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Wave radiation stress

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Abstract There are differences in the literature concerning the vertically dependent equations that couple currents and waves. In this paper, currents are purposely omitted until the end. Isolating waves from currents allows one to focus on two main topics: an explanation of Stokes drift with apparent mean vorticity obtained from an otherwise irrotational flow and the determination of vertically dependent wave radiation stress which, when vertically integrated, conforms to that obtained by Longuet-Higgins and Stewart (1964) and Phillips (1977) nearly 50 years ago and, more recently, by Smith (2006). Discussion begins with the simple case of nonlinear flow beneath a stationary wavy wall.

Keywords Surface gravity waves · Coupled wave-circulation ocean models · Stokes drift · Radiation stress

1 Introduction

Radiation stress terms appear in the phase-averaged momentum equation of motion describing oceanic fluid flow and were identified by Longuet-Higgins and Stewart (1964). Much of this paper may be considered an extension of that tutorial paper wherein radiation stress terms were derived as phase-averaged, vertical integrals of velocity components and pressure due to surface waves. However, derivation of the depth-independent radiation stress terms is simpler than derivation of their depth-dependent counterparts.

If one believes, as I do, that the vertically integrated Longuet-Higgins and Stewart (1964) equations are correct, a worthy criterion is that any set of depth-dependent equations, when integrated, should correspond to the results obtained by Longuet-Higgins and Stewart (1964) and Phillips (1977) for small sea surface slope and other nondimensional current-related parameters. It will be seen that the three terms comprising the wave stress in Longuet-Higgins and Stewart (1964) are directly related to the corresponding vertically dependent terms derived herein.

This paper is partially motivated by recent papers (e.g., McWilliams et al. 2004; Ardhuin et al. 2008) that diverge from the Longuet-Higgins and Stewart (1964) findings. It is also motivated by the paper by Mellor (2003) much of which I believe is correct except for its detailed treatment of pressure terms resulting in error of the final form of the wave radiation stress terms. In order to focus on the important role of pressure in the derivation of radiation stress, currents will be neglected until the end of the paper; "currents" are defined as fluid velocities that vary on space and time scales which are large relative to inverse frequency and wave number. Except for Section 4, the development is restricted to deep water.

Before addressing the vertically dependent radiation stress terms and the corresponding vertically independent terms according to Longuet-Higgins and Stewart (1964), it is useful to derive the irrotational nonlinear solution of fluid flow under a stationary wavy wall in Section 2. In Section 3, the stationary solutions are transformed to progressive wave solutions. The resolved (instantaneous) solutions are phased-averaged in Section 4. To simplify and focus the discussion on Stokes drift and radiation stress, the relevant equations were restricted to deep water and two-

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dimensional (x,z) flow. However, extension to three-dimensional flow and shallow water is included in Section 5. A summary of the results are in Section 6.

2 Steady wavy wall flow

The inviscid, two-dimensional equations of motion are

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}, \tag{2a}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}-g. \tag{2b}$$

The velocity components (u, w) are in the (x, z) directions with z pointing upward; g is the gravity constant and p is kinematic pressure, i.e., dynamic pressure divided by constant density.

From Eqs. 1, 2a, and 2b, one can derive the vorticity transport equation, $\partial\omega_z/\partial t+u\partial\omega_z/\partial x+w\partial\omega_z/\partial z=0$, so that for a field initially irrotational,

$$\omega_z\equiv\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0 \tag{3}$$

thereafter. From the same equations, one can obtain Bernoulli's equation which for steady flow ($\partial u/\partial t=\partial w/\partial t=0$) is

$$p+\frac{u^2+w^2}{2}+gz=B \tag{4}$$

where B = constant.

