

Foreword

The following is an amendment to the Mellor (2003) paper which has undergone several revisions in my head including a Corrigenda which is hereby superseded. This amended 2003 paper has been prepared for my own satisfaction, but it might be helpful to users.

The overall scheme of the paper was correct in my opinion. The primary need for the corrections resided in my slowly evolving understanding of the proper treatment of pressure. In Mellor (2011), I determined that pressure (relative to atmospheric pressure) should be represented by

$$p(x_a, z, t) = ga [\cosh k(z+h) / \cosh kD] [\cos \psi - \tilde{w}^2 + g(\hat{\eta} - z)]$$

resulting from an irrotational flow field. It does not exactly satisfy the condition, $p(x_a, \hat{\eta} + \tilde{\eta}, t) = 0$; in fact, $p(x_a, \hat{\eta} + \tilde{\eta}, t) = 2^{-1}(ka)^2 c^2 \cos 2\psi$ [plus higher order (ka) terms]; Thus, the phase-averaged $\overline{p(x_a, \hat{\eta} + \tilde{\eta}, t)} = 0$ whereas $\overline{p(x_a, z, t)} = \hat{p} = -\tilde{w}^2 + g(\hat{\eta} - z)$, the latter in agreement with Longuet-Higgins and Stewart (1964; hereafter L-HS), Phillips (1977), and Mellor (2011). In addition to the changes explicitly entered into the following pages, please delete the term, $s_a \tilde{p}$ as it appears 3 times following Eq. (37); the term is nil as corrected in (34d).

However, in Mellor (2011), a term which is part of the wave radiation stress was conceived to be singularly concentrated at the surface and represented by a Dirac delta function; this seemed to be a natural consequence of the methodology of L-HS. This singular behavior has been criticized in a recent unpublished paper by Hidenori Aiki and Richard Greatbatch. There are similarities in their basic approach and mine, but I confess that I don't understand other parts of their paper. Nevertheless, the paper has caused me to rethink various aspects of the 2003 paper and I now agree that the term in question is not singular. This part of the radiation stress, now corrected, is distributed, but still localized near the surface as in $\exp(2kz)$. The paper is now more internally consistent.

The final results of the paper are equations (50), (51) and (52) representing coupled wave-circulation dynamics.

I regret my slow evolution of understanding which may be assuaged by the fact that, at least, I am persistent and, if I **now** have this last bit right – and I believe it to be so – then other coupled wave-circulation papers may still be incorrect.

New References

Longuet-Higgins, M. S. and R. W. Stewart, 1964: Radiation stresses in water waves; a physical discussion with applications. *Deep-Sea Research*, **11**, 529-562.

Mellor, G. L., 2003: The three dimensional, current and surface wave equations. *J. Phys. Oceanogr.*, **33**, 1978-1989.

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The Three-Dimensional Current and Surface Wave Equations

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ABSTRACT

Surface wave equations appropriate to three-dimensional ocean models apparently have not been presented in the literature. It is the intent of this paper to correct that deficiency. Thus, expressions for vertically dependent radiation stresses and a definition of the Doppler velocity for a vertically dependent current field are obtained. Other quantities such as vertically dependent surface pressure forcing are derived for inclusion in the momentum and wave energy equations. The equations include terms that represent the production of turbulence energy by currents and waves. These results are a necessary precursor for three-dimensional ocean models that handle surface waves together with wind- and buoyancy-driven currents. Although the third dimension has been added here, the analysis is based on the assumption that the depth dependence of wave motions is provided by linear theory, an assumption that is the basis of much of the wave literature.

1. Introduction

In the last several decades there has been significant progress in the understanding and numerical modeling of ocean surface waves (e.g., Longue-Higgins 1953; Phillips 1977; WAMDI Group 1988; Komen et al. 1994) and similar progress has been made in ocean circulation and modeling (e.g., Bryan and Cox 1968; Blumberg and Mellor 1987; Bleck and Boudra 1986). However, the two streams are separate. More often than not, wave models do not recognize vertical current structure and ocean circulation models do not recognize surface waves as having any influence on the ocean.

In the wave modeling literature, before derivation of the surface wave equations, the equations are generally integrated from the bottom, $z = -h$, to the wave surface, $z = \eta$, and then are phase averaged. Furthermore, the slow (wind-, tide-, and density-driven) horizontal velocities are often stipulated to be independent of z a priori. The result is a mismatch between the wave models and three-dimensional circulation ocean models, which necessarily involve z -dependent horizontal velocities and other properties. For example, a recent paper by Xie et al. (2001) suggests including the conventional vertically integrated stress radiation terms from waves as forcing terms in the vertically dependent momentum equations, a strategy that is obviously incorrect. Dolata and Rosenthal (1984) did attempt to derive three-dimensional radiation stress terms but left out effects from

pressure so that their results differ from mine and, after vertical integration, differ from the corresponding terms in Phillips (1977). They did not address three-dimensional effects on the wave energy equation.

To contrast the developments in this paper with conventional logic and to simplify discussion, I will temporarily address deep water ($kh \gg 1$) propagating waves such that $\eta(x, t, \psi) = a(x, t) \cos \psi$, where $\psi = kx - \sigma t$; k is the wavenumber, σ is the frequency, $c \equiv \sigma/k$ is the phase speed, and a is wave amplitude. The associated horizontal and vertical velocities are $\tilde{u} = kac e^{kz} \cos \psi$ and $\tilde{w} = kac e^{kz} \sin \psi$. Reviewing conventional logic, I define the phase-averaging operator

$$\overline{(\quad)} = \frac{1}{2\pi} \int_0^{2\pi} (\quad) d\psi. \quad (1)$$

Whereas $\tilde{u} = 0$, vertical integrals such as

$$M \equiv \overline{\int_{-h}^{\eta} \tilde{u} dz} = \overline{\int_{-h}^{\eta} \tilde{u} dz} \quad (2)$$

are nontrivial; thus, approximately,

$$M = ac |e^{kz}|_0^{\eta} \cos \psi = \frac{ka^2 c}{2} \quad (3)$$

is the Stokes transport. This is said to be an Eulerian result.

One can alternately obtain the lowest-order Lagrangian velocity according to

$$u_L = \tilde{u} + \frac{\partial \tilde{u}}{\partial x} \tilde{z} = \frac{\partial \tilde{u}}{\partial x} \tilde{z} + \frac{\partial \tilde{u}}{\partial z} \tilde{z}, \quad (4)$$

where the particle displacement $(\tilde{x}, \tilde{z}) = \int_0^t (\tilde{u}, \tilde{w}) dt = ae^{kz} (-\sin \psi, \cos \psi)$, so that the Stokes drift is

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$$u_L = k^2 a^2 c e^{2kz} \quad (5)$$

and, in a similar way, $w_L = 0$. When (5) is integrated from $-h$ (where $kh \gg 1$) to 0, the result is identical to (3).

a. An alternate procedure

However, in this paper, I do not wish to first integrate vertically, because I seek z -dependent equations to complement the z -dependent circulation equations of three-dimensional ocean models. Furthermore, Lagrangian formulations are awkward when they involve equations more complicated than (4). Instead, let $\tilde{u}[x, z + \tilde{z}(t), t] = \tilde{u}(x, z, t) + \tilde{z}\partial\tilde{u}/\partial z$; then multiply $\tilde{u}$ by $1 + \partial\tilde{z}/\partial z$, which is the wave-distorted, normal flow area relative to the undistorted flow area. Thus,

$$\begin{aligned} \tilde{u}(x, z + \tilde{z}, t) \left(1 + \frac{\partial\tilde{z}}{\partial z} \right) &= \left(\tilde{u} + \tilde{z} \frac{\partial\tilde{u}}{\partial z} \right) \left(1 + \frac{\partial\tilde{z}}{\partial z} \right) \\ &= \tilde{u} \frac{\partial\tilde{z}}{\partial z} + \tilde{z} \frac{\partial\tilde{u}}{\partial z}. \end{aligned} \quad (6)$$

The first term on the right is due to the velocity-flow area correlation, and the second term is due to particle wave motion-velocity gradient correlation and is the same as the second term in (4). Thus, the Stokes drift velocity is

$$u_s(x, z, t) = \frac{\partial\tilde{z}\tilde{u}}{\partial z} = k^2 a^2 c e^{2kz}, \quad (7)$$

a result identical to (5). The Stokes transport, $M = \int_{-1}^0 u_s dz$, can be obtained directly from

$$M = \int_{-h}^0 \frac{\partial\tilde{z}\tilde{u}}{\partial z} dz = u(0)\tilde{z}(0) = \frac{ka^2c}{2}. \quad (8)$$

The correspondence between u_s and u_s is not accidental because, using the continuity equation, it can be shown that $\tilde{z}\partial\tilde{u}/\partial x = \tilde{u}\partial\tilde{z}/\partial z$.

