Joint Distribution of Wave Heights and Periods in Waters of Any Depth

By Constantine D. Memos,[1] Fellow, ASCE, and Kyriakos Tzanis[2]

Abstract: The short-term joint distribution of wave heights and periods gives an informative description of a sea state that is well suited for many practical applications in the coastal zone. This paper deals with modifications and extensions of recently published results, expressed in terms of the joint probability density function of wave heights and periods. The performance of the present model is shown to be improved as compared with experimental data. Waves of any bandwidth are treated in deep water as well as over a uniformly sloping bottom. The modifications of the joint distribution in deep water refer to the estimation of wave periods, the separation of swell from sea waves, and the accommodation of a breaking criterion. The modified joint distribution is then transformed as waves move in shallow water, through a wave-by-wave technique incorporating nonlinear shoaling of both Stokes third-order and cnoidal waves, depth-induced wave breaking, and wave reforming. Comparisons with experimental measurements show improved performance over past models.

INTRODUCTION

The description of a sea state in the frequency domain is usually provided through an energy density spectral representation in two or three dimensions. An alternative approach is to work in the time domain and describe a particular sea state in a probabilistic manner. The short-term joint distribution of wave heights and periods is most advantageous in this respect, since it contains substantial information about the geometric properties of the sea surface and is thus appropriate as input to coastal numerical models as well as for engineering design of coastal structures.

The problem of describing a sea state by the joint probability density function between wave heights and periods is quite perplexing. So far, a few treatments have yielded analytical results applicable to narrow-banded waves in deep water waves (e.g., Cavanié et al. 1976; Longuet-Higgins 1983). More recently, Memos and Tzanis (1994) produced numerical results of the joint distribution for deep water waves of any bandwidth. Their model, herein called the Deep Water Model (DWModel), is mainly based on a theoretical approach by Memos (1994a). An effort to expand our knowledge of the joint distribution of wave heights and periods in shallow water has produced a synthetic model employing a wave-by-wave treatment (Memos 1994b). That model, herein called the Shallow Water Model (SWModel), includes shoaling of cnoidal waves, depth-induced wave breaking, and wave decay after breaking. Results of the model compare well with experimental measurements taken by the first writer over a limited range of the outer surf zone (Memos 1994b).

The scope of the present paper is twofold. First, it seeks to apply a series of improvements to the representation of the joint distribution of wave heights and periods in deep water, i.e., to the DWModel. Second, it aims to modify the SWModel by allowing for a more realistic mathematical description of the shoaling waves controlled by the relative magnitude of the local water depth. The improved SWModel can be produced by the corresponding DWModel, thus providing a more complete picture of the joint probability density function (pdf) of wave heights and periods as it transforms from deep water through the surf zone. The improvements to the joint pdf in deep water refer to the following three aspects: the estimation of the individual wave periods; the separation of swell from sea waves; and the application of a deep-water breaking criterion. The SWModel was improved by incorporating the shoaling of Stokes higher order waves in the appropriate range of water depths. Results referring to the joint distribution of wave heights and periods for both deep and shoaling waters show improved agreement with existing data as compared with previous results (Longuet-Higgins 1983; Memos and Tzanis 1994).

BACKGROUND

Numerical results of the short-term joint distribution of wave heights and periods in deep water have been provided in the form of nondimensional graphs for any bandwidth of the sea state (Memos and Tzanis 1994). The main assumption employed is the Gaussian character of the sea-surface elevation as it develops in time. The joint probability density between wave heights H and periods T should be dependent on their correlation factor $r(H, T)$, defined as

$$r(H, T) = \frac{1}{\sigma_H \sigma_T n_o} \sum^{n_o} (H_i - H_m)(T_i - T_m) \qquad (1)$$

where σ_H and σ_T denote the standard deviation of wave heights and periods, respectively; n_o = number of individual waves; H_i and T_i = wave height and period, respectively, of the ith wavelet; and H_m and T_m = mean value of the wave heights and periods, respectively, for the n_o wavelets. This factor was selected because it can uniquely determine, from a mathematical point of view, the interrelation between the variables H and T. Indeed, the results produced in the previously mentioned paper are governed by the actual values of $r(H, T)$, calculated from the simulated data of each individual case and uniquely associated with the resulting probability structure. The above treatment was based on a decomposition of the wave record into zero up-crossing wavelets. A wave-generating model produced some 50,000 waveforms, which were then handled using statements from the statistical theory of random processes. Results of the DWModel for the joint pdf of wave heights and periods are encouraging, giving closer agreement with field measurements than other existing theories.

In shallow water, the problem of producing the joint distribution of wave heights and periods has been tackled by a simple mathematical model incorporating existing results for wave shoaling, depth-induced wave breaking, and wave decay (Memos 1994b). The model is based on the wave-by-wave

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approach proposed by Mase and Iwagaki (1982), Dally (1992), and others. This approach selects randomly deep water waves from a known joint distribution of wave heights and periods, transforms individual waves, and then reassembles them into probability distributions across the surf zone. Cnoidal wave theory has been assumed to apply for the shoaling process (Shuto 1974; Goda 1975). This synthetic model has been applied to a limited range of transitional waters and compared well with experimental results (Memos 1994b).

DEEP WATER WAVES

Estimation of Periods

In the DWModel, the estimation of periods was based on a proposition stating that, in a normal process, the time derivative of the main variable is statistically independent from the variable and follows a normal distribution as well. It was thus possible to express the period of a single waveform using the constraint that the slope of the free surface obeys a normal distribution. The individual values of this slope at various elevations of the waveform provided a mean slope, through which the time between two successive zero-upcrossings was obtained [for details, refer to the DWModel (Memos and Tzanis 1994)].

The present refinement of the above treatment relates to a further statement of the mathematical analysis of random noise [see, e.g., the review by Blake and Lindsey (1994)] providing that the time derivative of a normal process at any one crossing level is also normal and statistically independent from the underlying process. In the DWModel, the mean slope was deduced by taking into account the fact that the slopes at the crossing points at all levels of the grid of the waveform follow a normal distribution, whereas in the present treatment, the individual normal distribution at any single level were used to finally define the overall mean slope of the waveform. If a time series is assumed of the sea-surface elevation at a fixed point with the notation shown in Fig. 1, the above statement is equivalent to saying that the distribution of $d\eta/dt$ at the crossing points of the realization with any single level is Gaussian and that η and $d\eta/dt$ are statistically independent.

Now, denoting the total time span of the realization by t_o, the following relation is easily deduced:

$$t_a = \sum_{i=1}^{Na} dt_{ai} = N_a dt_{ma} \quad (2)$$

where dt_{ai} = time of the process spent at crossing i of level (a); t_a = total time spent at level (a); N_a = number of crossing

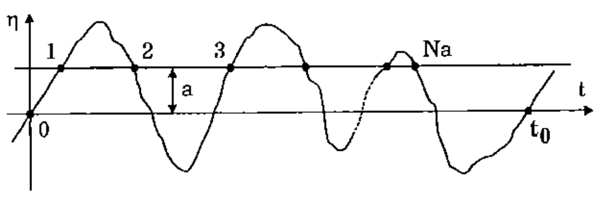

FIG. 1. Crossings of Free Surface With Level (*a*)

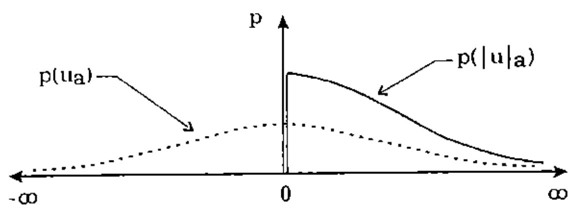

FIG. 2. Probability Density Function of Absolute Value of Slope at Crossings With Level (*a*)

points at level (a); and dt_{ma} = average time spent at one crossing at level (a). If t_m denotes the mean period of waves in record of time span t_o, and p_a = pdf of η at level (a), then t_a can also be expressed as [see, e.g., Newland (1984)]:

$$t_a = p_a N t_m da \quad (3)$$

where N = number of zero-upcrossing points within the time interval t_o. It is noted that the rigorous meaning of "time spent at level (a)" is "time that the process $\eta(t)$ lies within the open interval ($a - da/2$, $a + da/2$)."