Consider flow under a solid wavy wall with wave number k=2π/λ; λ is the wave length, a is the amplitude, and the depth is h. To simplify the following discussion, we stipulate that the water column is deep so that kh>>1 (in practice, $kh\geq 3$ is sufficiently deep). A solution satisfying Eqs. 1 and 3 is

$$u=-u_0+\widetilde{u};\quad \widetilde{u}(x,z)=kau_0\mathrm{e}^{kz}\cos kx, \tag{5a}$$

$$w=\widetilde{w};\quad \widetilde{w}(x,z)=kau_0\mathrm{e}^{kz}\sin kx, \tag{5b}$$

where the constant velocity, $-u_0$, is included. From $\partial\Psi/\partial z=u$, $\partial\Psi/\partial x=-w$, we obtain the stream function,

$$\Psi=-u_0z+au_0e^{kz}\cos kx. \tag{6}$$

Let the wall be set at z=$\eta(x)$ and Ψ=0 so that the nonlinear wall boundary is

$$\eta=ae^{k\eta}\cos kx. \tag{7}$$

Thus, Eqs. 5a and 5b is a flow where velocity at large depth is uniform and in the negative x-direction and bounded above by a wavy wall. In the limit as $ka\to 0$, a is the amplitude of the wavy boundary. From Eqs. 4, 5a, and 5b,

$$p(x,z)=-\frac{u_0^2}{2}+u_0^2kae^{kz}\cos kx-u_0^2(ka)^2\frac{e^{2kz}}{2}-gz+B. \tag{8}$$

At the wall,

$$p_\eta(x)=-\frac{u_0^2}{2}+u_0^2kae^{k\eta}\cos kx-u_0^2(ka)^2\frac{e^{2k\eta}}{2}-g\eta+B. \tag{9}$$

Figure 1 is a plot of $\eta(x)/a$ and $p_\eta(x)/u_0^2$ for the special case $u_0^2=g/k$ and for ka=0, 0.1, 0.2. The constant, B, has been set to $B=u_0^2/2$, but, of course, for a stationary solid wall, the "ambient" pressure could be any prescribed value.

Equations 5a, 5b, 7, and 8 are exact solutions for irrotational flow under a wavy wall. As will be seen below, the steady wave problem is simply related to progressive waves. The fact that pressure on the surface is not constant led to "stream function wave theory" (Dean 1965 and others) and a method of reducing the surface pressure variation.

2.1 Stokes drift

Particle trajectories were calculated numerically according to $dx_p/dt=u(x_p, z_p)$; $dz_p/dt=w(x_p, z_p)$ using Eqs. 5a and 5b. The time step was reduced until plots of successive trials showed no difference.

Notice that the particles, after traversing a wave length, have a Stokes drift component as illustrated in Fig. 2 for ka=0.2. The contribution of the wave portion of the flow yields higher average particle velocities near the surface than the particles in deeper water resulting in Stokes drift. A conclusion is that Stokes drift is not so much a result of propagating waves as it is the simple result of irrotational flow under a wavy boundary condition.

3 Progressive surface wave flow

Since c^2=g/k is the dispersion relation so that c=u_0, one can effect a Galilean transformation, $x\to x'-ct$; $u\to u'-c$. After dropping the prime superscripts, we have

$$u=kace^{kz}\cos\psi. \tag{10a}$$

$$w=kace^{kz}\sin\psi, \tag{10b}$$

$$\eta=ae^{k\eta}\cos\psi, \tag{11}$$

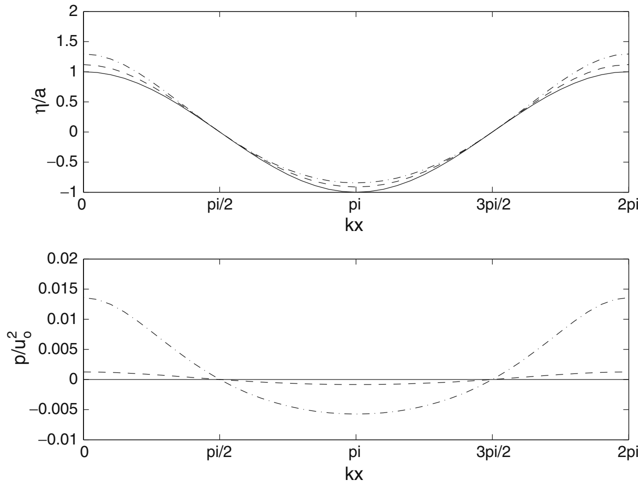

Fig. 1 *Top panel*: wall boundary for *ka*=0 (*solid line*), *ka*=0.1 (*dashed line*), *ka*=0.2 (*dot-dashed line*). *Bottom panel*: wall pressure for the same values of *ka*