A more formal procedure will be pursued below, but the essence is the same. I will obtain the above results along with other relations that are useful when incorporating wave effects into the three-dimensional, phase-averaged equations of motion; this is done by transforming the basic equations to a sigma-coordinate system. The motivation is not to obtain final sigma-coordinate equations. Rather it is a helpful first step in the process of deriving the three-dimensional wave interaction terms in the equations of motion. If one wishes final equations in Cartesian coordinates, the reverse transformation on the final equation set, derived later in this paper, is straightforward.

Note the fact that, in much of the literature, the mean wave momentum is conceptually thought to be trapped at the surface in consequence of (2) and the third term in (8). However, it is my opinion that it is more useful to conceive of the wave momentum as distributed con-

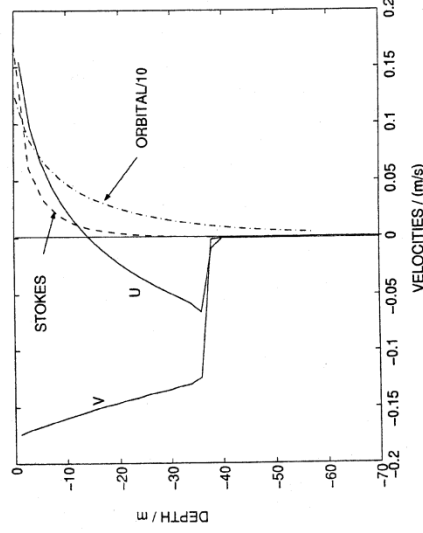


FIG. 1. Comparison of the wind-driven, mixed layer velocity components for a water side friction velocity of $u_{w*} = 0.02 \text{ m s}^{-1}$ and the Stokes drift velocity from integrals of the Pierson-Moskowitz spectrum for a wind, $U_{10} = 14.6 \text{ m s}^{-1}$; also shown are the rms wave orbital velocity $(\bar{u}^2 + \bar{w}^2)^{1/2}$ divided by 10. The friction velocity and wind are related according to $U_{10}/u_{w*} = \kappa^{-1} \ln(10 m/z_0)$, where Charney's wave roughness is taken to be $z_0 = 0.012 u_{w*}^2/g$, $\kappa = 0.40$, and $u_{w*}^2 = 860 u_{w*}^2$. The wind stress was initially ramped up from rest to the value u_{w*}^2 over one inertial period and then was held constant for 5 days. The initial temperature profile was linear with a vertical gradient of 0.05 K^{-1} .

tinuously down into the water column according to (7). Similar reasoning will be found in the paper by Longuet-Higgins (1969).

In Fig. 1, I show velocity profiles calculated from a one-dimensional turbulence closure model (Mellor and Yamada 1982) for a specific wind speed of 14.6 m s^{-1} (and kinematic stress of $0.33 \text{ m}^2 \text{ s}^{-2}$) and the Stokes drift velocity and wave orbital velocity corresponding to the same wind speed using the equilibrium wave spectrum according to Pierson and Moskowitz (1964). In the direction of the wind stress, the rms orbital velocities are an order of magnitude larger than the wind-driven velocity and the Stokes velocity. Following established convention, I refer to the wind-, baroclinic-, and Coriolis-driven velocities as "currents" that are distinct from the "Stokes drift." Figure 1 illustrates the point that, in general, the currents are not vertically constant—a seemingly trivial statement were it not for the fact that it is a requirement of present-day wave models. Although the relative magnitudes of the different velocity components are probably correct, the calculation of currents and Stokes drift, independent of each other as in Fig. 1, is not valid, as will be demonstrated in this paper.

b. This paper

In this paper, I specialize to monochromatic waves. Later extensions will undoubtedly generalize to a spectrum of waves for use in models.

The plan of this paper is to present the governing equations for the slowly changing variables, the wave variables, and the turbulence variables in section 2. The transformation to sigma coordinates is done in section 3. The well-known linear wave solutions are given in section 4. All terms appropriate to the continuity and momentum equations are obtained in section 5, and the wave energy equation is derived in section 6. The continuity and momentum equations are vertically dependent, whereas, by virtue of assuming the linear wave solutions to be valid to lowest order, one needs only the vertically integrated wave energy equation. Section 7 establishes the correspondence between the present results and the conventional momentum and energy equations. In section 8, additional terms that result from wind pressure processes are derived. The results are summarized in section 9 together with the equations for scalars and the turbulent kinetic energy. Appendix A considers interactions between the momentum and wave energy equations and appendix B defines the nomenclature of the velocities and scalars used in this paper.

2. The basic equations

The equations of motion are

$$\frac{\partial \mathcal{U}_i}{\partial x_j} = 0 \quad \text{and} \quad (9)$$

$$\frac{\partial \mathcal{U}_i}{\partial t} + \frac{\partial \mathcal{U}_i \mathcal{U}_i}{\partial x_i} + \epsilon_{ijk} f_j \mathcal{U}_k + \frac{\partial p}{\partial x_i} = -\frac{\rho g}{\rho_0} \delta_{i,2} \quad (10)$$

where $\mathcal{U}_i = (\mathcal{U}, \mathcal{V}, \mathcal{W})$, $x_i = (x, y, z)$, and $\mathcal{U}_i = \mathcal{U}_i(x, t)$. Here, $\mathcal{P} = \mathcal{P}(x, t)$ is the kinematic pressure—that is, the dynamic pressure divided by a reference density ρ_0 —and f_j is the Coriolis parameter. The coordinate z and velocity w are vertically upward. We have omitted viscous terms, but they can be reclaimed where they may be important locally such as next to a smooth bottom surface. The tensor ϵ_{ijk} is nil if any of the indices are repeated, is equal to 1 when i, j , and k are any triplet in the sequence xyz , and is -1 in any other sequence.

We now decompose $\mathcal{U}_i$ into three velocity components: a “slow” component $\hat{u}_i$, whose time- and space scales are L and T , respectively; a wave component $\tilde{u}_i$, whose smaller time- and space scales are k^{-1} and ω^{-1} , respectively; and a random turbulence component, $\hat{u}_i$, such that

$$\mathcal{U}_i = \hat{u}_i + \tilde{u}_i + \hat{u}_i \quad \text{and} \quad (11a)$$

$$\mathcal{P} = \hat{p} + \tilde{p} + \hat{p}. \quad (11b)$$

The density is split into a slow and fluctuating turbulence component

$$\rho = \hat{\rho} + \tilde{\rho}, \quad (11c)$$

therefore filtering out baroclinic interaction with surface waves. Using the above definitions, (10) may be written

$$\begin{aligned} & \frac{\partial}{\partial t}(\hat{u}_i + \tilde{u}_i + \hat{u}_i) \\ & + \frac{\partial}{\partial x_j}(\hat{u}_i \hat{u}_j + \hat{u}_i \tilde{u}_j + \hat{u}_i \hat{u}_j + \hat{u}_i \tilde{u}_j + \tilde{u}_i \hat{u}_j + \tilde{u}_i \tilde{u}_j \\ & \quad + \tilde{u}_j \hat{u}_i + \tilde{u}_j \tilde{u}_i + \hat{u}_i \hat{u}_i) \\ & + \epsilon_{ijk} f_j (\hat{u}_k + \tilde{u}_k + \hat{u}_k) + \frac{\partial}{\partial x_i}(\hat{p} + \tilde{p} + \hat{p}) \\ & = -\left(\frac{\hat{\rho}}{\rho_0} + \frac{\tilde{\rho}}{\rho_0}\right) g \delta_{i,2}. \end{aligned} \quad (12)$$

It is possible to write equations for the three components, and I have done so. However, as will be seen in appendix A, the slow and phase-averaged wave components interact in a somewhat complicated way. I therefore write the equations for $u_i \equiv \hat{u}_i + \tilde{u}_i$ and $p \equiv \hat{p} + \tilde{p}$ such that

$$\frac{\partial u_i}{\partial x_i} = 0 \quad \text{and} \quad (13)$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} + \epsilon_{ijk} f_j u_k + \frac{\partial p}{\partial x_i} = -\frac{\hat{\rho}}{\rho_0} g \delta_{i,2} - \frac{\partial}{\partial x_j} \langle \tilde{u}_i \tilde{u}_j \rangle. \quad (14)$$

The Reynolds stresses are $\langle \tilde{u}_i \tilde{u}_j \rangle$, and so it is assumed that $\tilde{u}_i$ can be extracted from u_i , a process simpler in laboratory experiments than in field experiments in which the waves are cyclic (Jensen et al. 1989; Cheung and Street 1988) and $u_i = \langle \mathcal{U}_i(t) \rangle \equiv N^{-1} \sum_{j=1}^N \mathcal{U}_i(t + j\tau_p)$ is a phase-conditioned average, where τ_p is the wave period and N is a large number; other dependent variables are similarly processed. Field observations present a greater challenge in separating wave and turbulence properties.

