable depth. *Phil. Trans. Roy. Soc. London*, **A286,** 48–49.

Harbard, J., 1977: Propagation of long waves over water of slowly varying depth, *J. Eng. Math.*, **11,** 97–119.

Jones, D. S., 1966: Fourier transforms and the method of stationary phase, *J. Inst. Math. Appl.*, **2,** 197–222.

Jonsson, I. G., O. Skovgaard and O. Brink-Kjaer, 1976: Diffraction and refraction calculations for waves incident on an island. *J. Mar. Res.*, **34,** 469–496.

Kajiura, K., 1963: On the partial reflection of water waves passing over a bottom of variable depth, *IUGG Monogr.*, No. 24, 206–230.

Kreisel, G., 1948: Surface waves. *Quart. Appl. Math.*, **7,** 21–44.

Langer, R. E., 1932: On the asymptotic solution of differential equations. *Trans. Amer. Math. Soc.*, **34,** 447–480.

Longuet-Higgins, M. S., 1967: On the trapping of wave energy around islands. *J. Fluid Mech.*, **29,** 781–821.

Lozano, C., and R. E. Meyer, 1976: Leakage and response of waves trapped by round islands. *Phys. Fluids*, **19,** 1075–1088.

Mahony, J. J., 1967: The reflection of short waves in a variable medium. *Quart. Appl. Math.*, **25,** 313–316.

Meyer, R. E., 1970: Resonance of unbounded water bodies. *Lectures in Applied Mathematics*, Vol. 13, Amer. Math. Soc., 189–227.

——, 1974: Adiabatic variation IV, action change of a pendulum for general frequency. *Z. Angew. Math. Phys.*, **25,** 651–654.

——, 1975: Gradual reflection of short waves, *SIAM J. Appl. Math.*, **29,** 481–492.

——, and A. D. Taylor, 1972: Run-up. *Waves on Beaches*, R. E. Meyer, Ed., Academic Press, 357–411.

——, and E. J. Guay, 1974: Adiabatic variation III, deep mirror. *Z. Angew. Math. Phys.*, **25,** 643–650.

Ogawa, K., and K. Yoshida, 1959: A practical method for a determination of reflection of long gravitational waves. *Rec. Oceanogr. Works Japan,* **5,** 38–50.

Olver, F. W. J., 1964: Error bounds for asymptotic expansion. *Asymptotic Solutions of Differential Equations and their Applications*, C. H. Wilcox, Ed., Wiley, 163–183.

Roseau, M., 1952: Contribution à la théorie des ondes liquides de gravité en profondeur variable. Publ. Sci. Tech. Ministère de l'Air, No. 275, Paris, 90 pp.

Shen, M. C., R. E. Meyer and J. B. Keller, 1968: Spectra of water waves in channels and around islands. *Phys. Fluids*, **11,** 2289–2304.

Stengle, G., 1977: Asymptotic estimates for the adiabatic invariance of a simple oscillator. *SIAM J. Math. Anal.*, **8,** 640–654.

Stoker, J. J., 1957: *Water Waves*, Interscience, 567 pp.

Titchmarsh, E. C., 1937: *Introduction to the Theory of Fourier Integrals*. Oxford University Press, 394 pp.

Ursell, F., 1952: Edge waves on a sloping beach. *Proc. Roy. Soc. London*, **A214,** 79–97.

Watson, G. N., 1944: *Treatise on the Theory of Bessel Functions*, 2nd ed. Macmillan, 74, 75, 198.