From (2) and (3), it follows $|u|_{ma} = da/dt_{ma} = M_1 N_a/p_a$, where $M_1 = 1/Nt_m$. The quantity $|u|_{ma}$ represents the mean absolute slope of the sea surface at level (a). Since the slope u_a of the surface at level (a) follows a normal distribution, the following is derived with the help of Fig. 2:

$$|u|_{ma} = 2\int_o^\infty \frac{x}{\sqrt{2\pi}\sigma_{ua}} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma_{ua}^2}\right) dx = (2/\pi)^{1/2}\sigma_{ua} \quad (4)$$

where σ_{ua} = standard deviation of the normal distribution of the free surface slope at the crossings with level (a). Thus, σ_{ua} can be expressed as follows:

$$\sigma_{ua} = M_2 N_a/p_a \quad (5)$$

with $M_2 = (\pi/2)^{1/2}/Nt_m$.

Since da/dt follows a normal distribution, the absolute values of the free surface slope $|u|$, where $u = da/dt$, can be expressed at any level (a) in terms of a randomly selected slope of the free surface at level (a) $|u_i|_a$:

$$|u_i|_a = \sigma_{ua} f(RN_1, RN_2) \quad (6)$$

where f is a generator of random numbers. In the applications, the function $f = |RN_1\sqrt{-2\log r/r}|$ was used, where $r^2 = (2RN_1 - 1)^2 + (2RN_2 - 1)^2 \leq 1$; and RN_1 and RN_2 are random numbers distributed uniformly over the interval $(-1, +1)$.

The wavelength (or period) of the ith wavelet in a wavetrain can be expressed as

$$t_i = c_p da/|u|_m \quad (7)$$

Here, c_p = total number of crossing points of the ith wavelet with all levels at increments da; and $|u|_m$ = mean absolute slope of the free surface at all crossing points with these levels. This quantity can be expressed through use (5) and (6) as follows:

$$|u|_m = (M_2/cp)\sum_a N_a f/p_a \quad (8)$$

with the summation sign covering all levels between the two extreme elevations η_{min} and η_{max}.

Comparing (3) and (7) and making use of the relation $M_1 = (2/\pi)^{1/2}M_2$ and of (8), one arrives at the following result:

$$\frac{t_i}{t_m} = C_p^2(2/\pi)^{1/2}Nda \bigg/ \left(\sum_a N_a f/p_a\right) \quad (9)$$

Since the value of the right-hand side of (9) is known for a given record of the model, the ratio t_i/t_m is independent of the actual value of the mean period t_m of the wave record; consequently, the statistical structure of the individual wave periods in a wavetrain does not depend on any characteristic values of these periods. Therefore, M_2 can be taken for convenience as $M_2 = 1$, since it is deleted in the nondimensional ratio t_i/t_m. The expression of $|u|_m$ in (8) depends on the distribution of the free surface slope at the crossing points at any one level (a) and constitutes a refinement over the definition adopted in DWModel, where $|u|_m$ was estimated by treating the slopes of the surface record at all levels indiscriminately, rather than by associating the individual levels (a) to their corresponding density p_a. In this way the period of any individual waveform of the model record is determined. Since the repetition number of each waveform in the record is estimated in the DWModel, the joint pdf and the associated $r(H, T)$ can now be calculated.

Exclusion of Swell

This modification of the original DWModel allows the sea waves to be treated separately from swell. The criterion $t_i/t_m \leq M$ was imposed, where M is a real number greater than unity. In the applications the values of $M = 2.0$ and $M = 2.5$ have been used; these are based on real-life data pertaining to wind waves. If during the generation process of t_i, the above criterion was not satisfied, a new value of the wave period was sought until the criterion was met. Thus, t_m does not have a constant value and the process requires an iteration technique. Convergence was assumed when the variation in t_m was less than 3% of the final value. This is achieved relatively quickly, even when only the swell criterion is applied without that of deep-water breaking, due to the small number of periods that do not initially satisfy the former.

The underlying provision in the above method was to comply with the wave height distribution produced by the widely accepted assumption of the normal distribution of the surface elevation, as given by the DWModel. The same assumption was also valid in the application of the wave breaking criterion, to be dealt with in the next paragraph.

Wave Breaking

The original DWModel did not account for wave breaking. Therefore, it was decided to include this effect by applying an appropriate filtering procedure. The breaking criterion adopted is the standard one applicable to Stokes waves, i.e., $H/L < 0.143$. The above value of limiting steepness is close to results by other, more refined, treatment of the onset of breaking. For asymmetric waves in deep water, Vinje and Brevig (1981) give a numerical result for the limiting steepness of 0.15, quite close to the adopted value of 0.143. Experiments carried out by Ochi and Tsai (1983) indicated a limiting steepness of deep-water random waves of 0.13, though this is based on down-crossing wave heights and wavelengths between successive maxima. In the DWModel, the joint pdf of wave heights and periods appeared in dimensionless graphs of the variables H_i/H_m, T_i/T_m. In order to be able to apply the above filter, an additional parameter is required interrelating the scales of wave heights and wavelengths. The following linear relation was assumed for this purpose:

$$H_m = \alpha L_m \tag{10}$$

where H_m and L_m = mean wave height and wavelength of the record, respectively; and α = steepness parameter assumed in the following equal to 0.05, a value supported by real-life data and relevant hindcasting methods. It is worthwhile to note that the model is quite insensitive to the actual value of α, which ranges usually between 0.04 and 0.07.

In this manner, the parameters needed to define a single joint pdf representation in deep water are the correlation coefficient $r(H, T)$ and the wave steepness factor α. The translation of deep water wavelengths into wave periods for use in the present model is performed through the relation

$$L_m = (g/2\pi) \sum_N (T_i^2/N) \tag{11}$$

The populations of the periods of the wavelets T_i, which so far were measured in an arbitrary scale, are, therefore, multiplied by a constant factor K_T calculated through (10) and (11) as follows:

$$K_T = \left[(2\pi/g) H_m/\alpha \sum_N T_i^2/N) \right]^{1/2} \tag{12}$$

The swell criterion can now be rewritten as

$$H_i \leq (g/14\pi)(K_T T_i)^2 \tag{13}$$

where H_i and T_i = values of wave height and period, respectively, of the ith wavelet produced initially by the model.

The filtering process that accounts for deep water wave breaking has been applied in the following manner. When (13) is not satisfied for a particular pair (H_i, T_i), a new period T_m is sought, through the random process exposed in a previous section, that satisfies (13). This treatment, however, affects the value of K_T by an increase in the denominator of (12). Thus, a new run is required to check the periods by applying the value of K_T produced in the previous run. These loops are carried on until the value of K_T does not vary between two successive runs by more than 10% of the final value. It should be noted that a maximum population of only 3% of the waveforms were affected by this filtering process. Thus, the Gaussian distribution of the surface elevation was regarded to be applicable, as is the usual practice when dealing with offshore waves.

The above technique leads to a slow converging process when only the breaking criterion is applied. However, when both the swell and breaking criteria are treated simultaneously, convergence is quickly achieved due to the fact that these criteria have opposing effects on the variation of K_T. In this case an acceptable cumulative error of 5% instead of 10% was assumed.

In summary, the offshore joint pdf of wave heights and periods is obtained through the following main steps:

1. Generation of a model record, i.e., a form dimensionless in time wavelets and number of repetitions (see DW-Model)
2. Estimation of each individual period for the above model record
3. Exclusion of swell, if required
4. Application of wave breaking filtering
5. Production of joint pdf graph from record processed as above and calculation of the associated value of $r(H, T)$

SHOALING WAVES

The probability structure of a wave field is modified as waves propagate into shallow water. In order to quantify the modifications incurred on the joint pdf of wave heights and periods, we refer to techniques that utilize information for regular waves of predetermined frequencies. Such an approach does not aim at improving our understanding of the underlying physical processes of these modifications, but rather at obtaining acceptable results pertaining to specific quantities, such as the wave height. Indeed, it has been shown that in shallow water the statistical behavior of the wave field can be predicted within acceptable limits by a wave-by-wave analysis of the incoming spectrum (Mase and Iwagaki 1982; Goda 1985; Mase and Kobayashi 1991; Dally 1992). In the present study, three processes were taken into account: shoaling, wave breaking, and wave reforming after breaking. It has to be emphasized that the present model of the joint distribution of wave heights and periods is not based on a site-specific numerical algorithm, where, e.g., Boussinesq-type equations could be incorporated. Rather, it predicts uniquely the joint pdf given a few parameters of the problem. The present refinement refers to the shoaling process which, in the SWModel, is described solely in terms of cnoidal theory. Now, in order to cover the whole area from deep water to the surf zone, an additional wave theory has been used, that of the Stokes third order. This wave theory covers a wider range of deeper water than cnoidal theory.