and

$$p(x,z,t) = c^2 kae^{kz}\cos\psi - c^2(ka)^2\frac{e^{2kz}}{2} - gz + B. \qquad (12)$$

At the surface,

$$p_\eta(x,t) = c^2 kae^{k\eta}\cos\psi - c^2(ka)^2\frac{e^{2k\eta}}{2} - g\eta + B. \qquad (13)$$

In Eqs. 10a through 13, $\psi \equiv kx-\sigma t=k(x-ct)$ and $\sigma=kc$ is the frequency.

Note that in Section 2 and in Figs. 1 and 2, the special case of $u_0^2 = g/k$ was not an arbitrary choice; in the small *ka* limit, any other choice would result in $p_\eta(x)$ significantly departing from a constant independent of *x*.

3.1 Stokes drift

Figure 3 shows the numerically calculated Stokes drift for particles initially at $x=0$ and after one period for $ka=0$, 0.1, 0.2. All of the curves in Fig. 3 have been normalized such that $x_p/(k^2a^2cT)$ is plotted as functions of kz; $T=2\pi/\sigma$ is the period. The Stokes velocity is x_p/T. The curve for $ka=0$ (solid line) is the theoretical formula, $x_p/(k^2a^2cT) = e^{2kz}$; see "Appendix 1". The curve for $ka=0.2$ yields the same Stokes drift "deviation" as in Fig. 2 after multiplication by $2\pi(ka)^2$.

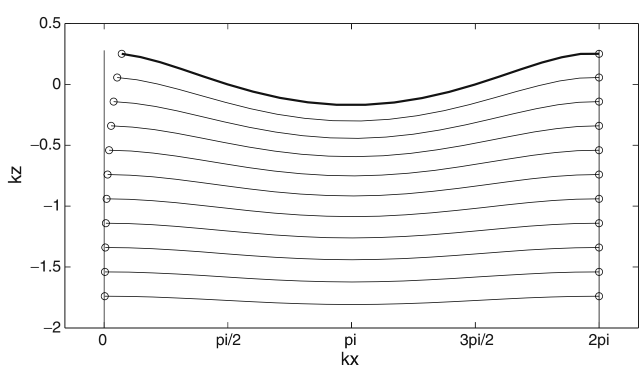

Fig. 2 Particle trajectories (and streamlines) under a wavy wall for particles launched at $kx=2\pi$ and after a time, $2\pi/(ku_0)$. In this example, $ka=0.2$. The arrival deviation at $kx=0$ is Stokes drift

Whereas an "Eulerian" velocity is the "Lagrangian" displacement of a particle divided by the time interval of the displacement as the interval limits to zero, the Stokes velocity is the same except that the time interval, the period, remains finite.

3.2 Linearization

The linearized wave solutions are often used to represent the wave portion of theory which couple waves with currents (e.g., Phillips 1977). To lowest order in *ka*, the wavy portions of Eqs. 11 and 12 are

$$\widetilde{\eta}(x,t) = a\cos\psi, \qquad (14a)$$

$$\widetilde{p}(x,z,t) = gae^{kz}\cos\psi \qquad (14b)$$

where, in Eq. 14b, the deep water identity, $c^2k=g$, is used. Thus, at $z=0$, the "interior" wave pressure, Eq. 14b, is balanced hydrostatically at the surface by $\int_0^\eta g dz = g\widetilde{\eta}$, i.e., $\widetilde{p}(x,0,t) = g\widetilde{\eta}(x,t)$. It can be remarked that in Eqs. 10a and 10b, $(u,w) = (\widetilde{u},\widetilde{w})$ and remains an exact solution of Eqs. 1 and 3. However, the linearized elevation and pressure are approximations when applied to finite *ka* cases.

