Some Consequences of the Three-Dimensional Current and Surface Wave Equations

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ABSTRACT

Three-dimensional, interacting current and surface gravity wave equations have recently been derived and compared with their counterpart vertically integrated equations; they are in the form of sigma-coordinate equations. The purpose of this paper is to examine some of the consequences of these equations including energy transfer between mean energy, wave energy, and turbulence energy, to frame some outstanding research issues, to provide a Cartesian version of the sigma-coordinate equations, and to compare with other formulations of wave–current interaction. In general, the paper is intended to set the stage for the development of numerical coupled surface wave and three-dimensional general circulation models. These models often include a flow-dependent turbulence-based viscosity.

1. Introduction

There have been a number of special solutions for wave–current interaction including viscosity, generally a constant (e.g., Lamb 1932; Weber 1983) or spatially varying (Jenkins 1987). Although instructive, they do not currently provide a basis for three-dimensional numerical ocean models. The equations in Phillips (1977), modified to include the Coriolis parameter, are appropriate to vertically integrated models.

In a recent paper (Mellor 2003, henceforth M03), depth-dependent, phase-averaged continuity and momentum equations that include current–wave interaction terms were derived through a wave-following sigma-coordinate system. The surface wave energy equation is, as before, depth independent but contains vertical integrals of the newly found depth-dependent wave radiation stress terms. As in M03, we deal only with monochromatic waves, but it is the future intention to extend these findings to a spectral description of wave fields.

The linear wave velocities are on the order of the wave slope (ak), where a is the elevation amplitude and k is the wavenumber. In the phase-averaged nonlinear equations presented in section 2, a couple of terms of order $(ak)^4$ were neglected in the derivation. In this regard, the M03 derivation does not differ from that found in Phillips (1977).

Some terms like wave dissipation must of course be modeled based on available data. To assist the modeling process, this paper provides background information by establishing the connections between mean energy, wave energy, and turbulence kinetic energy. It is, of course, intellectually satisfying to identify the flow of energy between the different energy modes, and, as a practical matter, this reduces somewhat the number of unknowns that must be modeled.

As shown in M03, the present equations, when vertically integrated, agree with the corresponding depth-independent equations in Phillips (1977), which were derived in an independent manner.

The three-dimensional continuity, momentum, turbulence energy, and the wave energy equations from M03 are repeated in section 2. In section 3, the mean energy equations are obtained and the energy budget among the three components is closed. In section 4, some outstanding research issues are addressed. In appendix A, the phase-averaged sigma-coordinate equations are transformed to Cartesian coordinates. To help in understanding the present equations, in appendix B they are contrasted with the three-dimensional wave–current interaction equations of Craik and Leibovich (1976), Leibovich (1980), and McWilliams and Restrepo (1999), which include a Stokes drift Coriolis term and a Stokes vortex force.

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2. The depth-dependent equations

By means of a sigma-like coordinate transformation and phase averaging, depth-dependent momentum and continuity equations have very recently been derived in M03 and include wave–current interaction terms. The wave energy equation is, as before, in vertically integrated form, but integrals of depth-dependent velocity are substituted for terms that previously had required an assumption that currents were independent of depth. The present equations should enable three-dimensional ocean circulation models to be coupled properly to surface wave models.

In this paper, $x_i = (x, y, z)$ and $x_a = (x, y)$. The continuity equation is

$$\frac{\partial DU_\alpha}{\partial x_\alpha} + \frac{\partial \Omega}{\partial \varsigma} + \frac{\partial \hat{\eta}}{\partial t} = 0. \tag{1}$$

The phase-averaged or mean elevation is $\hat{\eta}$ and $D \equiv h + \hat{\eta}$ is the mean water column depth. The horizontal coordinates are x_α and $\varsigma = (z - \hat{\eta})/D$ is a "sigma" coordinate (reserving σ for frequency) such that $\varsigma = 0$ and -1 at the surface ($z = \hat{\eta}$) and bottom ($z = -h$), respectively. The horizontal velocity is $U_\alpha = u_{S\alpha} + \hat{u}_\alpha$, where $u_{S\alpha}$ is the Stokes drift and $\hat{u}_\alpha$ is the wind stress, tide, density-driven current, henceforth "the current." The sigma, nearly vertical velocity is Ω (see appendix A for its definition) such that $\Omega = 0$ at $\varsigma = 0$ and $\varsigma = -1$.