The equations governing turbulence are

$$\frac{\partial \hat{u}_k}{\partial x_k} = 0 \quad \text{and} \quad (15)$$

$$\begin{aligned} & \frac{\partial}{\partial t} \hat{u}_i + \frac{\partial}{\partial x_j} (u_j \hat{u}_i + u_i \hat{u}_j + \hat{u}_i \hat{u}_i - \langle \hat{u}_i \hat{u}_i \rangle) \\ & + \frac{\partial \hat{p}}{\partial x_i} + \frac{\hat{\rho}}{\rho_0} g \delta_{i,2} = \nu \nabla^2 \hat{u}_i. \end{aligned} \quad (16)$$

Notice that $\langle \hat{u}_i \rangle = 0$ and that the averages of all terms in (16) are nil. I have added the viscous term (ν is kinematic viscosity) because, when (16) is converted to an energy equation, the term becomes the important turbulence kinetic energy dissipation.

3. A transformation

There is need for separate nomenclature for the horizontal and vertical coordinates; thus, $x_\alpha \equiv (x, y)$, $x_i \equiv (x_\alpha, z)$, and $u_i \equiv (u_\alpha, w)$. Next transform the dependent variables in (13) and (14) according to

$$\phi(x_\alpha, z, t) = \phi^*(x_\alpha^*, \zeta, t^*) \quad (17)$$

and the independent variables such that

$$x_\alpha = x_\alpha^*, \quad (18a)$$

$$t = t^*, \quad \text{and} \quad (18b)$$

$$z = s(x_\alpha^*, \zeta, t^*), \quad (18c)$$

where ζ and s are general but will be constrained shortly; see (23a,b). From (17), one has

$$\frac{\partial \phi}{\partial x_\alpha} = \frac{\partial \phi^*}{\partial x_\alpha^*} + \frac{\partial \phi^*}{\partial \zeta} \frac{\partial \zeta}{\partial x_\alpha}, \quad \frac{\partial \phi}{\partial z} = \frac{\partial \phi^*}{\partial \zeta} \frac{\partial \zeta}{\partial z},$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi^*}{\partial t^*} + \frac{\partial \phi^*}{\partial \zeta} \frac{\partial \zeta}{\partial t},$$

and from $\partial z / \partial x_\alpha = (\partial s / \partial x_\alpha^*) (\partial x_\alpha^* / \partial x_\alpha) + (\partial s / \partial \zeta) (\partial \zeta / \partial x_\alpha) = 0$, I obtain $\partial \zeta / \partial x_\alpha = -s_\alpha / s_\zeta$; in a similar way, $\partial \zeta / \partial z = 1 / s_\zeta$ and $\partial \zeta / \partial t = -s_t / s_\zeta$, where I define $s_\alpha \equiv \partial s / \partial x_\alpha^*$, $s_t \equiv \partial s / \partial t^*$, and $s_\zeta \equiv \partial s / \partial \zeta$. Putting these relations together yields

$$\frac{\partial \phi}{\partial x_\alpha} = \frac{\partial \phi^*}{\partial x_\alpha^*} - \frac{\partial \phi^*}{\partial \zeta} \frac{s_\alpha}{s_\zeta}, \quad (19a)$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi^*}{\partial \zeta} \frac{1}{s_\zeta}, \quad \text{and} \quad (19b)$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi^*}{\partial t^*} - \frac{\partial \phi^*}{\partial \zeta} \frac{s_t}{s_\zeta}. \quad (19c)$$

Using (19a,b), I now transform (13) (written as $\partial u_\alpha / \partial x_\alpha + \partial w / \partial z = 0$) and, at the same time, drop the asterisks so that

$$\frac{\partial u_\alpha}{\partial x_\alpha} - \frac{\partial u_\alpha}{\partial \zeta} \frac{s_\alpha}{s_\zeta} + \frac{\partial w}{\partial \zeta} \frac{1}{s_\zeta} = 0.$$

Next define

$$w = \tilde{w} + u_\alpha s_\alpha + s_t \quad (20)$$

so that, after some rearrangement,

$$\frac{\partial s_\alpha u_\alpha}{\partial x_\alpha} + \frac{\partial \tilde{w}}{\partial \zeta} + \frac{\partial s_t}{\partial t} = 0. \quad (21)$$

Equation (21) could have been derived from a control volume formulation (visualized with help from the top panel of Fig. 2); then $\tilde{w}$ is the component of (the nearly vertical) velocity normal to surfaces of constant ζ .

Equation (14), for the horizontal velocity components, can be similarly manipulated and transforms to

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial}{\partial x_\alpha} (s_\alpha u_\alpha) + \frac{\partial}{\partial x_\beta} (s_\beta u_\alpha u_\beta) + \frac{\partial}{\partial \zeta} (\tilde{w} u_\alpha) - \epsilon_{\alpha\beta\gamma} f_\gamma s_\alpha u_\beta \\ + \frac{\partial}{\partial x_\alpha} (s_t p) - \frac{\partial}{\partial \zeta} (s_\alpha p) \\ = - \frac{\partial}{\partial x_\beta} (s_\beta \langle \dot{u}_\alpha \dot{u}_\beta \rangle) + \frac{\partial}{\partial \zeta} (s_\beta \langle \dot{u}_\alpha \dot{u}_\beta \rangle) - \frac{\partial}{\partial \zeta} \langle \dot{w} \dot{u}_\alpha \rangle. \end{aligned} \quad (22)$$

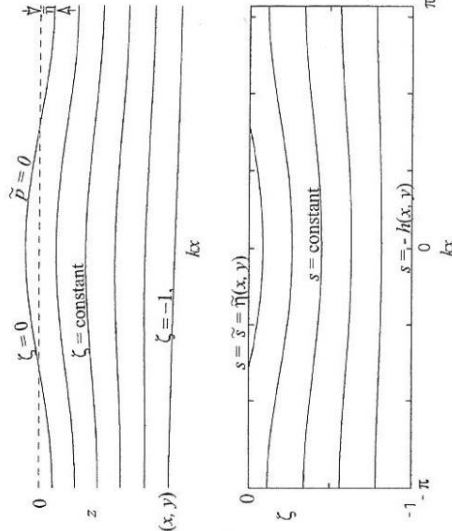


FIG. 2. An instantaneous sketch of wave particle position in (top) x, z space and (bottom) as transformed to x, ζ space according to (23) for $\dot{\eta} = 0$.

The transformation thus far is general. Now, however, (anticipating the linear wave solutions in the next section) define

$$s(x, y, \zeta, t) \equiv \hat{\eta} + \zeta D + \tilde{s} \quad \text{and} \quad (23a)$$

$$\tilde{s} \equiv a \frac{\sinh k D (1 + \zeta)}{\sinh k D} \cos \psi. \quad (23b)$$

The mean elevation is $\hat{\eta}(x, y)$; $h(x, y)$ is the bottom depth, and $D \equiv \hat{\eta} + h$ is the mean water column depth; $\hat{\eta}$, a , k , and D are slow functions of x, y , and t that vary on length scales and timescales of L and T , respectively, and $\psi \equiv k_\alpha x_\alpha - \omega t$, where k_α and ω are large in comparison with L^{-1} and T^{-1} . The “sigma” variable ζ ranges from -1 where $s = -h$ to 0 where $s = \eta = \hat{\eta} + \tilde{s}$. Figure 2 illustrates the transformation process for cases in which $\dot{\eta} = 0$ and there is zero atmospheric pressure.

The vertical derivatives of (23a,b) are

$$s_\zeta = \frac{\partial s}{\partial \zeta} = D + \tilde{s}_\zeta \quad \text{and} \quad (24a)$$

$$\tilde{s}_\zeta = k D a \frac{\cosh k D (1 + \zeta)}{\sinh k D} \cos \psi, \quad (24b)$$

and, in consequence of (20) at $\zeta = 0$ or $s = \eta$, $\tilde{w}(0) = 0$ so that $w(0) = u_\alpha(0) \partial \eta / \partial x_\alpha + \partial \eta / \partial t$ and, at $\zeta = 0$ or $s = -h$, $\tilde{w}(-1) = 0$ so that $w(-h) = -u_\alpha(-h) \partial h / \partial x_\alpha$.

The horizontal derivatives are

$$s_\alpha = \frac{\partial \hat{\eta}}{\partial x_\alpha} + \zeta \frac{\partial D}{\partial x_\alpha} + \tilde{s}_\alpha \quad \text{and} \quad (24c)$$

$$\tilde{s}_\alpha = -k_\alpha a \frac{\sinh k D (1 + \zeta)}{\sinh k D} \sin \psi \quad (24d)$$

4. The wave relations

The well-known wave relations are obtained from solutions of $\partial \tilde{u}_i / \partial x_j = 0$ and $\partial \tilde{u}_i / \partial t + \tilde{u}_\sigma \partial \tilde{u}_i / \partial x_\sigma + \partial / \partial x_i (p + gz) = 0$ wherein only the fast terms of order kL or ωT are retained in (13) and (14). The turbulence terms are assumed to be small. Solutions whose z -dependent arguments are $k(h + z)$ transform to $kD(1 + \xi + \tilde{s}/D)$ in accordance with (17), (18c) and (23a). The mean sea surface is at $z = \tilde{\eta}$. Thus, the wave velocities are

$$\tilde{u}_\alpha(x_\alpha, z, t, \psi) = k_\alpha c \frac{\cosh kD(1 + \xi + \tilde{s}/D)}{\sinh kD} \cos \psi \quad (25a)$$

and

$$\tilde{w}(x_\alpha, z, t, \psi) = kac \frac{\sinh kD(1 + \xi + \tilde{s}/D)}{\sinh kD} \sin \psi. \quad (25b)$$

Notice that $\tilde{w} = 0$ where $\xi = -1$.