Notice that Eq. 12 has a phase-averaged, nonlinear, $O(ka^2)$ component. Thus,

$$\overline{p} = -c^2(ka)^2\frac{e^{2kz}}{2} - gz + B, \qquad (14c)$$

about which more will be discussed below. Phase averaging is defined as an average taken over a wave length, $\lambda=2\pi/k$, or period, $T=2\pi/\sigma$ or $\overline{(\ )} \equiv (2\pi)^{-1}\int_0^{2\pi}(\ )d\psi$.

4 Phase-averaged pressure

Consider Eq. 2b the vertical component of momentum. After phase averaging,

$$\rho_w\frac{\partial\overline{\widetilde{w}^2}}{\partial z} = -\frac{\partial\overline{p}_w}{\partial z} - \rho_w g.$$

We have reverted to dynamic pressure without change in nomenclature where ρ_w is the water density. Upon vertical integration within the water column,

$$\overline{p}_w + \rho_w\overline{\widetilde{w}^2} + \rho_w gz = B(x). \qquad (15)$$

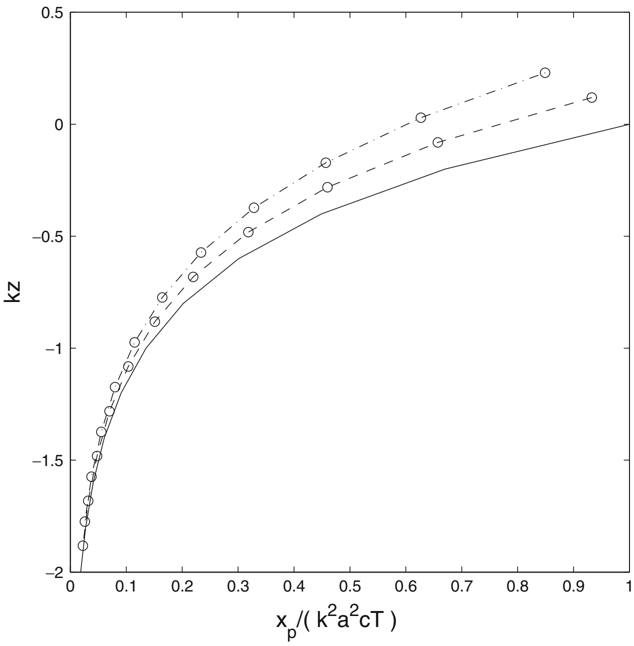

Fig. 3 Stokes drift for a progressive wave showing the distance traveled by particles after a wave period. The curves are labeled as in Fig. 1. The curve for ka=0.2 yields the Stokes drift "deviation" in Fig. 2 after multiplication by $2\pi(ka)^2$. The *solid line* is the theoretical Stokes drift, $x_p=(ka)^2cTe^{2kz}$ as determined in "Appendix 1"

To evaluate B, consider Fig. 4 which illustrates a control volume enclosing the air–sea interface and bounded vertically by z=−b and z=b where b is slightly larger than the wave amplitude and horizontally by x and x+λ. After integration from x to x+λ, we have

$$\overline{p_w(-b)} + \rho_w\overline{\widetilde{w}^2(-b)} - \overline{p_a(b)} - \rho_a\overline{\widetilde{w}^2(b)} = -\rho_w g\overline{(b+\widetilde{\eta})} - \rho_a g\overline{(b-\widetilde{\eta})}.$$

The subscripts a and w refer to the air and water sides of the air–sea interface. The tendency and lateral transport terms vanish after phase averaging. Next neglect ρ_a relative to ρ_w. After letting $b\rightarrow 0$, we obtain

$$\overline{p}_w + \rho_w\overline{\widetilde{w}^2} - \overline{p}_a = 0; \quad z = 0. \tag{16}$$

Thus, comparing Eqs. 15 and 16, one obtains the result that $B = \overline{p}_a \equiv p_{atm}$. Independent derivations of Eq. 16 will be found in Longuet-Higgins and Stewart (1964) and Smith (2006). For still another derivation, see "Appendix 2". These details are presented here since it may be counter-intuitive that there is a discontinuity in the *phase-averaged* pressure across the z=0 interface due to the $\overline{\widetilde{w}^2}$ term.