A note should be made at this point regarding the use of these mathematical wave theories in the present context of the probability structure of the wave field. Both Stokes and Cnoidal waves are waves of permanent form and thus cannot represent adequately the evolution of a wavetrain propagating into shallow waters, where a vertical asymmetry of the wave form is developed. However, it should be borne in mind that the use of these theories in the present study aims only at predicting with acceptable accuracy the main wave characteristics —i.e., wave height and wavelength—at any location in shallow waters. The detailed geometry of the free surface does not affect our results provided the wave heights and periods are correctly estimated. A parameter for the asymmetry of wave profiles is the skewness of the time derivative of surface elevation, as proposed by Goda (1986), who showed that wave height parameters are not affected by the degree of wave asymmetry. Another commonly used parameter measuring the degree of nonlinearity associated with the vertical wave asymmetry, or the wave steepness in either deep or shallow water, is the statistical skewness based on the third order statistical moment of the surface elevation record. Experiments conducted by Stansberg (1994) for deep water waves indicate that, although the skewness has some relation to the wave crest elevation, it does not affect the wave height distribution. The criteria assumed to determine the two areas of applicability of these wave theories is that the following inequalities hold in the range of cnoidal wave theory and vice versa (Laitone 1963):

$$d/L < 1/8 \quad \text{and} \quad \mathrm{Ur} > 26 \tag{14}$$

where Ur = Ursell parameter ($=L^2H/d^3$); and d, H, and L = local values of water depth, wave height, and wavelength, respectively.

Starting from deep water the waves undergo a shoaling transformation described by the Stokes third order wave theory. This transformation is nonlinear and can be estimated by the following relation based on results by Le Méhauté and Webb (1964):

$$L_o^4\lambda_o^2[4(1 + 3\lambda_o^2/4)] = L_o^4\lambda_d^2A(\lambda_d, d/L_d)/s^2 \tag{15}$$

$$(L_d/T)^2 = (g/k_d)(1 + B_d\lambda_d^2)\tanh(k_d d) \tag{16}$$

where λ_o = positive root of the equation $3\lambda_o^3/8 + \lambda_o = \pi H_o/L_o$; the subscripts o and d denote deep water and shallow water of depth d, respectively; and

$$B_d = (8c^4 - 8c^2 + 9)/8s^4; \quad c = \cosh k_d d; \quad s = \sinh k_d d$$

$$A = 4(sc + dk_d) + \lambda_d^2[(sc + k_d d)(-20c^6 + 16c^4 + 4c^2 + 9)/4s^6 + sc(16c^4 + 2c^2 + 9)]$$

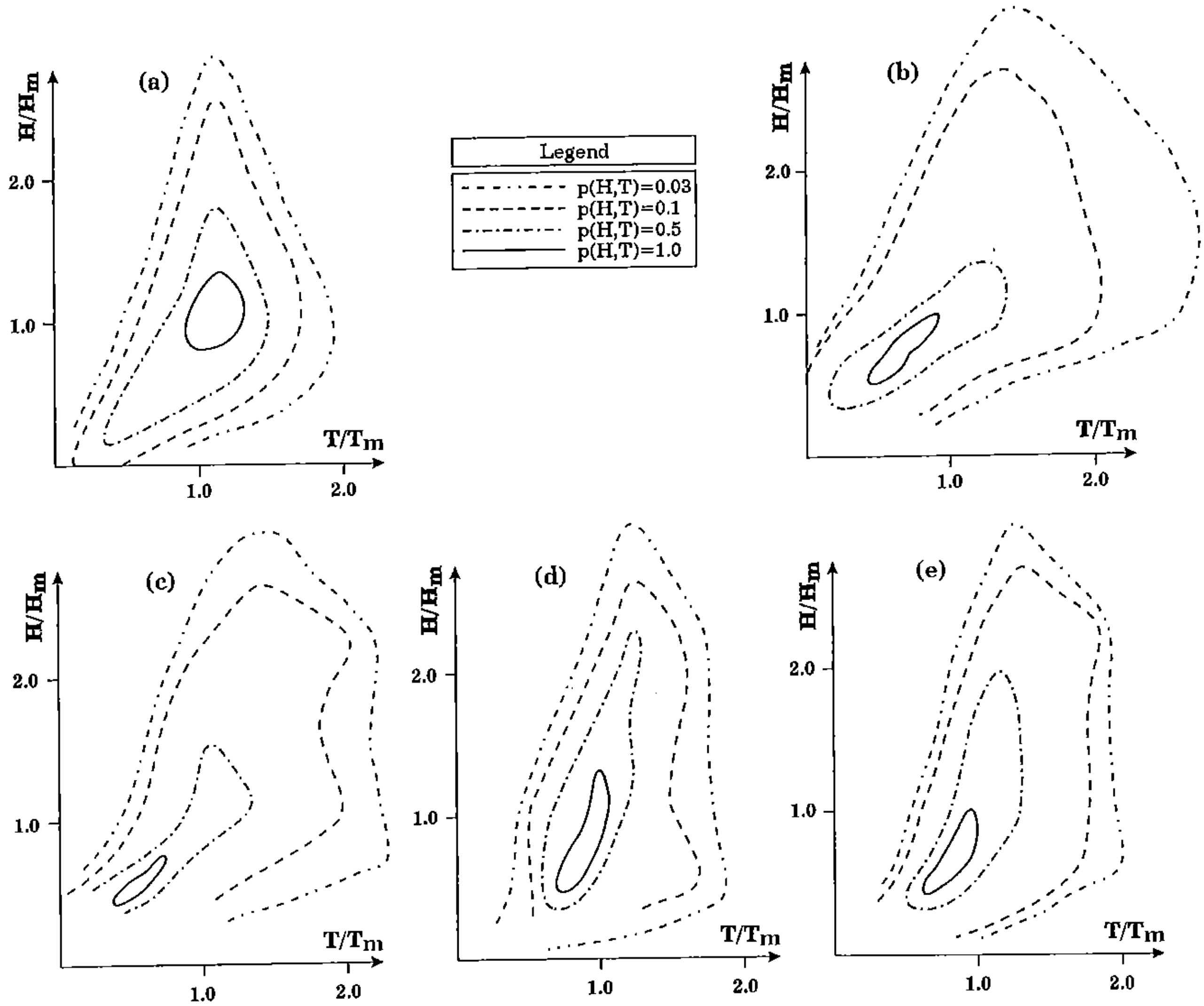

FIG. 3. Deep Water Modifications: (a) Measurements, from Goda (1978); (b) Original DWModel; (c) Wave Period Modifications; (d) Wave-Breaking Plus (c); (e) Exclusion of Swell ($T/T_m \leq 2$) plus (d)

From (15) and (16) one gets L_d and λ_d; the wave height H_d is then calculated through $\lambda_d^3 + B + \lambda_d = \pi H_d/L_d$, where $B = 3(8c^6 + 1)/64s^6$.

The local values of the wave characteristics determined by the above procedure are checked in two ways, one of which refers to the depth-induced wave breaking, the other to the applicability of the cnoidal wave theory. The former is effected through the frequency dependant criterion (Weggel 1972) $H_b/L = (1/7)\{7h_b b/L[1 + a(h_b/gT^2)]\}$, where H_b = wave height at breaking; T = wave period; $a = 43.75[1 - \exp(-19m)]$; $b = 1.56[1 + \exp(-19.5m)]$; m = bed slope; and h_b = still water depth at incipient breaking.

The transition into cnoidal theory is verified by (14). Depth-induced wave breaking is followed by the decay process after Dally et al. (1985). When for a particular water depth it is found that the cnoidal wave theory is applicable, the calculation returns stepwise offshore to define the limit between the Stokes third and cnoidal theories for the particular characteristics of the wavelet. In the sequel depth-induced wave breaking and wave decay transformations are incorporated, as stated earlier. The actual implementation of the above described procedure is quite complicated, mainly due to the nonlinearity of the adopted shoaling process. A computer code has been developed to cope with the entire transformation from deep water through Stokes third order wave shoaling, cnoidal shoaling, and depth-induced wave breaking to wave decay after breaking in very shallow waters.

As regards the distribution of periods, it is assumed that the individual periods of the waveforms remain unaltered through the shoaling process described previously.