The momentum equation is

$$\frac{\partial DU_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}[D(U_\beta U_\alpha + S_{\alpha\beta})] + \frac{\partial}{\partial \varsigma}(\Omega U_\alpha - S_{p\alpha}) + \epsilon_{\alpha\beta z} f_z DU_\beta + D\frac{\partial}{\partial x_\alpha}(g\hat{\eta} + \hat{p}_{\text{atm}}) + D\frac{\partial \hat{p}}{\partial x_\alpha} - \varsigma\frac{\partial D}{\partial x_\alpha}\frac{\partial \hat{p}}{\partial \varsigma} = \frac{\partial \tau_{t\alpha}}{\partial \varsigma} + \frac{\partial \tau_{p\alpha}}{\partial \varsigma} \quad \text{and} \tag{2a}$$

$$\frac{\partial \hat{p}}{\partial \varsigma} = -Dg\frac{\hat{\rho}}{\rho_o}, \tag{2b}$$

where $\hat{p}(x_\alpha, \varsigma)$ and $\hat{\rho}(x_\alpha, \varsigma)$ are the mean pressure and density, ρ_o is a reference density, f_z is the vertical component of the Coriolis parameter, ϵ_{ijk} is the Levi–Civita symbol, and

$$S_{\alpha\beta} \equiv kE\left[\frac{k_\alpha k_\beta}{k^2}F_{\text{CS}}F_{\text{CC}} + \delta_{\alpha\beta}(F_{\text{CS}}F_{\text{CC}} - F_{\text{SS}}F_{\text{CS}})\right] \tag{3a}$$

and

$$S_{p\alpha} \equiv (F_{\text{CC}} - F_{\text{SS}})\left[\frac{F_{\text{SS}}}{2}\frac{\partial E}{\partial x_\alpha} + F_{\text{CS}}(1 + \varsigma)E\frac{\partial kD}{\partial x_\alpha}\right] \tag{3b}$$

are the three-dimensional, wave radiation stresses [(3a) differs from the definition in M03 by relocating D from (3a) to (2)]; the term $\partial S_{p\alpha}/\partial\varsigma$ had not been seen prior to M03 since it vertically integrates to zero. The functions

$$F_{\text{SS}} \equiv \frac{\sinh kD(1 + \varsigma)}{\sinh kD}, \; F_{\text{CS}} \equiv \frac{\cosh kD(1 + \varsigma)}{\sinh kD}, \tag{4a,b}$$

$$F_{\text{SC}} \equiv \frac{\sinh kD(1 + \varsigma)}{\cosh kD}, \quad \text{and} \quad F_{\text{CC}} \equiv \frac{\cosh kD(1 + \varsigma)}{\cosh kD} \tag{4c,d}$$

are recurring depth-dependent functions; for deep water (large kD) all of these functions asymptotically approach exponentials; for shallow water (small kD) the functions asymptotically approach constants or linear functions of ς. They are plotted in M03.

One of the two vertical momentum transfer terms in (2a) is

$$\tau_{t\alpha} = -\overline{\langle w' u'_\alpha\rangle} + \frac{\nu}{D}\frac{\partial U_\alpha}{\partial z}. \tag{5a}$$

The primes denote turbulent fluctuating quantities (uncorrelated with mean and wave motion); the angle brackets represent an average of an ensemble of values at a specific phase, and the overbar represents averaging over all phase values. Whereas (5a) is the familiar Reynolds momentum flux, the momentum transfer due to the wind pressure fluctuating component, $\tilde{p}_{w\eta} = a_w \sin\psi$, which is correlated with the wave surface slope, $\partial\tilde{\eta}/\partial x_\alpha = ak_\alpha \sin\psi$, where $\psi = k_\alpha x_\alpha - \omega t$ (Donelan 1999), is

$$\tau_{p\alpha} = \overline{\tilde{p}_{w\eta}\frac{\partial \tilde{\eta}}{\partial x_\alpha}} F_{\text{SS}}F_{\text{CC}} \quad \text{and}$$

$$\tau_{p\alpha}(0) = \overline{\tilde{p}_{w\eta}\partial\tilde{\eta}/\partial x_\alpha} = a_w a k_\alpha/2. \tag{5b}$$

The wave energy is

$$E \equiv \int_{-1}^{0} D(\overline{\tilde{u}_i^2}/2)\, d\varsigma + g\overline{\tilde{\eta}^2}/2, \tag{6}$$

where $\tilde{u}_i$ and $\tilde{\eta}$ are the wave orbital velocities and wave surface elevation, respectively, and the right side is the sum of the kinetic and potential energies, which are equal so that $E = g\overline{\tilde{\eta}^2}$.