The phase is defined as

$$\psi \equiv k_\sigma x_\sigma - \omega t. \quad (26)$$

The dispersion relation and intrinsic frequency are

$$\omega = \sigma + k_\alpha \tilde{u}_{\alpha\sigma} \quad \text{and} \quad (27a)$$

$$\sigma^2 = gk \tanh kD. \quad (27b)$$

The "Doppler velocity" $\tilde{u}_{\alpha\sigma}$ is, according to the analysis of Kirby and Chen (1989; and see other references in their paper), a weighted average of the vertically nonuniform current:

$$\tilde{u}_{\alpha\sigma} = 2 \int_{-1}^0 \tilde{u}_\sigma \frac{kD \cosh 2kD(1 + \xi)}{\sinh 2kD} d\xi. \quad (27c)$$

The intrinsic phase speed and group speed are

$$c \equiv \frac{\sigma}{k} = \sqrt{\frac{g}{k} \tanh kD} \quad \text{and} \quad (28a)$$

$$c_g \equiv \frac{d\sigma}{dk} = \frac{c}{2} \left(1 + \frac{2kD}{\sinh 2kD} \right). \quad (28b)$$

The surface elevation and pressure from waves are

$$\tilde{\eta}(x_\alpha, t, \psi) = a(x_\alpha, t) \cos \psi \quad \text{and} \quad (29a)$$

from Mellor (2011)

$$\tilde{p} = kac^2 \frac{\cosh kD(1 + \xi + \tilde{s}/D)}{\sinh kD} \cos \psi - g\tilde{s} - \tilde{w}^2 \quad (29b)$$

where, subsequently, it is useful to define

$$\tilde{p}_\psi \equiv ga \frac{\cosh kD(1 + \xi)}{\cosh kD} \cos \psi \quad (29c)$$

Although equations for k_α will not be needed in this paper, they are included for completeness. Thus, from (26), one has $k_\alpha \equiv \partial \psi / \partial x_\alpha$ and $\omega \equiv -\partial \psi / \partial t$ from

which we obtain $\partial k_\alpha / \partial t + \partial \omega / \partial x_\alpha = 0$ and $\partial k_\alpha / \partial x_\beta - \partial k_\beta / \partial x_\alpha = 0$ so that the wave number vector is irrotational. These equations and $\partial \sigma / \partial x_\alpha = (\partial \sigma / \partial k)(\partial k / \partial x_\alpha) + (\partial \sigma / \partial \omega)(\partial \omega / \partial x_\alpha)$ yield the well-known relation

$$\frac{\partial k_\alpha}{\partial t} + (c_{\sigma\beta} + \tilde{u}_{\beta\sigma}) \frac{\partial k_\alpha}{\partial x_\beta} = - \left(\frac{\partial \sigma}{\partial c_\sigma} \right)_k - k_\beta \frac{\partial \tilde{u}_{\beta\sigma}}{\partial x_\alpha},$$

where $c_{\sigma\beta} = (\partial \sigma / \partial k)(k_\beta / k)$. If the right side were nil (depth and Doppler velocity are constant), then the wave number vector is invariant along a trajectory prescribed by the combined group velocity and Doppler velocity.

5. The transformed continuity and momentum equations

The next step is to phase average the terms in (21) and (22) and to define a mean velocity $\bar{U}_\alpha$ such that

$$\begin{aligned} DU_\alpha &\equiv \overline{s_\alpha^2 \bar{U}_\alpha} = (D + \tilde{s}_\ell)(\bar{u}_\alpha + \tilde{u}_\alpha) \\ &= D\bar{u}_\alpha + D\bar{u}_{\alpha\sigma}, \end{aligned} \quad (30a)$$

where $u_{\alpha\sigma} \equiv (D + \tilde{s}_\ell)\bar{u}_\sigma / D$ is the Stokes drift velocity. Referring to (25a) and $\bar{u}_\alpha = \bar{u}_\alpha(x, y, 1 + \xi + \tilde{s}/D) = \bar{u}_\alpha(x, y, 1 + \xi) + \tilde{s}(\partial \bar{u}_\alpha / \partial \xi) / D$, we obtain (see section 1)

$$\begin{aligned} u_{\alpha\sigma} &= \overline{(1 + \tilde{s}_\ell/D)\bar{u}_\sigma} = \frac{1}{D} \frac{\partial \tilde{s}_\ell \bar{u}_\sigma}{\partial \xi} \\ &= \frac{k_\alpha}{k} \frac{(ka)^2}{2} c \frac{\cosh 2kD(1 + \xi)}{\sinh^2 kD} \\ &= \frac{2k_\alpha E \cosh 2kD(1 + \xi)}{c \sinh 2kD}. \end{aligned} \quad (30b)$$

Note that $u_{\alpha\sigma}/c$ is of order $(ka)^2$. Here and throughout the paper, I use the fact that phase averages of odd powers of $\cos \psi$ and products of $\cos \psi$ and $\sin \psi$ are nil. The expression on the right of (30b) uses (27b) and the definition, $E \equiv ga^2/2$; E will later be shown to be the sum of the kinetic and potential wave energies.

The product, $s_\ell u_{\alpha\sigma} \bar{u}_\beta$ in (22), when averaged, is algebraically complicated. Thus,

$$\begin{aligned} \overline{s_\ell u_{\alpha\sigma} \bar{u}_\beta} &= \overline{s_\ell (\bar{u}_\alpha + \tilde{u}_\alpha)(\bar{u}_\beta + \tilde{u}_\beta)} \\ &= D\bar{u}_\alpha \bar{u}_\beta + \bar{u}_\alpha (D + \tilde{s}_\ell) \bar{u}_\beta \\ &\quad + \bar{u}_\beta (D + \tilde{s}_\ell) \bar{u}_\alpha + D\bar{u}_\alpha \bar{u}_\beta \\ &= DU_\alpha \bar{U}_\beta + D\bar{u}_\alpha \bar{u}_\beta, \end{aligned} \quad (31)$$

where the terms $-D^{-1}(D + \tilde{s}_\ell)\bar{u}_\alpha (D + \tilde{s}_\ell)\bar{u}_\beta = DU_{\alpha\sigma} \bar{u}_\beta$ are of order $(ka)^2$ and are neglected.

Next define $\bar{\omega} \equiv \Omega(1 + \tilde{s}_\ell/D)$ so that

$$\bar{\omega} = \Omega \quad \text{and} \quad (32a)$$

$$\frac{\partial \bar{u}_\alpha}{\partial t} = \Omega \bar{U}_\alpha, \quad (32b)$$

where $\Omega(-1) = \Omega(0) = 0$. Using (30), (31), and (32), (21) and (22), after phase averaging, may be written

$$\frac{\partial DU_a}{\partial x_a} + \frac{\partial \Omega}{\partial \zeta} + \frac{\partial \hat{\eta}}{\partial t} = 0 \quad \text{and} \quad (33)$$

$$\frac{\partial}{\partial t}(DU_a) + \frac{\partial}{\partial x_\beta}(DU_a U_\beta) + \frac{\partial}{\partial \zeta}(\Omega U_a) + \epsilon_{\alpha\beta} f_\beta DU_\beta$$

$$- \frac{\partial \hat{p}}{\partial \zeta} \frac{\partial \hat{\eta}}{\partial \zeta} + \frac{\partial}{\partial x_a}(D\hat{p}) - \frac{\partial}{\partial \zeta} \left(\zeta \frac{\partial D}{\partial x_a} \hat{p} \right)$$

$$= - \frac{\partial}{\partial x_\beta} (D\bar{u}_\beta \bar{u}_\beta) - \frac{\partial}{\partial x_a} (s_\beta \bar{p}) + \frac{\partial}{\partial \zeta} (s_a \bar{p}) \quad (34a)$$

wherein we invoke the boundary layer approximation and neglect horizontal gradients of the Reynolds stresses. From the vertical hydrostatic relation and adding the buoyancy term

$$\hat{p} = -gD\zeta - gD \int_0^z (\hat{\rho} - \rho_0) / \rho_0 d\zeta. \quad (34b)$$

From (24a) and (29b,c) evaluate

$$\overline{s_\beta \bar{p}} = (D + \tilde{s}_\beta)(\tilde{p}_\beta + \frac{\partial \tilde{p}_\beta}{\partial \zeta} \tilde{s} - g\tilde{s} - \tilde{w}^2)$$

or

$$\overline{s_\beta \bar{p}} = \frac{\partial}{\partial \zeta} \left(\tilde{p}_\beta \tilde{s} - \frac{g\tilde{s}^2}{2} \right) - D\tilde{w}^2 \quad (34c)$$

Also,

$$\overline{s_a \tilde{p}} = (\eta_a + \zeta D_a + \tilde{s}_a) \left(\tilde{p}_a + \frac{\partial \tilde{p}_a}{\partial \zeta} \tilde{s} - g\tilde{s} - \tilde{w}^2 \right)$$

or

$$\overline{s_a \tilde{p}} = (\eta_a + \zeta D_a) \left(\frac{\partial \tilde{p}_a}{\partial \zeta} \tilde{s} - \tilde{w}^2 \right) = 0 \quad (34d)$$

The two terms in the last brace conveniently cancel. Note that on the left side of (34c) and (34d), the arguments of $\tilde{p}$ is $kD(1 + \zeta + \tilde{s}/D)$ whereas on the right side the argument of $\tilde{p}_\nu$ is $kD(1 + \zeta)$.