RESULTS AND DISCUSSION

Deep Water

The improvements effected on the representation of the joint pdf between wave heights and periods in deep water can be assessed through the resulting modified graphs of the joint distribution. As the basis for the comparisons has been taken the nondimensional standard deviation of the wave heights. This was kept constant in the results of both the original DWModel and the modified versions examined in the present paper. Fig. 3 presents measured and calculated results of the considered joint pdf. The measured results were obtained during field experiments by Goda (1978). His data come from 89 records exhibiting a single spectral peak and a correlation factor $r(H, T)$ ranging from 0.40 to 0.59. In total 2,593 individual waves formed the records of these measurements. The calculated results refer to the original DWModel and to the modified theory presented in this paper. All three modifications have been included one-by-one, namely, the improved calculation of the wave periods, then in addition the deep water wave breaking, and finally on top of the previous two modifications the separation of swell from seas.

The iso-density contours of the joint pdf in Fig. 3 have been kept to a minimum for clarity and ease of comparison. It is noted that the theoretical results of this figure were produced by the same original population of wavelets as modified subsequently, exhibiting a constant standard deviation of surface elevation $\sigma'_\eta = 1$, nondimensionalized with respect to half the mean wave height. As described earlier in the paper, the main concern at the application of both criteria of swell exclusion and of wave breaking was to adhere to the assumption of Gaussian distribution of water elevation, which yields a unique wave height distribution. Thus, inevitably, the correlation factor $r(H, T)$ varies through representations (b)–(e) of Fig. 3. A marked reduction of this factor appears after application of the breaking criterion; indeed, the deep water breaking process can be viewed, from the probability standpoint, as excluding "expected" (H, T) pairs from the wavelets population, thus reducing the correlation between H and T. It should be added, however, that this reduction diminishes as one moves to a higher σ'_η. Table 1 shows the variation of r-values corresponding to graphs (b)–(e) of Fig. 3. As expected, the impact of

TABLE 1. Modifications of $r(H, T)$

σ'_η (1)	Fig. 3(b) (2)	Fig. 3(c) (3)	Fig. 3(d) (4)	Fig. 3(e) (5)
1	0.44	0.34	0.10	0.51
2	0.69	0.70	0.65	0.68
3	0.78	0.80	0.79	0.79

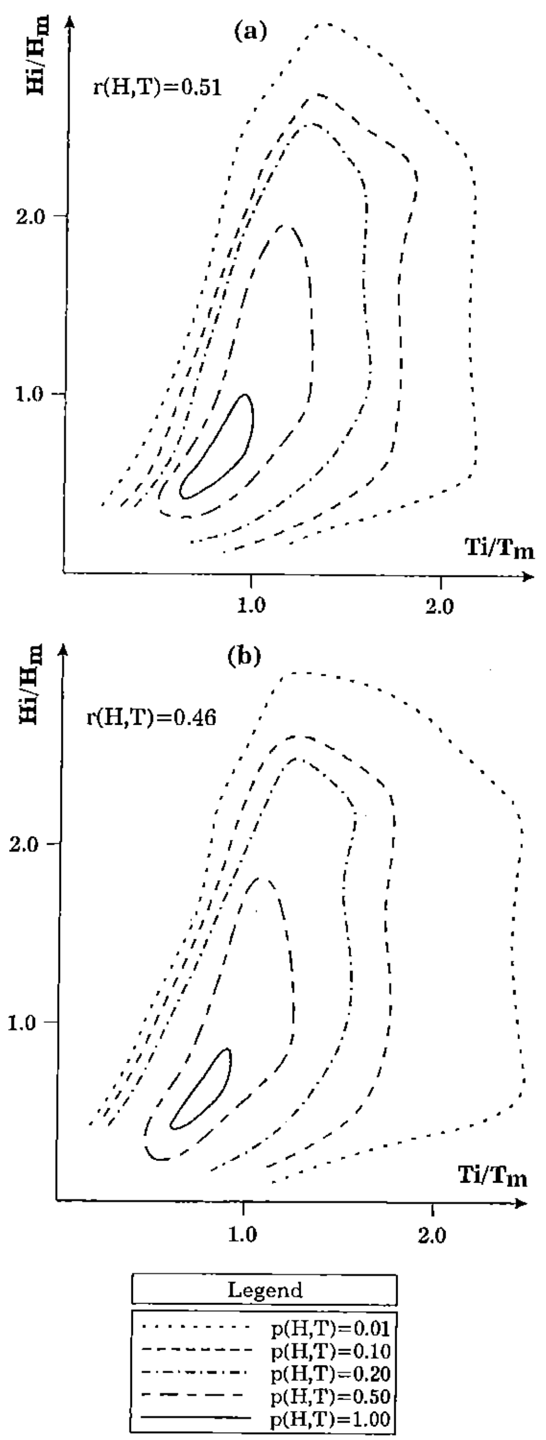

FIG. 4. Exclusion of Swell for $\sigma'_\eta = 1$: (a) $T/T_m \leq 2.0$; (b) $T/T_m \leq 2.5$

swell on the joint pdf diminishes with increasing r, i.e., for a wider frequency spectrum.

Inspection of Fig. 3 shows that the presented modifications improve the behavior of the DWModel. A noticeable improvement results after application of the deep water wave breaking criterion [graph (d)], whereas the exclusion of swell produces a relatively weaker effect. This was to be expected since in the original DWModel the presence of swell was small, anyway. Indeed, there is no significant variation between the results of swell exclusion if one applies the criterion $T/Tm \leq 2.5$ instead of $T/Tm \leq 2.0$, as shown in Fig. 3. This invariability is presented in Fig. 4, where a more complete (as compared with the previous figure) set of iso-density contours of the joint pdf have been drawn. Graph (a) of this figure corresponds to graph (e) of Fig. 3. The slight reduction in r of graph (b) with respect to (a) can be explained by the fact that, in the more diverse population of wavelets in the former graph, the correlation between wave heights and periods becomes looser.

It is evident from the previous discussion that the correlation factor $r(H, T)$ is a key parameter to the description of a sea state by use of the joint pdf, like the width parameter ε is to the energy spectral representation. The two parameters are interrelated in a manner not yet fully investigated. In order to study this topic for deep waters, a model has been developed that can produce the one-dimensional energy density spectrum of wave heights and periods. To this effect an assumption was made pertaining to the wave energy of any wavelet of the record constructed as described in the DWModel. This energy E was taken to be proportional to the wave height squared,

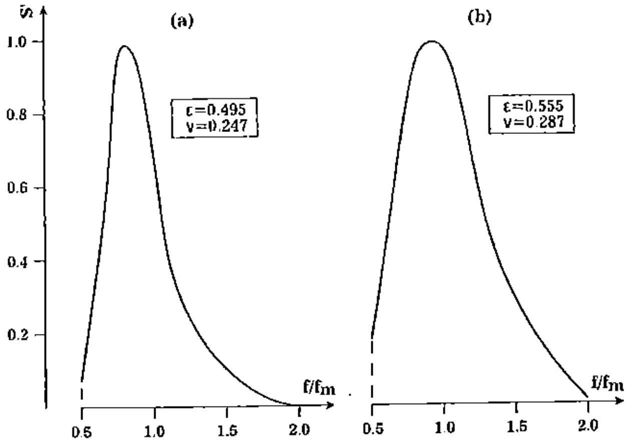

FIG. 5. Nondimensional Tower Spectra Produced by Corresponding Joint pdf Representations: (a) $\sigma'_\eta = 1$, $r(H, T) = 0.51$; (b) $\sigma'_\eta = 2$, $r(H, T) = 0.68$