The Stokes drift is

$$u_{S\alpha} = \frac{k_\alpha}{k}\frac{E}{cD}\frac{\partial F_{\text{CC}}F_{\text{SS}}}{\partial \varsigma} = k_\alpha\frac{2E}{c}\frac{\cosh 2kD(1 + \varsigma)}{\sinh 2kD}. \tag{7}$$

It has been shown [appendix A in M03] that, in the absence of currents, viscous effects, Coriolis terms, and horizontal advective and buoyancy terms, for a horizontally homogeneous wave field propagating in the x direction (2a) reduces to

$$\partial Du_S/\partial t = -\overline{\tilde{p}_{w\eta}\partial\tilde{\eta}/\partial x}\,\partial F_{\text{SS}}F_{\text{CC}}/\partial\varsigma.$$

Unidirectional Stokes drift is $u_S = (E/cD)\partial F_{CC}F_{SS}/\partial\varsigma$, where c is the phase speed. Thus, the last term on the right of (2a) is now seen to be a source term for Stokes drift and, in integral form, was shown in M03 to coincide with the corresponding result obtained from the wave energy equation. However, in view of expected enhanced wave breaking turbulence at the surface and the discussion of bottom boundary layers in section 4, it may be difficult to separate the Stoke part of U_α from the current. On the other hand, is it necessary (albeit comforting) to do so?

Equation (1) and all of the terms on the left of (2) are deterministic. However, the turbulence Reynolds stress $\overline{\langle w'u'_\alpha\rangle}$ (a function of x_α and ς) and the correlation between the wind surface pressure fluctuations and wave slope $\overline{\tilde{p}_{w\eta}\partial\tilde{\eta}/\partial x_\alpha}$ require empirical information and modeling.

The vertically integrated wave energy is needed in (2) and (3a, b) for which

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}[(c_{g\alpha} + \hat{u}_{A\alpha})E] + \int_{-1}^{0} DS_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}\,d\varsigma - \int_{-1}^{0} S_{p\alpha}\frac{\partial U_\alpha}{\partial\varsigma}\,d\varsigma = S_{\text{in}} + S_{\text{dis}}. \tag{8}$$

All of the terms on the left are deterministic. In (9), $c_{g\alpha}$ is the group velocity and the Doppler velocity $\hat{u}_{A\alpha}$ is defined by

$$\hat{u}_{A\alpha} \equiv \int_{-1}^{0} U_\alpha(\varsigma)r(\varsigma)\,d\varsigma, \tag{9}$$

where $r(\varsigma) = \partial F_{SS}F_{CC}/\partial\varsigma$ is a weighting function such that $\int_{-1}^{0} r(\varsigma)\,d\varsigma = 1.0$ (Kirby and Chen 1989). Note that, heretofore, velocities in the integrands of the last two terms on the left of (8) and the integrand of (9) had previously been assumed to be independent of ς (or z), which is inappropriate in the context of three-dimensional models of the ocean.

The source terms as derived in M03 are

$$S_{\text{in}} = c_\alpha\overline{\tilde{p}_{w\eta}\frac{\partial\tilde{\eta}}{\partial x_\alpha}} = c_\alpha\tau_{p\alpha}(0), \tag{10a}$$

where $c_\alpha = k_\alpha c/k$ is the phase velocity vector, k_α is the wavenumber vector, and $k = |k_\alpha|$. The total dissipation is

$$S_{\text{dis}} = \int_{-1}^{0} \overline{(\tilde{u}_i - u_{Si})\frac{\partial\langle w'u'_i\rangle}{d\varsigma}}\,d\varsigma, \tag{10b}$$

which, the expressions on the right notwithstanding, has to be modeled and experimentally supported. It can be divided into surface and bottom dissipation as discussed below.

It can be shown (M03) that when the barotropic simplification of (1) and (2a,b) are vertically integrated, one obtains the corresponding equations in Phillips (1977) [sans the Coriolis term and the second term on the right of (2a)] and in other references.

