Reply

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ABSTRACT

In response to the comments of Ardhuin et al., the formulation of Mellor has been revised. Solutions of the model equations are now consistent with known deep-water behavior and agree with the shallow water, analytical–numerical experiment put forward by Ardhuin et al.

1. Introduction

Mellor (2003, henceforth M03) produced an analysis providing depth-dependent wave–current interaction equations that, when vertically integrated, were in agreement with the depth-integrated equations of Longuet-Higgins and Stewart (1960, 1964), Phillips (1977), and others—an encouragement then, but now considered a consequence of canceling errors. A commentary by Ardhuin et al. (2008, henceforth AJB) pointed to a discrepancy in M03 for shallow water ($kD \cong 1$, where k is the wavenumber and D is the local mean water depth) compared with a two-dimensional, unidirectional case of unforced waves progressing over a bottom with variable depth. Of greater concern was my discovery that the M03 formulation and variable depth produced mean currents even for deep water (say, $kD \cong 10$), a physically unacceptable finding.

2. A revision

The derivation process was revisited and a new formulation is available (ftp://aden.princeton.edu/pub/glm) that, although containing elements of M03, abandons the a priori use of sigma coordinates; characterization of linear irrotational waves derived for a flat bottom was misinterpreted in the sigma domain. The new analysis in Cartesian coordinates and here simplified to two-dimensional, unidirectional, steady flow resulted in the continuity equation

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0, \tag{1}$$

and the momentum equation

$$\frac{\partial}{\partial x}(UU) + \frac{\partial}{\partial z}(WU) + \frac{\partial}{\partial x}(g\hat{\eta}) = -\frac{\partial}{\partial x_\beta} S_{xx}, \tag{2}$$

where

$$S_{xx} = \overline{\tilde{u}\tilde{u}} - \overline{\tilde{w}^2} + E_D. \tag{3}$$

A modified Dirac delta function is defined such that

$$E_D = 0 \quad \text{if} \quad z \neq \hat{\eta} \quad \text{and} \quad \int_{-h}^{\hat{\eta}+} E_D\, dz = E/2.$$

[In a finite difference rendering of E_D, the top vertical layer of incremental size δz (and only the top layer) would be forced by $E_D \delta z = E/2$.] The Cartesian coordinates are (x, z), where x is the horizontal coordinate and z the vertical coordinate, positive upward from the sea surface; U (current plus Stokes drift) and W are the corresponding velocities; and $\tilde{u}$, $\tilde{w}$, and $\tilde{p}$ are the instantaneous wave velocities and pressure. The overbars represent phase averaging. The mean sea level is $\hat{\eta}$ and g is the gravity constant.

It is next assumed that the terms in (3) can be evaluated using Airy functions for which, as in M03 and AJB, it is convenient to define the following terms:

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$$F_{SS} \equiv \frac{\sinh k(z+h)}{\sinh kD}, \quad F_{CS} \equiv \frac{\cosh k(z+h)}{\sinh kD},$$

$$F_{SC} \equiv \frac{\sinh k(z+h)}{\cosh kD}, \quad \text{and} \quad F_{CC} \equiv \frac{\cosh k(z+h)}{\cosh kD}.$$

Then

$$\tilde{\eta} = a \cos\psi, \tag{4a}$$

$$(\tilde{u}, \tilde{w}) = \frac{ga}{c}(F_{CC}\cos\psi, F_{SC}\sin\psi), \quad \text{and} \tag{4b, c}$$

$$\tilde{p} = gaF_{CC}\cos\psi. \tag{4d}$$

In (4), $\psi \equiv kx - \omega t$; k and ω are directional wavenumber and frequency such that $\omega = \sigma + ku_D$, where σ is the intrinsic frequency and u_D is the Doppler velocity [in (4), ku_D is a constant]; a is wave amplitude; $c = \sigma/k$ is the phase speed; $z = -h$ is the bottom depth; and $\tilde{\eta}$ is the surface wave elevation given by (4a) so that $\hat{\eta} + \tilde{\eta}$ is the instantaneous surface elevation above mean sea level. Also, $D \equiv \hat{\eta} + h$ is the mean water column depth. Associated with (4c) is the location of material surfaces,

$$s = z + \tilde{s}, \quad \text{and} \quad \tilde{s} = aF_{SS}\cos\psi, \tag{4e}$$

where $\partial\tilde{s}/\partial t = \tilde{w}$. Notice that $s(\hat{\eta}) = \hat{\eta} + \tilde{\eta}$ and $s(-h) = -h$ are the two bounding material surfaces.