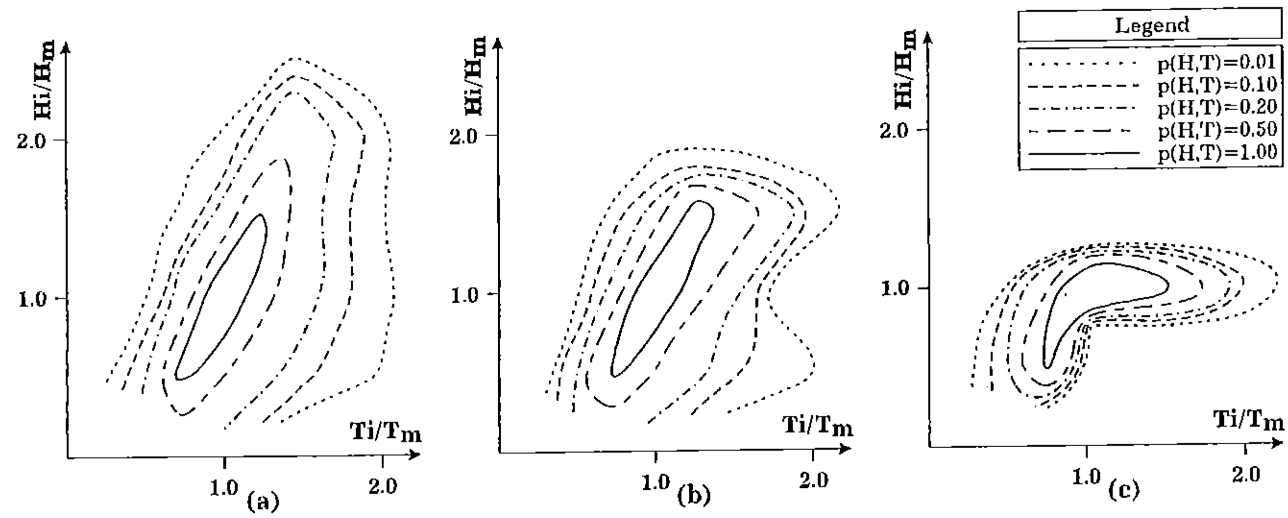

FIG. 6. Joint pdf (H, T) in Shoaling Water for $m = 5\%$: (a) $d/H_m = 3$; (b) $d/H_m = 2$; (c) $d/H_m = 1$

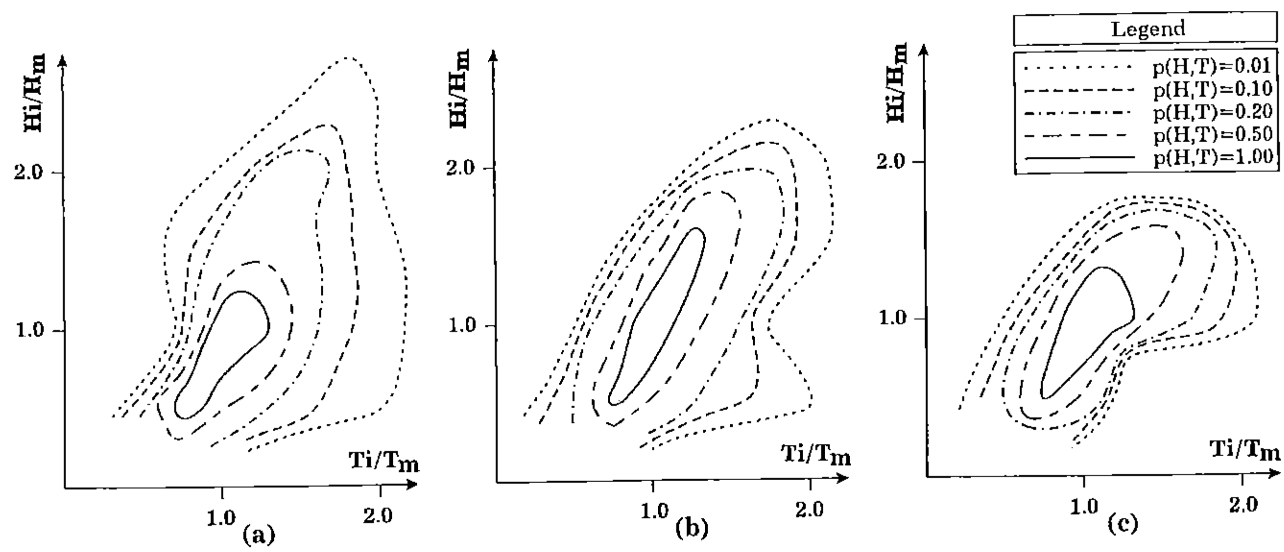

FIG. 7. Joint pdf (H, T) in Shoaling Water for $m = 10\%$: (a) $d/H_m = 3$; (b) $d/H_m = 2$; (c) $d/H_m = 1$

i.e., $E \sim H^2$, which is a plausible assumption. Also, the frequency f_i of any spectral component was assumed to be equivalent to the inverse of the time base T_i of the associated wavelet.

Let us suppose a sea state described by a correlation factor $r(H, T)$ and a wave steepness α, taking in the following the value of 0.05. For any given period T_i a set of waves of different wave heights H_{ij} can be determined. These wave heights are associated to the corresponding probability densities p_{ij} through the relevant joint pdf graph. The actual energy of waves with period T_i can be calculated by the densities p_{ij} as follows:

$$E_i = \Delta t_i N_o \sum_j H_{ij}^2 p_{ij} \Delta H_{ij} \tag{17}$$

where N_o = total number of waves; and H_{ij} = wave height corresponding to the density p_{ij}.

The power spectra of sea states characterized by their joint pdf can thus be determined through use of (17). Examples of these spectra in nondimensional form are given in Fig. 5 for two correlation factors $r(H, T)$, or equivalently for two nondimensional standard deviations of surface elevation σ'_η. In this figure, the resulting spectral width parameters ε and ν are shown. It is remembered that $\varepsilon = (1 - m_2^2/m_o m_4)^{1/2}$ and $\nu = (m_o m_2/m_1^2 - 1)^{1/2}$. The spectra were nondimensionalized with respect to the maximum spectral power calculated at the discrete values f_i, while as cut-off frequencies the values of $f/fm = 0.5$ and $f/fm = 2.0$ were taken.

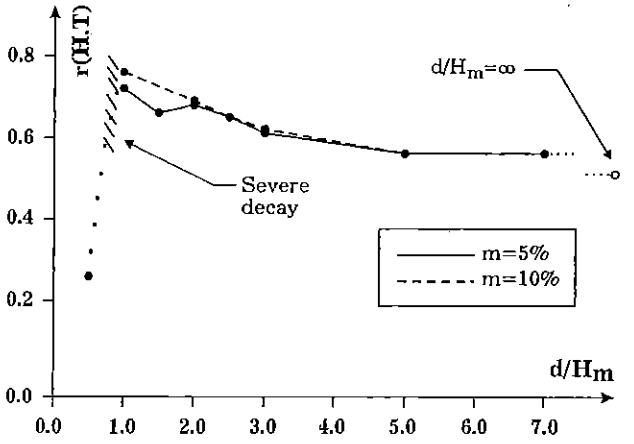

FIG. 8. Variation of $r(H, T)$ across Two Sloping Beaches

Shallow Water

The SWModel incorporating the modifications presented in the section on shoaling waves demands tedious and time-consuming calculations. The results, however, can be put in a nondimensional form with respect to d/H_m, where d = water depth and H_m = mean wave height in deep water. This is quite an important feature of the model, as it allows the unique representation of the joint pdf of wave heights and periods in a situation of wave shoaling perpendicular to shore, given the