As we will see, the terms on the right of (34a) together with (34c) comprise the so-called radiation stress derived in their vertically integrated form by Longuet-Higgins and Stewart (1961).

6. The wave energy equation

To close the momentum equation derived in section 5, one needs to know $E(x_a, t)$ which is the vertically integrated or two-dimensional wave energy. Obtaining an equation for $E(x_a, t)$ involves complicated algebra, and I have followed Phillips (1977) but also utilized the

method developed above. Thus, multiply (14) by u_i to form a kinetic energy equation:

$$\frac{\partial}{\partial t} \left(\frac{u_i^2}{2} + gz \right) + \frac{\partial}{\partial x_\beta} \left(\frac{u_i^2}{2} u_\beta + p + gz \right) + \frac{\partial}{\partial z} \left(\frac{u_i^2}{2} + p + gz \right) = -u_i \frac{\partial}{\partial x_j} \langle \dot{u}_i \dot{u}_j \rangle. \quad (35)$$

The buoyancy term is excluded from (14) because I assume that buoyancy does not affect the near surface wave motion. Now transform (35) according to (17), (19), and (20) and phase average so that

$$\overline{\frac{\partial}{\partial t} \left(\frac{u_i^2}{2} + gs \right)} + \frac{\partial}{\partial x_\beta} \overline{\left(\frac{u_i^2}{2} + p + gs \right)} + \frac{\partial}{\partial \zeta} \overline{\left(\frac{u_i^2}{2} + p + gs \right)} = -\overline{u_i \frac{\partial}{\partial \zeta} \langle \dot{w} \dot{u}_i \rangle}. \quad (36a)$$

Because it is known that the Reynolds stress tensor components are of the same order, we have used the boundary layer approximation in the last term in (36a). Next, expand

$$\overline{s_\beta u_i^2/2} = s_\beta (\bar{u}_\beta^2/2 + \bar{u}_\beta \hat{u}_\beta + \bar{\hat{u}}_\beta^2/2) = D(\bar{u}_\beta^2/2 + u_{\beta\alpha} \hat{u}_\alpha + \bar{\hat{u}}_\beta^2/2)$$

and use $\bar{\hat{u}}_\alpha = U_\alpha - \bar{u}_{\alpha\alpha}$ to obtain

$$\overline{s_\beta u_i^2/2} = D(U_\beta^2/2 - u_{\beta\alpha}^2/2 + \bar{\hat{u}}_\beta^2/2) = D(U_\beta^2/2 + \bar{\hat{u}}_\beta^2/2). \quad (36b)$$

The deleted term is of order $(ka)^4$. In a similar way,

$$\overline{s_\beta u_\beta u_i^2/2} = DU_\beta U_\beta^2/2 + DU_\beta \bar{\hat{u}}_\beta^2/2 + DU_\beta \bar{\hat{u}}_\beta \hat{u}_\beta \quad (36c)$$

and

$$\overline{\omega u_i^2/2} = \Omega(U_\beta^2/2 + \bar{\hat{u}}_\beta^2/2). \quad (36d)$$

Neglecting buoyancy terms, one has $p = \hat{p} + \bar{p} = -\zeta gD + \bar{p}$ and, of course, $s = \hat{\eta} + \zeta D + \bar{s}$ so that $p + gs = \bar{p} + g\bar{s} + g\hat{\eta}$. Completing the operations involving these terms in (36a) yields

$$g\bar{s}\bar{s} = gD(\hat{\eta} + \zeta D) + g\bar{s}\bar{s}, \quad (36e)$$

$$\overline{s_\beta u_\beta(p + gs)} = DU_\beta g\hat{\eta} + U_\beta \bar{s}_\beta (\bar{p} + g\bar{s}) + D\bar{\hat{u}}_\beta (\bar{p} + g\bar{s}), \quad \text{and} \quad (36f)$$

$$\overline{\omega(p + gs)} = \Omega[g\hat{\eta} + \bar{s}_\beta (\bar{p} + g\bar{s})/D]. \quad (36g)$$

Inserting (36b–g) into (36a) yields

$$\frac{\partial}{\partial t} \left[\frac{DU_\beta^2}{2} + \frac{\bar{\hat{u}}_\beta^2}{2} + gD(\hat{\eta} + \zeta D) + g\bar{s}\bar{s} \right]$$

$$\begin{aligned}
& + \frac{\partial}{\partial x_\beta} \left[\frac{DU_\beta U_\alpha^2}{2} + DU_\alpha \bar{u}_\beta \bar{u}_\alpha + \frac{U_\beta D \bar{u}_\alpha^2}{2} \right. \\
& + \hat{u}_\beta \bar{s}_\zeta (\bar{p} + g\bar{s}) + U_\alpha g\hat{\eta} + D\bar{u}_\beta (\bar{p} + g\bar{s}) \left. \right] \\
& + \frac{\partial}{\partial \zeta} \left\{ \Omega \left[\frac{U_\alpha^2}{2} + \frac{\bar{u}_\alpha^2}{2} + g\hat{\eta} + \frac{\bar{s}_\zeta (\bar{p} + g\bar{s})}{D} \right] \right\} + \frac{\partial}{\partial \zeta} \bar{s}_\zeta p \\
& = -u_i \frac{\partial}{\partial \zeta} \langle \hat{w} u_i \rangle. \quad (37)
\end{aligned}$$

Now from (37), subtract U_α times (34a) and add $U_\alpha^2/2 - g\hat{\eta}$ times (33). After complicated but straightforward algebra, one obtains

$$\begin{aligned}
& \frac{\partial}{\partial t} \left[\frac{\bar{u}_\alpha^2}{2} + gD(\hat{\eta} + \zeta D) + g\bar{s}_\zeta \bar{s} - \frac{g\hat{\eta}^2}{2} \right] \\
& + \frac{\partial}{\partial x_\beta} \left[\frac{DU_\beta \bar{u}_\alpha^2}{2} + U_\beta \bar{s}_\zeta \bar{s} + D\bar{u}_\beta (\bar{p} + g\bar{s}) \right] + S_{\alpha\beta} \frac{\partial U_\alpha}{\partial x_\beta} \\
& + \frac{\partial}{\partial \zeta} \left\{ \Omega \left[\frac{\bar{u}_\alpha^2}{2} + \frac{\bar{s}_\zeta (\bar{p} + g\bar{s})}{D} \right] \right\} + \frac{\partial}{\partial \zeta} \bar{s}_\zeta p \\
& = -\hat{u}_\alpha \frac{\partial}{\partial \zeta} \bar{s}_\zeta \bar{p} - u_i \frac{\partial}{\partial \zeta} \langle \hat{w} u_i \rangle + \hat{u}_\alpha \frac{\partial}{\partial \zeta} \langle \hat{w} u_\alpha \rangle.
\end{aligned}$$

A term, $-u_\alpha \bar{s}_\zeta \partial \bar{p} / \partial \zeta$, on the right is of order $(ka)^4$ and has been discarded.

Next integrate from $\zeta = -1$ to 0 and obtain

$$\begin{aligned}
& \frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\beta} \left[c_{g\beta} E + \int_{-1}^0 \hat{u}_\beta \left(\frac{D\bar{u}_\alpha^2}{2} + g\bar{s}_\zeta \bar{s} \right) d\zeta \right] \\
& = - \int_{-1}^0 S_{\alpha\beta} \frac{\partial U_\alpha}{\partial x_\beta} d\zeta - \int_{-1}^0 \hat{u}_\alpha \frac{\partial}{\partial \zeta} \bar{s}_\zeta \bar{p} d\zeta - \int_{-1}^0 \bar{s}_\zeta p|_{-1} \\
& + \int_{-1}^0 u_{s\alpha} \frac{\partial}{\partial \zeta} \langle \hat{w} u_\alpha \rangle d\zeta - \int_{-1}^0 \hat{u}_i \frac{\partial}{\partial \zeta} \langle \hat{w} u_i \rangle d\zeta, \quad (38)
\end{aligned}$$

where the wave energy is defined according to

$$\begin{aligned}
E & \equiv \int_{-1}^0 [D(\bar{u}_\alpha^2/2) + g\bar{s}_\zeta \bar{s}] d\zeta \\
& = \int_{-1}^0 D(\bar{u}_\alpha^2/2) d\zeta + g\bar{\eta}^2/2. \quad (39)
\end{aligned}$$