From the dispersion relation $\sigma^2 = gk \tanh kD$, obtain $ga/c = kac/\tanh kD$ to convert coefficients of (4) to those proportional to ka. In the derivation of the above equations, as in most precursor derivations, wave slopes ka are assumed to be small, as are the derivatives $\partial a/\partial x$, $\partial k/\partial x$, and vertical current shear (properly nondimensionalized on a representative k and σ). Furthermore, a major element in the derivation of (4)—and in the AJB commentary—is that the bottom boundary condition is $\partial h/\partial x = 0$. Terms additional to (4) that account for bottom slope are proportional to $(ka)c(\partial h/\partial x_\alpha)$. To obtain this scaling, start with the linear irrotational wave equations; then expand the potential function using the small parameter $\varepsilon = \partial h/\partial x$. The lowest-order solutions are (4a)–(4c) and the next order that satisfies a nonzero but small bottom slope yields the aforementioned scaling. Further analysis, or indeed intuition, reveals that a more descriptive parameter is $(ka)[(\partial h/\partial x)/\sinh kh]$ since for deep water, the bottom slope should not be a factor in the interaction of waves and currents. The nonlinear equations discussed below might incur errors on the order of $(ka)^4$ for flat bottom applications and errors on the order of $(ka)^2$ $[(\partial h/\partial x)/\sinh kh]^2$ for shallow water with bottom topography.

Evaluating (3) using (4) in the two-dimensional, unidirectional case results in

$$S_{xx} = kE(F_{CS}F_{CC} - F_{SC}F_{SS}). \tag{5}$$

3. The AJB calculation

Using the multimode National Technical University of Athens (NTUA) model by Athanassoulis and Belibassakis (1999) and Belibassakis et al. (2001), AJB provide an analytical–numerical calculation for the problem shown in their Fig. 3a; for $kD \approx 1$, the group velocity and the wave energy changes are small. They analyze their results according to

$$\frac{\partial DU^2}{\partial x} + \frac{\partial \Omega U}{\partial \varsigma} + gD\frac{\partial \hat{\eta}}{\partial x} = -\frac{\partial S_{XX}}{\partial x} + \frac{\partial S_{X3}}{\partial \varsigma} \tag{6}$$

[I use capital subscripts to distinguish (3) and (5) from AJB's definitions]. They claim that the "mean current U and the Stokes drift are of the same order" and therefore, the two advective terms are an order of $(ka)^2$ smaller than the last three terms of (9). I think that argument is specious since, if the right side of (9) has vertical structure, the left side must respond. Nevertheless the calculations yield the somewhat surprising result that

$$-\frac{\partial S_{XX}}{\partial x} + \frac{\partial S_{X3}}{\partial \varsigma} = gD\frac{\partial \hat{\eta}}{\partial x}, \tag{7}$$

so that the "radiation stress" forcing has no vertical structure (one presumes in the limit of an infinite number of modes); that is, there is very little wave–current interaction except to effect wave setup. I do not know how they separately evaluate the two terms on the left of (7) (it is stated that S_{XX} is the same as that in M03, which I now believe is not correct!), but clearly this is an interesting numerical experiment that deserves analysis. One would like to see the evaluation of fluxes in their more primitive form, as in (3).

4. The new result

Eq. (5) may be written

$$S_{xx} = kE\frac{\cosh^2 k(z+h) - \sinh^2 k(z+h)}{\cosh kD \sinh kD} = kE\frac{2}{\sinh 2kD} \tag{8}$$

so that its horizontal derivative has no vertical structure and is balanced by $-gD\partial\hat{\eta}/\partial x$.

5. Vertical momentum transport

An important finding in M03 and the revised version is that surface wave stress or form drag $\overline{\tilde{p}\partial\tilde{\eta}/\partial x}|_{s=0}$ is projected into the water column according to $\overline{\tilde{p}\partial\tilde{\eta}/\partial x}|_{s=0}\partial F_{SS}F_{CC}/\partial\varsigma$ and thus competes with turbulent momentum flux; this is contrary to (all?) surface boundary layer models (e.g., Mellor and Yamada 1982; Large et al. 1994) that assume that vertical momentum transport is entirely supported by turbulence. Sandwiched between discussions of bottom slope, AJB describe a phase-resolved solution to a wind-forced ocean resulting in an exponentially growing wave field, which corroborates the aforementioned M03 finding. Another solution wherein surface form drag is balanced by viscous dissipation will be part of a future paper on surface boundary layers.

6. Conclusions

The commentary of AJB alerted me to a serious problem with the derivation in M03. For example, $\overline{\tilde{w}^2}$ was missing from (3) and $\partial s/\partial x \neq 0$ at $z = -h$ entered into the derivation; this is inconsistent with (4c), which is based on the boundary condition $\partial h/\partial x = 0$. Significant unforced wave–current interaction occurred even for deep water, which is physically unacceptable.

In the new formulation, since the Airy functions are used, it is expected that errors in the equations discussed here will be on the order of $(ka)^2\ [(\partial h/\partial x)/\sinh kh]^2$. However, when compared with the AJB shallow-water ($kD \cong 1$) case, there is now agreement with their result. The M03 difficulty with unforced wave–current interactions for, say, $kD \cong 3$ or larger, when there should be none, is absent in the revised formulation.

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