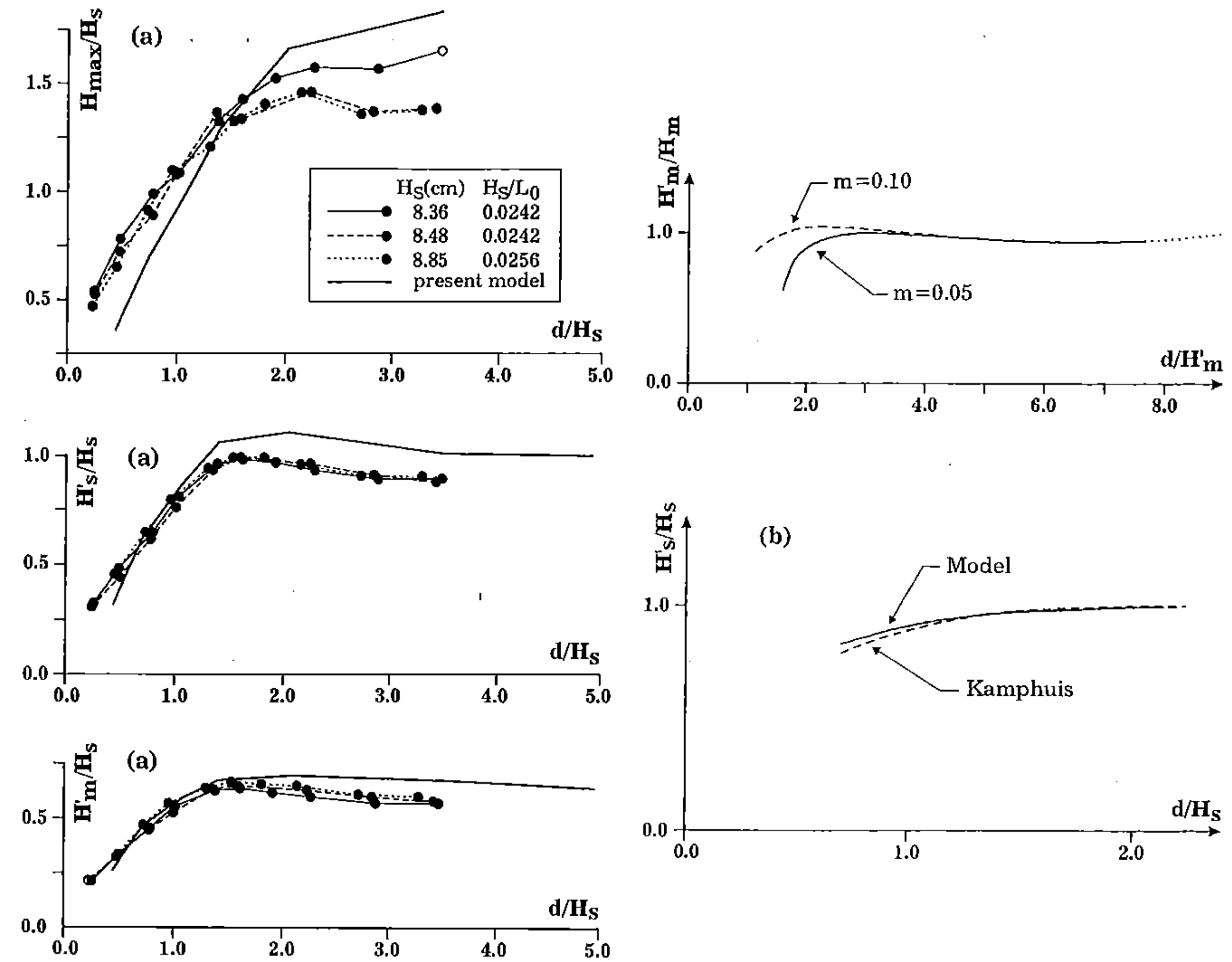

FIG. 9. Variation of Mean Wave Height across Beach, for m = 5% and m = 10%; Comparisons with: (a) Mase m = 5%; (b) Kamphuis m = 10%

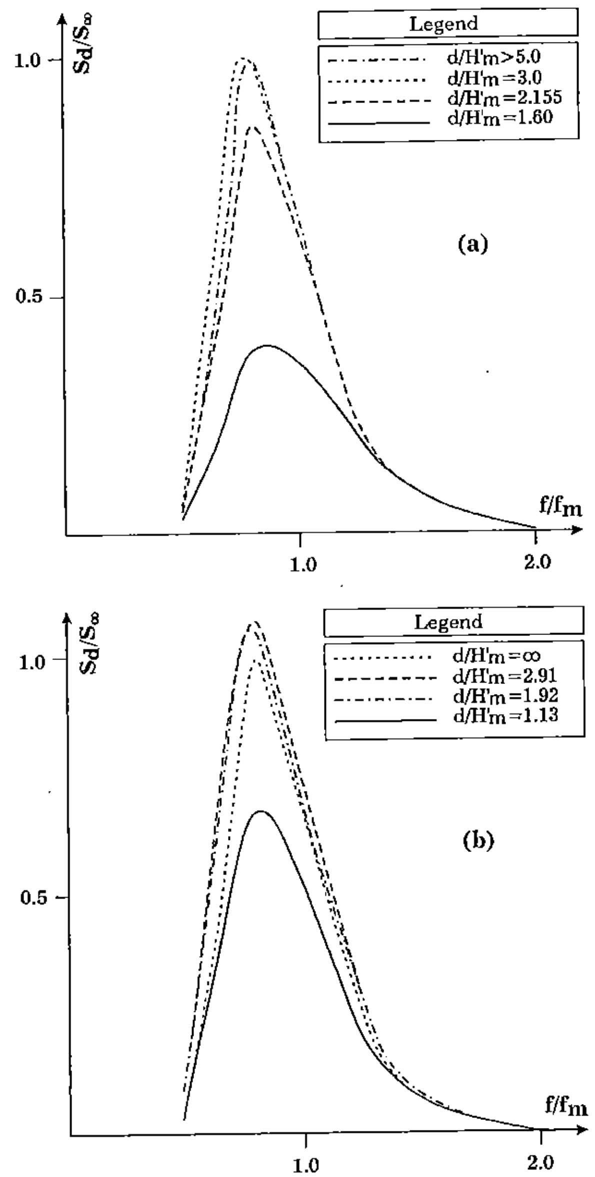

FIG. 10. Spectral Transformation across Beach Sloping Uniformly at (a) $m = 0.05$; (b) $m = 0.10$

TABLE 2. Variation of r and ε with Depth

d/H_m (1)	$m = 0.05$: r (2)	$m = 0.05$: ε (3)	$m = 0.10$: r (4)	$m = 0.10$: ε (5)
∞	0.51	0.50	0.51	0.50
3	0.62	0.50	0.61	0.48
2	0.68	0.50	0.69	0.51
1	0.72	0.52	0.78	0.50

Note: d = local depth; H_m = deep water wave height.

local depth, the bed slope, and the corresponding joint pdf offshore. In Figs. 6 and 7, results for the joint pdf over uniform slopes of 5 and 10%, respectively, are shown, for an offshore pdf with $r(H, T) = 0.51$, as shown in Fig. 4(a). H_m denotes the deep water mean wave height and d is the local depth.

Figs. 6 and 7 show the transformation of the joint pdf (H, T) as the waves travel over a uniform slope into shallower waters. The evolution of the joint distributions of wave heights and periods tends to densify the probability contours around the line $H_i/H_m = 1$ as the waves move inshore, for both slopes. This tendency is to be expected due to the filtering of wave heights imposed by depth-induced wave breaking, and it has been verified experimentally by others (Doering and Donelan 1993). It can be seen, also, that the correlation coefficient $r(H, T)$ tends to increase as the waves move inshore. This implies in general a rather wider spectrum in shallow rather than in deep water, a fact not yet widely accepted. The actual variation of $r(H, T)$ as the waves travel across the beach is depicted in Fig. 8.

This graph shows for d/H_m greater than about 3.0 a slight modification in $r(H, T)$, which does not necessarily provide a corresponding monotonic variation of ε, as will be seen in the following. The general trend, however, is an increasing r as the waves travel inshore. This trend has been, also, verified experimentally for d/H_m greater than about 4, with smaller actual values of $r(H, T)$ (Memos 1994b). The graph in Fig. 8 indicates invariability of the correlation factor with respect to the bed slope, provided that the wavetrains are associated with respect to the same sea state in deep water. Nevertheless, a region within the surf zone can be detected, where a severe decrease of r takes place. In this zone a relation between r and ε has not been detected so far.

The variation of mean wave height across the shore is also reproduced correctly by the model, as can be verified by the results given in Fig. 9. These are produced again from an offshore pdf with $r = 0.51$, a value characterizing medium to broad-banded seas. The results are in agreement with various sets of measured data (see, e.g., the laboratory measurements of Isobe 1985), of which some with similar offshore characteristics and a joint pdf of $r(H, T) = 0.51$ have been included in Fig. 9 (Mase 1989; Kamphuis 1994). Mase's data refer to laboratory experiments with random waves of varying groupiness factor fitting a Pierson-Moskowitz spectrum with peak frequency $Tp = 1.67$ s. Kamphuis' results are based on hydraulic-model data of long-crested irregular waves obeying a Jonswap spectrum with $H_s = 0.14$ m, $T_p = 1.15$ s. The primed wave heights denote local values, H_{max} is the local maximum wave height, and L_o is the deep water significant wavelength. As can be seen from Fig. 9, the present model predicts very well the wave height transformation in shallow waters and in particular in the surf zone. The deviations observed in the $m = 5\%$ case are probably due to somewhat different offshore conditions between the present model and the experimental input. These deviations could be accounted for by using linear shoaling theory as shown by Mase and Kobayashi (1991).

Referring to the transformation of the spectral representation produced by the joint pdf of wave heights and periods as described in the previous section, some results are presented in Fig. 10, where d denotes the water depth and H'_m = local mean wave height.