Using (25) and (29), both terms on the right of (39) may be shown to be equal to $ga^2/4$ and their sum, $E = ga^2/2$. The term $|\bar{s}_\zeta \bar{p}|_{-1} = 0$, but it is left as a place holder, useful in section 8. To obtain (38), I have also used

$$\frac{\partial}{\partial x_\beta} \int_{-1}^0 D\bar{u}_\beta (\bar{p} + g\bar{s}) d\zeta = \frac{\partial}{\partial x_\beta} \left(c_{g\beta} \frac{ga^2}{2} \right) = \frac{\partial}{\partial x_\beta} (c_{g\beta} E),$$

where $c_{g\beta} = k_\beta c_g/k$ is the group velocity and

$$\frac{\partial}{\partial t} \left[gD \int_{-1}^0 (\hat{\eta} + \zeta D) d\zeta - \frac{g}{2} \frac{\partial \hat{\eta}^2}{\partial t} \right] = -\frac{g}{2} \frac{\partial h^2}{\partial t} = 0.$$

7. Correspondence with conventional equations

To compare with formulas in Phillips (1977) and others, neglect baroclinicity, so that $\bar{p} = -gD\zeta$, and turbulence terms; then integrate (33) and (34a) from $\zeta = -1$ to $\zeta = 0$ and obtain

$$\frac{\partial \hat{\eta}}{\partial t} + \frac{\partial M_\beta}{\partial x_\beta} = 0 \quad \text{and} \quad (40)$$

$$\begin{aligned}
& \frac{\partial}{\partial t} (M_\alpha) + \frac{\partial}{\partial x_\beta} \left(M_\alpha \frac{M_\beta}{D} + \bar{S}_{\alpha\beta} \right) + \epsilon_{\alpha\beta\gamma} f_\gamma M_\beta \\
& + gD \frac{\partial \hat{\eta}}{\partial x_\alpha} = 0, \quad (41)
\end{aligned}$$

where the transport is

$$M \equiv D \int_{-1}^0 U_\alpha d\zeta \quad \text{and} \quad (42)$$

$$\bar{S}_{\alpha\beta} \equiv E \frac{c_\alpha c_\beta}{c^2} + \delta_{\alpha\beta} E \left(\frac{c_\alpha}{c} - \frac{1}{2} \right). \quad (43)$$

The conventional simplification of (38) requires that $\hat{u}_\alpha$ be independent of ζ , in which case (38) reduces to

$$\frac{\partial}{\partial t} E + \frac{\partial}{\partial x_\alpha} [(c_{g\alpha} + \hat{u}_\alpha) E] = -\bar{S}_{\alpha\beta} \frac{\partial U_\alpha}{\partial x_\beta}. \quad (44)$$

Phillips had $\hat{u}_\alpha$ in place of U_α , but the difference is a term of order $(ak)^4$.

Except for the fact that I include the Coriolis terms in (41), I recover the depth-independent results obtained by Phillips (1977) and others.

8. Wind pressure forcing

Waves are driven by wind pressure fluctuations acting on the sloping sea surface (Miles 1954; Phillips 1954; Donelan 1999, etc.).

I will continue to represent the free wave pressure field by $\bar{p}(x, \zeta, t)$ to which I now add the pressure field driven by atmospheric pressure forcing. This pressure field may be written

$$p_w = \hat{p}_{\text{atm}} + \tilde{p}_w; \quad \tilde{p}_w = a_w \frac{\cosh kD(1 + \zeta)}{\cosh kD} \sin \psi,$$

where $\hat{p}_{\text{atm}}$ is the slowly varying atmospheric imposed pressure and $\tilde{p}_w$ is the solution for imposed surface fluctuation for the Fourier constituent in phase with $\partial \hat{\eta} / \partial x_\alpha$. Therefore,

$$\overline{\tilde{s}_a \tilde{p}_w} = \overline{\tilde{p}_{w\eta}} \frac{\partial \tilde{\eta}}{\partial x_\alpha} \frac{\cosh kD(1 + \zeta)}{\cosh kD} \frac{\sinh kD(1 + \zeta)}{\sinh kD},$$

where $\tilde{p}_{w\eta} = a_w \sin \psi$ is the wind pressure at the surface. Differentiation yields

$$\frac{\partial \tilde{s}_a \tilde{p}_w}{\partial \zeta} = \overline{\tilde{p}_{w\eta}} \frac{\partial \tilde{\eta}}{\partial x_\alpha} \frac{2kD \cosh 2kD(1 + \zeta)}{\sinh 2kD}, \quad (45)$$

which in the next section is a term that will be added to $\partial \tilde{s}_a \tilde{p} / \partial \zeta$ in (34a).

In a similar way,

$$\begin{aligned} \int_{-1}^0 \frac{\partial \tilde{s}_a \tilde{p}_w}{\partial \zeta} d\zeta + |\overline{s p_w}|_{-1}^0 \\ = \overline{\tilde{p}_{w\eta}} \frac{\partial \tilde{\eta}}{\partial x} \int_{-1}^0 \hat{u}_\alpha \frac{2kD \cosh 2kD(1 + \zeta)}{\sinh 2kD} d\zeta + \overline{\tilde{p}_{w\eta}} \frac{\partial \tilde{\eta}}{\partial t} \\ = \overline{\tilde{p}_{w\eta}} \frac{\partial \tilde{\eta}}{\partial x} \hat{u}_{\alpha a} + \overline{\tilde{p}_{w\eta}} \frac{\partial \tilde{\eta}}{\partial t}. \end{aligned} \quad (46)$$

Last, because $\overline{\tilde{p}_{w\eta}} \partial \tilde{\eta} / \partial t = (k_\alpha \omega / k^2) \overline{\tilde{p}_{w\eta}} \partial \tilde{\eta} / \partial x_\alpha$ and using (27a)

$$\int_{-1}^0 \hat{u}_\alpha \frac{\partial \tilde{s}_a \tilde{p}_w}{\partial \zeta} d\zeta + |\overline{s p_w}|_{-1}^0 = -\frac{k_\alpha}{k} c \overline{\tilde{p}_{w\eta}} \frac{\partial \tilde{\eta}}{\partial x_\alpha}, \quad (47)$$

which is a term that will be added to the corresponding free wave pressure terms in (38).

In potential flow over a solid wavy wall, Donelan (1999) reminds us that the pressure and wall slope are in antiphase so that, at the wave surface, the existence of a correlation between the two quantities depends on the existence of a nonzero relative phase shift. He advances specific laboratory-derived relations for $\overline{\tilde{p}_{w\eta}} \partial \tilde{\eta} / \partial x_\alpha$ as a function of the "wave age" defined as the ratio of wind speed to dominant phase speed, which, when considering only monochromatic waves, is equal to $c = \sigma / k$.

9. Summary and final comments

Here I summarize the final equations for which it is useful to define

$$F_{ss} = \frac{\sinh kD(1 + \zeta)}{\sinh kD}, \quad (48a)$$

$$F_{cs} = \frac{\cosh kD(1 + \zeta)}{\sinh kD}, \quad (48b)$$

$$F_{sc} = \frac{\sinh kD(1 + \zeta)}{\cosh kD}, \quad \text{and} \quad (48c)$$

$$F_{cc} = \frac{\cosh kD(1 + \zeta)}{\cosh kD}, \quad (48d)$$

after which (30b) may be written

$$u_{sa} = \frac{k_\alpha}{k} \frac{E}{cD} \frac{\partial F_{ss} F_{cc}}{\partial \zeta}. \quad (49)$$

The functions in (48) are plotted in Fig. 3. (The product kDF_{cs} is plotted because it is finite in the shallow water limit, $kD \rightarrow 0$.) The factor $\partial F_{ss} F_{cc} / \partial \zeta$ is plotted in Fig. 4.