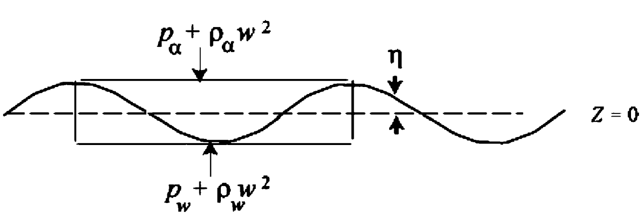

Fig. 4 A schematic of a control volume enclosing the air–sea interface

Reverting to kinematic pressures ($\overline{p}_w$ and p_{atm} are divided by ρ_w), Eq. 15 may now be written

$$\overline{p} + \overline{\widetilde{w}^2} + g(z-\widehat{\eta}) = p_{atm}; \quad z \leq \widehat{\eta} \tag{17}$$

where $\widehat{\eta}$ is the mean elevation. In going from Eq. 16 to Eq. 17, we have changed the origin of z from that relative to the mean elevation—so as to simplify the foregoing algebra—to z relative to a geoid. After insertion of Eq. 10b, Eq. 17 is identical to Eq. 14c for $B = p_{\mathrm{atm}} + g\widehat{\eta}$.

To obtain an expression for the radiation stress, one could now insert Eq. 17 into the phase-averaged version of Eq. 2a and obtain $\partial(\overline{\widetilde{u}^2} - \overline{\widetilde{w}^2})/\partial x = -\partial(g\widehat{\eta} + p_{\mathrm{atm}})/\partial x$, but then an important wave radiation component would be missing as first determined by Longuet-Higgins and Stewart (1964). Note that the phase-averaged portion of $\widetilde{p}$ given by Eq. 14b is nil. However, the internal hydrostatic pressure represented by Eq. 14a is $\widetilde{p} = g(\widehat{\eta} + \widetilde{\eta} - z)$ so that

$$\overline{\int_0^{\widetilde{\eta}} \widetilde{p}\,dz} = \overline{\int_{\widehat{\eta}}^{\widehat{\eta}+\widetilde{\eta}} g(\widehat{\eta} + \widetilde{\eta} - z)dz} = \overline{\int_0^{\widetilde{\eta}} g(\widetilde{\eta} - z')dz'} = \frac{g\overline{\widetilde{\eta}^2}}{2} = \frac{E}{2} \tag{18}$$

where E is the wave energy. If Eq. 12 is used instead of the more simple hydrostatic pressure, one obtains $(E/2)\times[1 + 1.75(ka)^2 + O(ka)^4]$ on the right side of Eq. 18 in agreement with Eq. 18 for small ka.

Obviously, Eq. 18 is localized to the surface so that one obtains

$$S_{xx} \equiv \overline{\widetilde{u}^2} - \overline{\widetilde{w}^2} + \frac{g\overline{\widetilde{\eta}^2}}{2}\delta(z) \tag{19}$$

[Note that the quantity, $\overline{\widetilde{u}^2} - \overline{\widetilde{w}^2}$, in Eq. 19 is independent of depth and is only nonzero in finite depth water.] Now use Eq. 19 together with the phase-averaged version of Eq. 2a to obtain

$$\frac{\partial}{\partial x}S_{xx} = -\frac{\partial}{\partial x}(g\widehat{\eta} + p_{\mathrm{atm}}) \tag{20}$$

In Eq. 19, Dirac function $\delta(z)$=0 if $z\neq 0$, but $\int_{-h}^{\varepsilon}\delta(z)dz = 1.0$ for indefinitely small ε. After vertical integration, Longuet-Higgins and Stewart (1964) labeled the three terms on the right of Eq. 19 as $S_{xx}^{(1)}, S_{xx}^{(2)}$, and $S_{xx}^{(3)}$.