The nondimensionalized power spectra are seen to be smoothly reduced, as expected, with diminishing water depth. It was found previously that the correlation factor r between wave heights and periods increases as the waves travel inshore. However, the spectral width parameter ε remains insensitive to the depth variation. Table 2 contains values of both r and ε for bed slopes $m = 0.05$ and $m = 0.10$. It is noted that the water depths indicated in Table 2 are equal to the corresponding depths depicted in Fig. 10. The above values suggest that the interrelation of r-ε becomes quite weak in shallow water. Nevertheless this cannot be regarded yet as a general conclusion.

A direct comparison of the present model with experimental results by Doering and Donelan (1993) is quite interesting. Their experiments were carried out with random waves traveling over a uniformly sloping beach and the results were compared in the referenced paper with those of Longuet-Higgins (1983). Since the latter theory provides an analytical expression for the joint pdf in deep water dependent only on one parameter, namely ν, Doering and Donelan parameterized this

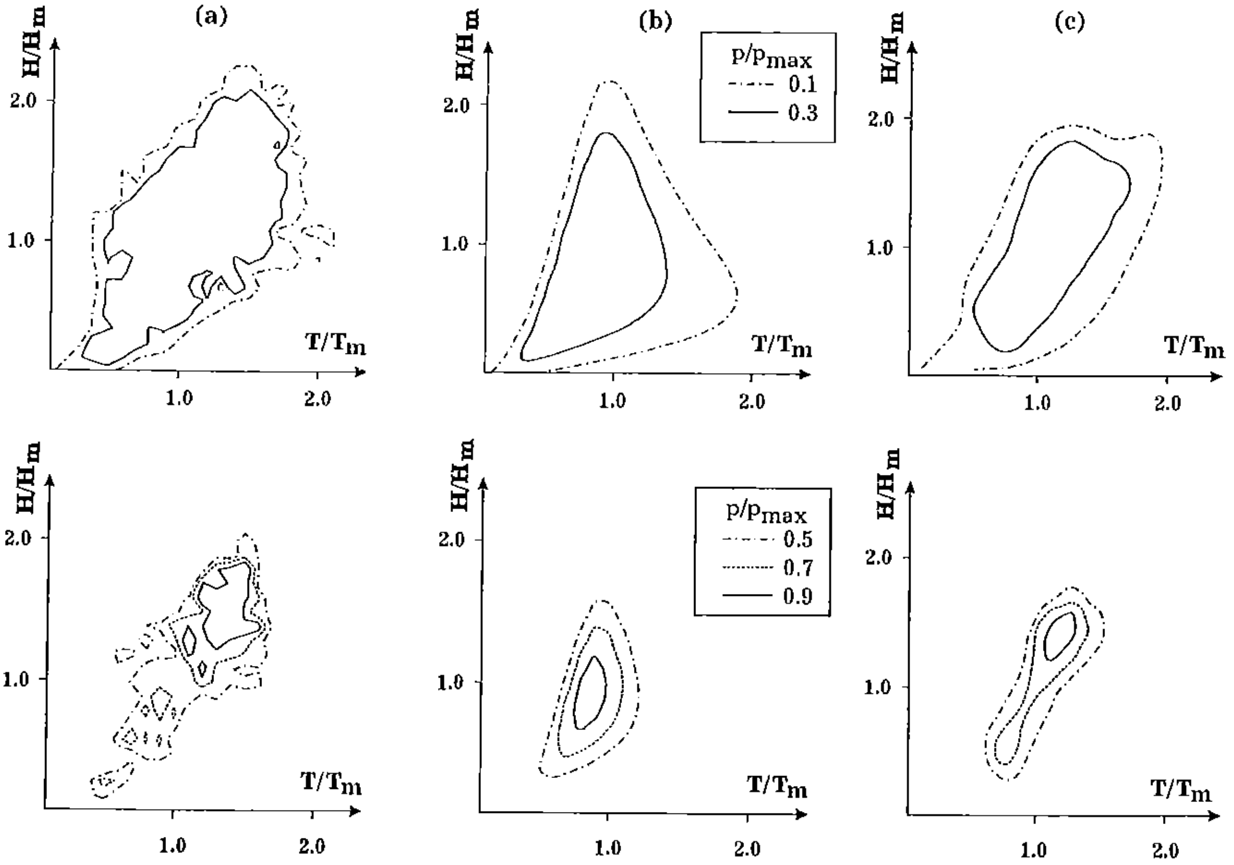

FIG. 11. Comparison of Experimental Data with Longuet-Higgins' Results and with Present Model: (a) Experimental Results; (b) Longuet-Higgins' Results; (c) Present Model, $d/H'_m = 2.0$, Beach Slope 1:20

with respect to Ursell's parameter and were thus able to extend the applicability of Longuet-Higgins' results into shoaling waters. Alternatively, they provided a linear shoaling transformation of the original Longuet-Higgins' deep water result. The target spectra used in the experiments were DHH spectra, after Donelan et al. (1985), with two peak enhancement values leading to either fully developed or strongly forced seas. For comparison with the present model, the former state was naturally chosen. There are two sets of results for fully developed seas presented in Doering and Donelan's paper, one referring to a beach slope 1:40 and the other to a slope 1:20. Since both sets present the same degree of deviation from Longuet-Higgins' results, only the 1:20 slope has been used for comparison with the present model. From the data given in that paper, the spectral width parameter ν has a value of 0.30 at an intermediate water depth of $d/L = 0.24$. Thus, the comparison was performed with a realization of the present model giving a lower value of ν in deep waters, say $\nu = 0.25$, and displaying a similar overall shape of the joint pdf graph at the "deep water" probe. The joint pdf calculated by the present model were found to compare better than those of Longuet-Higgins with the experimental measurements. In particular, this is clearly evident for small water depths where the joint distribution of broken waves is not well predicted by Longuet-Higgins' method. Fig. 11 depicts the said comparison at a relative depth $d/H'_m = 2.0$, where H'_m = local mean wave height, beach slope 1:20, and fully developed DHH target spectrum at the experiments. In this figure, the experimental data are compared (1) against Longuet-Higgins' results produced by the technique of ν-parameterization, which gives better agreement than the linearly shoaled joint distribution mentioned previously; and (2) with results of the present model based on joint distribution of wave heights and periods in deep water giving $\nu = 0.25$, a value assumed to be close to that of the corresponding target spectrum in deep water. The spectrum yielded by this joint distribution of the present model, incorporating swell filtering at $T/T_m = 2.0$, displays cut-off frequencies at $f/f_p = 0.5$ and 2.4 compared with the corresponding values of 0.5 and 3.0 of the spectrum used at the experiments. The ratio d/L, where L = wavelength, corresponding to the data of this figure, has a value of 0.08, close to the shallow water limit. It is evident from Fig. 11 that the present model behaves better than Longuet-Higgins' in shoaling water. This is mainly due to the fact that the present model incorporates the physics of the governing processes in shoaling waters including nonlinear wave shoaling and reforming, whereas Longuet-Higgins' theory depends solely on a mathematical treatment of random noise.

Another set of comparisons was conducted against experimental data covering the deeper part of the intermediate water depth, ranging at $0.25 < d/L < 0.50$. These experiments, presented and discussed by Memos (1994b), are similar to those previously mentioned in that they provide data at probes on a sloping beach that can yield the joint probability of wave heights and periods of a random wavetrain. The comparison was performed with measured results at the most innershore probe of the experimental setup. Fig. 12 shows both experimental and model results. The latter were produced by applying the wave-by-wave transformations upon the measurements, taken at the offshore probe. Inspection of the graphs of Fig. 12 shows good agreement between model and experimental results of the joint pdf of wave heights and periods. It is worth noting that the prediction of the mean wave height at the same location was 9.8 cm and the measured 10.3 cm a deviation of only 5%.