In summary, I first repeat (33):

$$\frac{\partial DU_\alpha}{\partial x_\alpha} + \frac{\partial \Omega}{\partial \zeta} + \frac{\partial \tilde{\eta}}{\partial t} = 0. \quad (50)$$

Equation (34a), after addition of (45) may be written

$$\begin{aligned} \frac{\partial}{\partial t} (DU_\alpha) + \frac{\partial}{\partial x_\beta} (DU_\alpha U_\beta) + \frac{\partial}{\partial \zeta} (\Omega U_\alpha) + \epsilon_{\alpha\beta\gamma} f_\gamma DU_\beta \\ + D \frac{\partial}{\partial x_\alpha} (g\tilde{\eta} + \tilde{p}_{am}) + D^2 \int \left(\frac{\partial b}{\partial x_\alpha} - \zeta \frac{\partial D}{\partial x_\alpha} \frac{\partial b}{\partial \zeta} \right) d\zeta \\ = -\frac{\partial S_{\alpha\beta}}{\partial x_\beta} + \overline{\tilde{p}_{w\eta}} \frac{\partial \tilde{\eta}}{\partial x_\alpha} \frac{\partial F_{cc}}{\partial \zeta} - \frac{\partial}{\partial \zeta} \langle \tilde{w}^2 u'_\alpha \rangle \end{aligned} \quad (51a)$$

and from (34a,c,d)

$$S_{\alpha\beta} = D \left(\tilde{u}_\alpha \tilde{u}_\beta - \delta_{\alpha\beta} \tilde{w}^2 \right) + \delta_{\alpha\beta} \frac{\partial}{\partial \zeta} \left(\tilde{p}_{w\eta} \tilde{s} - g \tilde{s}^2 / 2 \right)$$

or

$$\begin{aligned} S_{\alpha\beta} = kDE \left(\frac{k_\alpha k_\beta}{k^2} F_{cs} F_{cc} - \delta_{\alpha\beta} F_{ss} F_{sc} \right) \\ + \delta_{\alpha\beta} \frac{E}{2} \frac{\partial}{\partial \zeta} \left(2F_{cc} F_{ss} - F_{ss}^2 \right) \end{aligned} \quad (51b)$$

The last term could be differentiated but in its present form it is easy to see that

$$\int_{-1}^0 \frac{\partial}{\partial \zeta} (2F_{cc} F_{ss} - F_{ss}^2) d\zeta = 1$$

and can be shown to behave as a Dirac delta function for large k as in Mellor (2011). In (51a), buoyancy is defined as

$$b = g(\rho - \rho_0) / \rho_0.$$

An important result is that the third term on the right of (51a), the wind forcing term, has the same vertical structure as the Stokes drift velocity in (49). Also, the second and third term have generally been lumped together in mixed layer modeling.

Upon insertion of (47), the surface wind pressure terms, the wave energy equation (38) becomes

$$\begin{aligned} \frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\alpha} [(c_{gea} + \hat{u}_{\alpha 2a})E] \\ = - \int_{-1}^0 S_{\alpha\beta} \frac{\partial U_\alpha}{\partial x_\beta} d\zeta - \int_{-1}^0 \tilde{s}_\alpha \tilde{p} \frac{\partial U_\alpha}{\partial \zeta} d\zeta \\ + \frac{k_\alpha}{k} c \overline{\tilde{p}_{w\eta}} \frac{\partial \tilde{\eta}}{\partial x_\alpha} - \tilde{u}_\eta \langle \tilde{w} u'_\eta \rangle_\eta + (\tilde{u}_{sa} \langle \tilde{w} u'_a \rangle)_\eta \\ + \int_{-1}^0 \langle \tilde{w} u'_i \rangle \frac{\partial \tilde{u}_i}{\partial \zeta} d\zeta - \int_{-1}^0 \langle \tilde{w} u'_a \rangle \frac{\partial u_{sa}}{\partial \zeta} d\zeta. \end{aligned} \quad (52a)$$

In (52a) we have defined

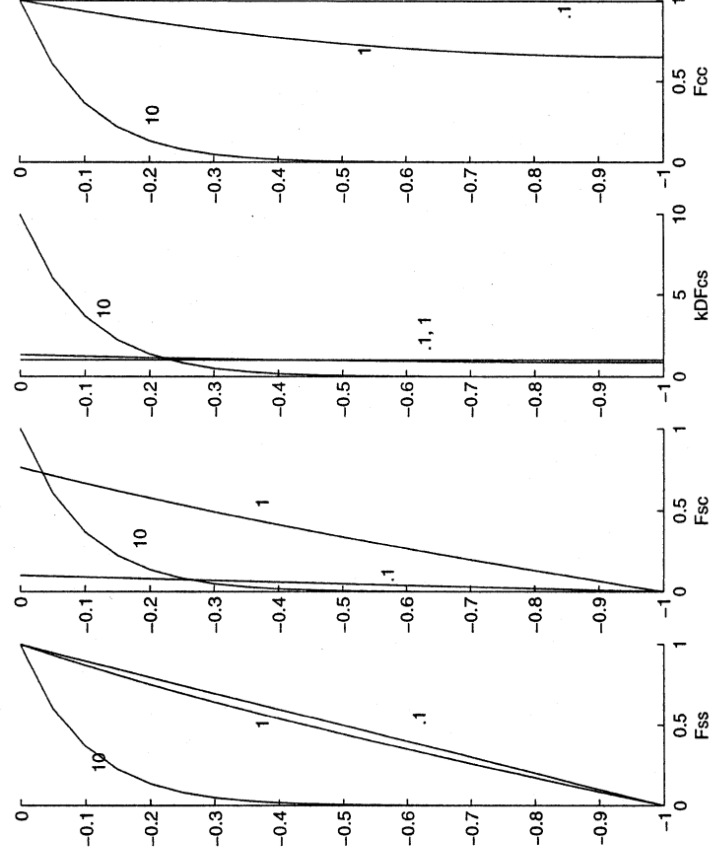


FIG. 3. Plots of the functions (48a-d). The labels on each curve are kD values. As $kD \rightarrow 0$, $F_{ss} \sim 1 + \xi$, $F_{sc} \sim 0$, $kDF_{cs} \sim 1$, and $F_{cc} \sim \infty$, F_{ss} , F_{cs} , and $F_{cc} \sim \exp(kD\xi)$.

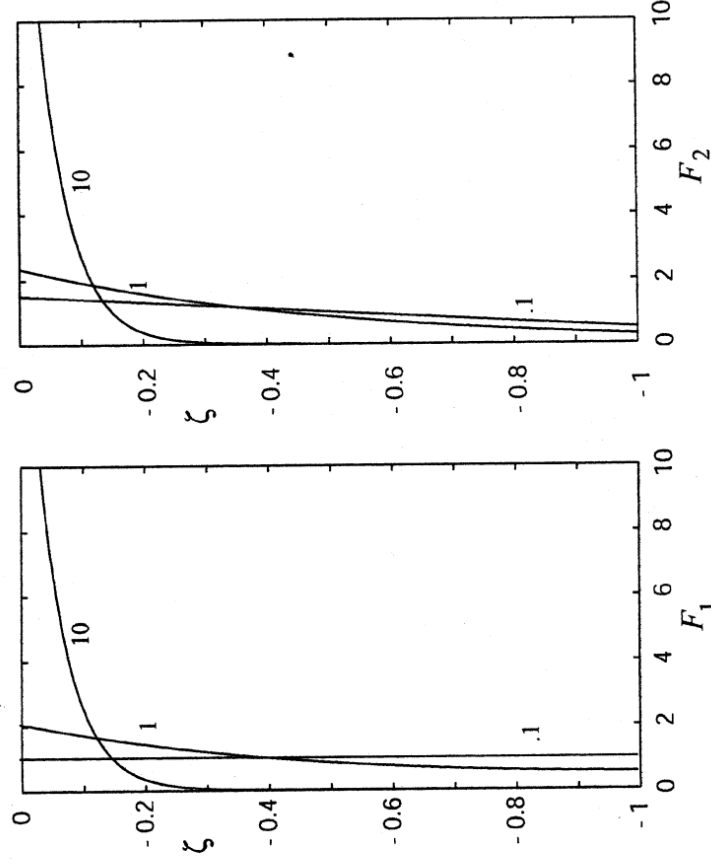


FIG. 4. Plots of the functions $F_1 \equiv \partial F_{ss} F_{cc} / \partial \xi = kD(F_{cs} F_{cc} + F_{ss} F_{sc})$ in (49) and (53) and $F_2 \equiv kD[(F_{cs} F_{cc} + F_{ss} F_{sc})/2 + F_{cs} F_{ss}]$ in (52b). Note that $u_{ss} \propto F_1$. The labels on each curve are kD values. As $kD \rightarrow 0$, $F_1 \sim 1$ and $F_2 \sim 2(1 + \xi)$. As $kD \rightarrow \infty$, F_1 and $F_2 \sim kD \exp(2kD\xi)$.

$$\begin{aligned}\hat{u}_{\theta\alpha} &\equiv \int_{-1}^0 \hat{u}_\alpha E^{-1} (D\hat{u}_i^2/2 + g\hat{s}\hat{s}_i) d\zeta \\ &= kD \int_{-1}^0 \hat{u}_\alpha [(F_{cs}F_{cc} + F_{ss}F_{sc})/2 + F_{cs}F_{ss}] d\zeta.\end{aligned}\quad (52b)$$

The phase and group speeds are as in (28a,b) and the Doppler velocity may be written

$$\hat{u}_{\theta\alpha} = kD \int_{-1}^0 \hat{u}_\alpha (F_{cs}F_{cc} + F_{ss}F_{sc}) d\zeta. \quad (53)$$

The weighting factors in (52b) and (53) are similar. They both integrate to unity cum sole and are identical in the short-wave limit but differ in the long-wave limit as shown in Fig. 4. Nevertheless, in the interest of simplicity, the approximation $\hat{u}_{\theta\alpha} \approx \hat{u}_{\alpha}$ is suggested. Furthermore, $\hat{u}_\alpha = U_\alpha$ is also an acceptable approximation because the difference $u_{\alpha\alpha}$ is of order $(ka)^2$ relative to the group velocity in (52a).

In (52a), the third, fourth, and fifth terms on the right are surface wind work terms, one from pressure and the others from turbulence. The sixth term is wave dissipation; any eddy viscosity model will yield a negative value.