4.1 Completing Eq. 20

The right side of Eq. 20 was included at the suggestion of a reviewer; however, inclusion of the term requires further comment. Upon vertical integration, Eq. 20 yields an expression for wave induced variation of mean sea level or wave setup, $\partial\widehat{\eta}/\partial x$. Longuet-Higgins and Stewart (1964) and Phillips (1977) address the problem of setup resulting from variations in wave energy due to variations of group velocity on a gently sloping beach. Currents and Stokes drift were assumed negligible. Although plausible using the vertical integral of Eq. 20, neglect of a depth-dependent advection term is not acceptable because the left side of Eq. 20 is dependent on z whereas the right side is not. In fact, borrowing from Mellor (2003) [wherein S_{xx} differed incorrectly from Eq. 19] instead of Eq. 20, we must write

$$\frac{\partial}{\partial x}DU^2 + \frac{\partial}{\partial\varsigma}\Omega U + \frac{\partial}{\partial x}DS_{xx} = -D\frac{\partial}{\partial x}(g\widehat{\eta} + p_{\mathrm{atm}}). \quad (21)$$

Equation 21 invokes "sigma" coordinates where $\varsigma \equiv (-\widehat{\eta}+z)/(\widehat{\eta}+h)$ such that, when $z=\widehat{\eta}$ and $z=-h$, $\varsigma=0$ and -1, respectively. Top and bottom boundary conditions are automatically included if $\Omega(0)=\Omega(-1)=0$. Thus, since the right side of Eq. 21 is independent of ς (or z), nonnegligible, vertically dependent variations in U must exist to balance the vertical variations in $\partial DS_{xx}/\partial x$. It should be noted that U is the sum of currents and Stokes drift. There are further complications; because of the Dirac term in Eq. 19, a resulting discontinuity in U cannot realistically exist requiring inclusion of a viscous-turbulence term in Eq. 21.

Sigma coordinates facilitate vertical integration since; thus,

$$\frac{\partial}{\partial x}\int_{-1}^{0} DU^2 d\varsigma + \frac{\partial}{\partial x}\int_{-1}^{0} DS_{xx} d\varsigma = -D\frac{\partial}{\partial x}(g\widehat{\eta} + p_{\mathrm{atm}}) \quad (22)$$

wherein the momentum advection term should not be neglected. Clearly, further research is needed to explore the consequences of these findings.

5 Extensions

The discussion above was simplified for unidirectional flow to aid comprehension. However, if instead of Eq. 2a one begins with the more general $\partial u_\alpha/\partial t + \partial(u_\beta u_\alpha)/\partial x_\beta + \partial(wu_\alpha)/\partial z = -\partial p/\partial x_\alpha$ (Greek indices refer to horizontal coordinates; repeated indices are to be summed), one would obtain

$$S_{\alpha\beta} = \overline{\widetilde{u}_\alpha\widetilde{u}_\beta} - \delta_{\alpha\beta}\left\{\overline{\widetilde{w}^2} - \frac{g\overline{\widetilde{\eta}^2}}{2}\delta(z)\right\} \quad (23)$$

instead of Eq. 19. Both the Kronecker delta ($\delta_{\alpha\beta}=1$ if $\alpha=\beta$ and $=0$ otherwise) and the Dirac delta functions are invoked.