CONCLUSIONS

Two recent models of short-term joint distribution of wave heights and periods in deep and shoaling waters were im-

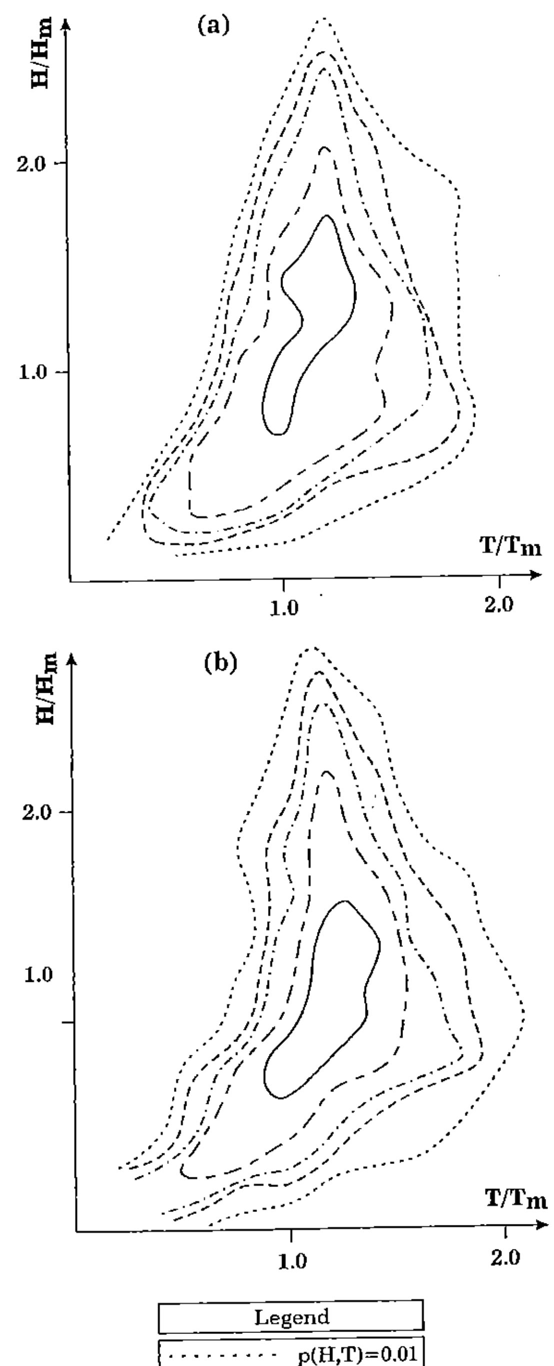

FIG. 12. Comparison of Model and Experimental Results (Memos 1994b): (a) Model; (b) Experiment

proved, extended, and compared with experimental and real-life data. There were also merged to form a single model spanning the whole coastal zone from the deep water well into the surf zone. To this end, coastal processes were incorporated pertaining to deep and shallow water wave breaking, separation of swell from seas, nonlinear wave shoaling, wave decay, and reforming after breaking. An improved estimation of wave periods was also included in the present model.

To identify a joint distribution of wave heights and periods in deep water, only two parameters are needed according to this model: the correlation factor $r(H, T)$ and an average wave steepness expressed by $\alpha = H_m/L_m$, where the subscript m denotes mean values in the considered sea state. The steepness parameter is required for the treatment of deep water wave breaking, while the correlation factor is a key parameter of sea states. This factor can be substituted by $\sigma'_\eta = \sigma_\eta/a_{min}$, where σ_η = standard deviation of surface elevation and a_{min} = threshold wave amplitude of the sea state under investigation. The nondimensional standard deviation σ'_η can be viewed as a characteristic wave height of the sea state, such as significant or mean wave height, nondimensionalized with respect to an assumed minimum or threshold wave height of the sea state, above which all waves are to be included in the sought joint distribution. It is noted that a similar threshold value appears also in long-term extreme wave statistics. Various runs with different mean steepness values α showed a small sensitivity of the model to this parameter, mainly because it relates only two representative measures of wave heights and wavelengths of the sea state without imposing any other condition to the many other pairs of the individual wavelengths and wave heights.

The merits of the joint distribution representation in deep water can be extended into shallow water by employing a wave-by-wave transformation. It was found that the results can be put in nondimensional form, thus being suitable for general application. This means that in practice only the local depth and the bed slope are needed, in addition to the parameters required in deep water, for the description of the sea state in shallow water through the joint distribution of wave heights and periods. Comparison of the present model with measurements show improved predictability over past models in all deep, transitional, and shallow waters.

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APPENDIX. REFERENCES

Blake, I. F., and Lindsey, W. C. (1973). "Level-crossing problems for random processes." *IEEE Trans. on Information Theory*, 19(3), 295–315.

Cavanié, A., Arhan, M., and Ezraty, R. (1976). "A statistical relationship between individual heights and periods of storm waves." *Proc., Behaviour of Offshore Struct.*, Norwegian Institute of Technology, Trondheim, Norway, 354–360.

Dally, W. R. (1992). "Random breaking waves: field verification of a wave-by-wave algorithm for engineering application." *Coast. Engrg.*, 16, 369–397.

Dally, W. R., Dean, R. G., and Dalrymple, R. A. (1985). "Wave height variation across beaches for arbitrary profile." *J. Geophys. Res.*, 90(6), 11917–11927.

Doering, J. C., and Donelan, M. A. (1993). "The joint distribution of heights and periods of shoaling waves." *J. Geophys. Res.*, 98(7), 12543–12555.

Donelan, M. A., Hamilton, J., and Hui, W. H. (1985). "Directional spectra of wind-generated waves." *Philosophical Trans. Royal Soc.*, London, 315, 509–562.

Goda, Y. (1975). "Irregular wave deformation in the surf zone." *Coast. Engrg. in Japan*, Tokyo, 18, 13–26.

Goda, Y. (1978). "The observed joint distribution of periods and heights of ocean waves." *Proc., 16th Conf. on Coast. Engrg.*, ASCE, New York, 227–246.

Goda, Y. (1985). *Random seas and design of maritime structures*. University of Tokyo Press, Tokyo.

Goda, Y. (1986). "Effect of wave tilting on zero-crossing wave heights and periods." *Coast. Engrg. in Japan*, Tokyo, 29, 79–90.

Isobe, M. (1985). "Calculation and application of first-order cnoidal wave theory." *Coast. Engrg.*, 9, 309–325.

Kamphuis, J. W. (1994). "Wave height from deep water through breaking zone." *J. Wtrwy., Port, Coast., and Oc. Div.*, ASCE, 120(4), 347–367.

Laitone, E. V. (1963). "Higher approximation to non-linear water waves and the limiting heights of cnoidal, solitary and Stokes waves." *Rep. TM-133*, U.S. Army Corps of Engineers, Washington, D.C.

Le Méhauté, B., and Webb, L. M. (1964). "Periodic gravity waves over

a gentle slope at a third order of approximation." *Proc., 9th Conf. on Coast. Engrg.*, ASCE, New York, 23–40.
Longuet-Higgins, M. S. (1983). "On the joint distribution of wave periods and amplitudes in a random wave field." *Proc., Royal Soc.*, London, 389, 241–258.
Mase, H. (1989). "Groupiness factor and wave height distribution." *J. Wtrwy., Port, Coast., and Oc. Div.*, ASCE, 115(1), 105–121.
Mase, H., and Iwagaki, Y. (1982). "Wave height distributions and wave grouping in surf zone." *Proc., 18th Conf. on Coast. Engrg.*, ASCE, New York, 58–76.
Mase, H., and Kobayashi, N. (1991). "Transformation of random breaking waves and its empirical numerical model considering surf beat." *Proc., Coast. Sediments '91*, ASCE, New York, 688–702.
Memos, C. D. (1994a). "On the theory of the joint probability of heights and periods of sea waves." *Coast. Engrg.*, 22, 201–215.
Memos, C. D. (1994b). "Experimental results of wave transformation across a sloping beach." *Proc., 24th Conf. on Coast. Engrg.*, ASCE, New York, 2350–2364.
Memos, C. D., and Tzanis, K. (1994). "Numerical results of the joint probability of heights and periods of sea waves." *Coast. Engrg.*, 22, 217–235.
Newland, D. E. (1984). *An introduction to random vibrations and spectral analysis*. Longman's, London.
Ochi, M. K., and Tsai, C. H. (1983). "Prediction of occurrences of breaking waves in deep water." *J. Physical Oceanogr.*, 13, 2008–2019.
Shuto, N. (1974). "Nonlinear long waves in a channel of variable section." *Coast. Engrg. in Japan*, Tokyo, 17, 1–12.
Stansberg, C. T. (1994). "Effects from directionality and spectral bandwidth on nonlinear spatial modulations of deep-water surface gravity wave trains." *Proc., 24th Conf. on Coast. Engrg.*, ASCE, New York, 579–593.
Vinje, T., and Brevig, P. (1981). "Numerical simulation of breaking waves." *Advanced Water Resour.*, 4, 77–82.
Weggel, J. R. (1972). "Maximum breaker height." *J. Wtrwy., Harb., and Coast. Engrg. Div.*, ASCE, 98(4), 529–548.