Appendix A provides further discussion, useful for understanding interaction of the momentum and wave energy equations.

To model an ocean with interacting waves and currents, one needs (50), (51), and (52). To solve for any scalar Θ (such as temperature and salinity after which an equation of state will yield b), one can write

$$\frac{\partial}{\partial t}(D\Theta) + \frac{\partial}{\partial x_\beta}(DU_\alpha\Theta) + \frac{\partial}{\partial \zeta}(\omega\Theta) = \frac{\partial}{\partial \zeta}(-\overline{w\theta}), \quad (54)$$

which neglects interaction between the scalar properties and surface waves.

To complete the set of equations initiated in section 2, I write the turbulence kinetic energy equation

$$\begin{aligned}\frac{\partial}{\partial t}\left(D\frac{q^2}{2}\right) + \frac{\partial}{\partial x_\alpha}\left(DU_\alpha\frac{q^2}{2}\right) + \frac{\partial}{\partial \zeta}\left(\Omega\frac{q^2}{2}\right) \\ = -\frac{\partial}{\partial \zeta}(\overline{wu_i^2}) + \overline{\langle w\dot{p} \rangle} - Dg\frac{\overline{\langle \rho w \rangle}}{\rho_0} - D\epsilon,\end{aligned}\quad (55)$$

which is derived from (16) after multiplication of every term by $\dot{u}_i$; then, the terms are operated on by $\langle \rangle$, transformed to sigma coordinates, and phase averaged. Thus, $q^2 \equiv \langle \dot{u}_i^2 \rangle$ is 2 times the turbulence kinetic energy, and the terms on the right are turbulence diffusion, shear production, buoyancy production, and dissipation; the first and last terms need to be modeled. The turbulence energy equation is the common basis of many current turbulence closure models. The turbulence dissipation

represents the final conversion of wind work into thermal (or internal) energy.

Left for the future are prescriptions for the Reynolds flux, wind pressure forcing, and turbulence energy production. There is a large literature on empirical ways of determining the Reynolds stresses and fluxes where surface waves are neglected. And, utilizing the turbulence kinetic equation, there are beginning attempts to include the effects of wave breaking on current structure (e.g., Craig and Banner 1994; Terray et al. 1996). Now, however, there is a need to scrutinize the existing empirical knowledge base and to enter into new research framed by the continuity, momentum, wave energy, and turbulence energy equations derived in this paper.

Acknowledgments. I made much use of the book by Owen Phillips; if my equations, when vertically integrated, did not initially agree with those in his book, then I knew I was wrong and renewed effort to find the mistake. I profited from discussions with Mark Donelan, Gene Terray, and Jurjen Battjes. The research was funded by the NOPP surf-zone project and by ONR Grant N00014-01-1-0170.

APPENDIX A

Some Applications

Associated with a given wave energy E is a Stokes drift velocity given by (49b). However, how does a given velocity field develop and do (51) and (52) produce compatible results?

Consider a thought experiment in which, initially, $\hat{u}_\alpha = 0$, turbulence and Coriolis terms are nil, and the wave field is horizontally homogeneous (infinite fetch). Let the wave field be given by $k_\alpha = (k_x, 0)$ and define $P_x \equiv \overline{\tilde{p}_{w\eta}}\partial\tilde{\eta}/\partial x$ so that (51) and (52) reduce to

$$\frac{\partial}{\partial t}(DU_x) = P_x \frac{\partial F_{ss}F_{cc}}{\partial \zeta} \quad \text{and} \quad (A1a)$$

$$\frac{\partial E}{\partial t} = cP_x. \quad (A1b)$$

Notice that the U_x profile has the same vertical distribution as the Stokes profile in (30b). Let $P_x = 0$ for $t < 0$ and $P_x > 0$ for $t \geq 0$. Now, define

$$M_x \equiv D \int_{-1}^0 U_x d\zeta = \hat{M}_x + M_{sx},$$

and because, from (49), $M_{sx} \equiv \int_{-1}^0 u_{sx} d\zeta = E/c$, the momentum and energy equations result in the same vertically integrated Stokes transport such that $M_{sx} > 0$ and $\hat{M}_x = 0$. We conclude that the wind pressure forcing creates a pure Stokes velocity response. In particular, if $P_x = \text{constant}$ for $t \geq 0$, then

$$\hat{M}_x = 0, \quad \hat{M}_y = 0, \quad M_{sx} = P_x t, \quad \text{and} \quad M_{sy} = 0.$$

The situation changes if one includes Coriolis terms. The same wind forcing is invoked, so that

$$\frac{\partial(M_x, M_y)}{\partial t} + f(-M_y, M_x) = [P_x(t), 0] \quad \text{and} \quad (\text{A2a})$$

$$\frac{\partial(M_{sx}, M_{sy})}{\partial t} = [P_x(t), 0]. \quad (\text{A2b})$$

For zero forcing and steady state, $(M_x, M_y) = (0, 0)$, and, if a Stokes drift exists, the current transport $\hat{M}_x$ just cancels the Stokes drift; this was first demonstrated by Ursell (1950) using a vorticity/circulation argument. This has been said to be a paradox (Huang 1970; Xu and Bowen 1994) because there seems to be a discontinuity in solution form between $f = 0$ and $f > 0$ no matter how small the value f . But let us now set up a specific example flow that is steady and with a Stokes drift. Set $M \equiv M_x + iM_y$; then (A2a,b) may be written

$$\frac{\partial M}{\partial t} + i f M = P_x \quad \text{and} \quad \frac{\partial M_{sx}}{\partial t} = P_x.$$

For $P_x = 0$ when $t \leq 0$ and $P_x = \text{constant}$ for $t > 0$, the solutions for M and M_{sx} are

$$M = \int_0^t P_x(t') e^{i f (t-t')} dt' = -\frac{i P_x}{f} (1 - e^{-i f t}), \quad (\text{A3a})$$

$$M_{sx} = P_x t, \quad \text{and} \quad (\text{A3b})$$

$$M_{sy} = 0. \quad (\text{A3c})$$

If we create a top hat forcing such that $P_x = 0$ for $t > T$, where $T = 2\pi/f$ is the inertial period, then $M = 0$ or

$$\begin{aligned} \hat{M}_x &= -M_{sx}, & \hat{M}_y &= 0, \\ M_{sx} &= P_x T, & \text{and} & \quad M_{sy} = 0. \end{aligned}$$

This is just one of many ways to generate a steady state with a Stokes drift and an equal and opposite current. Similar results have been obtained by Hasselmann (1970) and Xu and Bowen (1994) but from a specialized and different analytical route, whereas here the same results are imbedded in (51) and (52).

For small f , (A3a) yields $M = P_x t$ so that now

$$\hat{M}_x = 0, \quad \hat{M}_y = 0, \quad M_{sx} = P_x t, \quad \text{and} \quad M_{sy} = 0$$

as in the case of no Coriolis term, thus resolving the aforementioned paradox because the solution evolves smoothly as f increases from small to large values.

In all of the above, it is evident that all components $\hat{u}_x$ and u_{sx} have the same z -dependent, Stokes-like profile shape.

These examples and others (e.g., the progression of dissipating waves on a shore) also illustrate how the current and Stokes drift may be coupled, as discussed in section 2.

APPENDIX B

Nomenclature

$\hat{u}_i$ = current velocity, slowly varying in space and time, $= (\hat{u}, \hat{v}, \hat{w}) = (\hat{u}_\alpha, \hat{w})$
 $\hat{u}_\alpha = (\hat{u}, \hat{v})$
 $\hat{p}$ = pressure, slowly varying
 $\hat{\eta}$ = surface elevation, slowly varying
 $\hat{u}_i$ = wave velocity $= (\hat{u}, \hat{v}, \hat{w})$
 $\hat{p}$ = wave pressure
 $\hat{\eta}$ = surface wave elevation
 $u_i = \hat{u}_i + \hat{u}_i$
 $p = \hat{p} + \bar{p}$
 u_i = turbulence velocity
 $U_\alpha = \hat{u}_\alpha + \hat{u}_{s\alpha}$
 ζ = transformed vertical coordinate such that $\zeta = -1$ when $z = -h$ and $\zeta = 0$ when $z = \hat{\eta} + \bar{\eta}$
 $z = -h$, the ocean bottom; $D \equiv h + \hat{\eta}$
 $\hat{M}_\alpha \equiv D \int_{-1}^0 \hat{u}_\alpha d\zeta$
 $M_{s\alpha} \equiv D \int_{-1}^0 u_{s\alpha} d\zeta$
 $\hat{M}_\alpha \equiv \hat{M}_\alpha + M_{s\alpha}$
 $\hat{u}_{A\alpha} = \text{weighted vertical average of } \hat{u}_\alpha \text{ equal to } \int_{-1}^0 r(\zeta) \hat{u}_\alpha d\zeta, \text{ where } \int_{-1}^0 r(\zeta) d\zeta = 1$
 $E \equiv g \bar{\eta}^2 = g a^2/2$, where g is the gravity constant and a is wave elevation amplitude

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