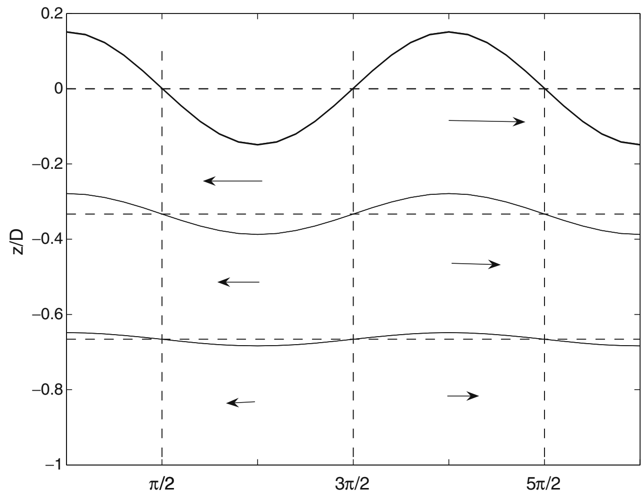

Fig. 5 The wave velocity field and streamlines at an instant in time. From Eqs. 10a and 10b. Notice that $D = \widehat{\eta} + h$

Using Eqs. 10a and 10b—except that $\widetilde{u}_\alpha = k_\alpha ace^{kz}\cos\psi$—and Eq. 14a, one obtains

$$S_{\alpha\beta} = E\left[\frac{k_\alpha k_\beta}{k}e^{2kz} - \delta_{\alpha\beta}\left\{ke^{2kz} - \frac{1}{2}\delta(z)\right\}\right]. \quad (24)$$

The wave energy, $E = ka^2c^2/2 = ga^2/2 = g\overline{\widetilde{\eta}^2}$.
If one allows for shallow depths, Eq. 23 still applies, but, as shown in Mellor (2008; Eq. 24b)

$$S_{\alpha\beta} = E\left[\frac{k_\alpha k_\beta}{k}\frac{\cosh^2 k(z+h)}{\sinh kh\cosh kh} - \delta_{\alpha\beta}\left\{k\frac{\sinh^2 k(z+h)}{\sinh kh\cosh kh} - \frac{\delta(\varsigma)}{2}\right\}\right]. \quad (25)$$

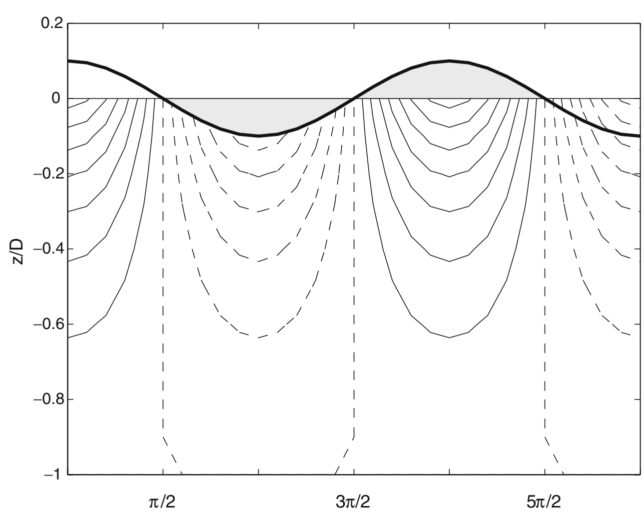

Fig. 6 The wave pressure field at an instant in time. In the range, $0>z>-D$, pressure is given by Eq. 14b. At $z=0$, the interior pressure (*solid* are positive, *dashed lines* are negative) is balanced hydrostatically by the product of g and the elevation, Eq. 14a

Equation 25 asymptotes to Eq. 24 for large kD. After vertical integration, Eq. 25 conforms to that derived by Phillips (1977) which is

$$\int_{-1}^{0} S_{\alpha\beta} D d\varsigma = \int_{-h}^{\widehat{\eta}} S_{\alpha\beta} dz = E\left[\frac{k_\alpha k_\beta}{k^2}\frac{c_g}{c} + \delta_{\alpha\beta}\left(\frac{c_g}{c} - \frac{1}{2}\right)\right] \tag{26}$$

where c_g and c are the group and phase speeds, respectively.

6 Summary

Flows under a stationary wall have been examined, and it is seen that Stokes drift is a consequence of irrotational flow under a wavy boundary and is not related to progressive waves per se. Deviations from linear theory have been derived for surface elevation and pressure. A Galilean transformation relates the wall flow to progressive wave flow.

Vertically dependent wave radiation stress terms have been derived in the absence of currents. Wave velocities given by Eqs. 10a and 10b are exact irrotational solutions throughout the region, $\eta(x)>z>-h$, as depicted in Fig. 5. However, the wave part of the pressure according to Eqs. 14a and 14b is an approximation when simplified for small wave slope, ka, as depicted in Fig. 6. The fact that the phase-averaged pressure is discontinuous across the air–sea interface requires intuitive accommodation. The vertically dependent radiation stress terms, which include a surface singularity, are directly related to their vertically integrated counterparts derived by Longuet-Higgins and Stewart (1964) and Phillips (1977).

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Appendix 1

Derivation of the Stokes drift

The conventional "Lagrangian" derivation of Stokes drift velocity is

$$u_S \cong \overline{\widetilde{u} + \widetilde{x}\frac{\partial \widetilde{u}}{\partial x} + \widetilde{z}\frac{\partial \widetilde{u}}{\partial z}}; \quad \widetilde{x} \equiv \int \widetilde{u}dt; \ \widetilde{z} \equiv \int \widetilde{w}dt.$$

After insertion of Eqs. 10a and 10b, one obtains

$$u_S = (ka)^2 ce^{2kz}$$

Another approach is to consider the flow through a fixed element bounded by z and $z+\Delta z$. The velocity is $\widetilde{u}(z) + \widetilde{z}(\partial \widetilde{u}/\partial z)_z$, and the wave distorted flow area relative to the undistorted area is $1 + (\partial \widetilde{z}/\partial z)_z$; the product of the two terms after phase averaging is

$$u_S = \overline{\left(\widetilde{u} + \widetilde{z}\frac{\partial \widetilde{u}}{\partial z}\right)\left(1 + \frac{\partial \widetilde{z}}{\partial z}\right)} = \frac{\partial \overline{\widetilde{z}\widetilde{u}}}{\partial z} = (ka)^2 ce^{2kz}$$

Appendix 2

Alternate derivation of Eq. 17 following Phillips (1977)

Adjusting the origin of z so that $\widehat{\eta} = 0$, the result of integrating Eq. 2b from arbitrary negative z to $z = \widetilde{\eta}$ is,

$$\frac{\partial}{\partial t}\int^{\widetilde{\eta}} \widetilde{w}dz - \widetilde{w}(\eta)\frac{\partial \widetilde{\eta}}{\partial t} + \frac{\partial}{\partial x}\int^{\widetilde{\eta}} \widetilde{u}\widetilde{w}dz - \widetilde{u}\widetilde{w}|_{\widetilde{\eta}}\frac{\partial \widetilde{\eta}}{\partial x} + \widetilde{w}^2(\widetilde{\eta})$$
$$- \widetilde{w}^2(z) = -p(\widetilde{\eta}) + p(z) - g(\widetilde{\eta} - z)$$

where $p(\widetilde{\eta}) = p_{\mathrm{atm}}$, i.e., the instantaneous pressure is continuous across the interface. Note that $\widetilde{w}(\eta) = \partial\widetilde{\eta}/\partial t + \widetilde{u}(\widetilde{\eta})\partial\widetilde{\eta}/\partial x$ so that the second, fourth, and fifth terms on the left cancel. After phase averaging, $\overline{\partial \int^{\widetilde{\eta}} \widetilde{w}dz/\partial t} = \overline{\partial \int^{\widetilde{\eta}} \widetilde{u}\widetilde{w}dz/\partial x} = 0$ leaving

$$\overline{p} + \overline{\widetilde{w}^2} + gz = p_{\mathrm{atm}}; \ z \leq 0.$$

Thus, whereas the instantaneous pressure is continuous across the air–sea interface at z=0, the phase-averaged pressure is discontinuous.

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