The turbulence energy equation is the basis of several turbulence closure models and is

$$\frac{\partial Dq^2/2}{\partial t} + \frac{\partial}{\partial x}\left(DU\frac{q^2}{2}\right) + \frac{\partial}{\partial\varsigma}\left(\Omega\frac{q^2}{2}\right) + \frac{\partial}{\partial\varsigma}(\overline{\langle w'u_i'^2\rangle} + \overline{\langle w'p'\rangle}) = (\tau_{p\alpha} + \tau_{t\alpha})\frac{\partial U_\alpha}{\partial\varsigma} + s_{\text{dis}} - D\epsilon, \tag{11}$$

where $q^2 \equiv \overline{\langle u_i'^2\rangle}$ is 2 times the turbulence kinetic energy. The last term on the left of (11) is turbulence diffusion, a relatively small term generally modeled as in Fickian diffusion. On the right is the mean shear production term and the wave dissipation term such that $\int_{-1}^{0} Ds_{\text{dis}}\,d\varsigma = S_{\text{dis}}$. The last term on the right of (11) is turbulence kinetic energy dissipation, the final depository for all of the work done by the wind acting on the ocean surface; it can be modeled according to $\epsilon = q^3/\Lambda$, where Λ is a length scale (Mellor and Yamada 1982). Justification for inclusion of the term $\tau_{p\alpha}\partial U_\alpha/\partial\varsigma$ will be found in the next section.

3. The energy budget

To complement the wave and turbulence energy equations, the mean energy equation is needed. Thus, multiply (2) by U_α and subtract (1) after multiplication by $U_\alpha^2/2$. After rearrangement, one obtains

$$\frac{\partial}{\partial t}\left(D\frac{U_\alpha^2}{2} + g\frac{\hat{\eta}^2}{2}\right) + \frac{\partial}{\partial x_\beta}\left[DU_\beta\left(\frac{U_\alpha^2}{2} + g\hat{\eta} + S_{\alpha\beta}\right)\right] + \frac{\partial}{\partial\varsigma}\left[\Omega\left(\frac{U_\alpha^2}{2} + g\hat{\eta}\right) + U_\alpha S_{p\alpha}\right] = \frac{\partial U_\alpha\tau_{t\alpha}}{\partial\varsigma} + \frac{\partial U_\alpha\tau_{p\alpha}}{\partial\varsigma} + DS_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} - S_{p\alpha}\frac{\partial U_\alpha}{\partial\varsigma} - \tau_{p\alpha}\frac{\partial U_\alpha}{\partial\varsigma} - \tau_{t\alpha}\frac{\partial U_\alpha}{\partial\varsigma}. \tag{12}$$

After integrating from $\varsigma = -1$ to $\varsigma = 0$, we obtain

$$\frac{\partial}{\partial t}\left(\int_{-1}^{0}\frac{U_\alpha^2}{2}D\,d\varsigma + g\frac{\hat{\eta}^2}{2}\right) + \frac{\partial}{\partial x_\beta}\left[\int_{-1}^{0}U_\beta\left(\frac{U_\alpha^2}{2} + S_{\alpha\beta}\right)D\,d\varsigma + g\hat{\eta}\right] = (U_\alpha\tau_{p\alpha})_0 + (U_\alpha\tau_{t\alpha})_0 + \int_{-1}^{0}\left(DS_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} - S_{p\alpha}\frac{\partial U_\alpha}{\partial \varsigma}\right)d\varsigma - \int_{-1}^{0}\left(\tau_{p\alpha}\frac{\partial U_\alpha}{\partial \varsigma} + \tau_{t\alpha}\frac{\partial U_\alpha}{\partial \varsigma}\right)D\,d\varsigma. \quad (13)$$

After vertical integration, the distinction between Cartesian and sigma coordinates virtually disappears because $\int_{-1}^{0}\Phi D\,d\varsigma = \int_{-h}^{\hat{\eta}}\Phi\,dz$ and $\int_{-1}^{0}\Phi(\partial U_\alpha/\partial\varsigma)\,d\varsigma = \int_{-h}^{\hat{\eta}}\Phi\,(\partial U_\alpha/\partial z)\,dz$, where Φ is any function of x, y, t, and z or ς. The first and second terms on the left of (13) are mean kinetic and potential energy tendency and flux divergence terms; the first and second terms on the right are wind to ocean work terms, the first due to the pressure slope correlation and second due to turbulence.

The vertical integral of (11) is

$$\frac{\partial}{\partial t}\int_{-1}^{0}\frac{q^2}{2}D\,d\varsigma + \frac{\partial}{\partial x_\beta}\int_{-1}^{0}U_\beta\frac{q^2}{2}D\,d\varsigma = \int_{-1}^{0}(\tau_{p\alpha} + \tau_{t\alpha})\frac{\partial U_\alpha}{\partial z}D\,d\varsigma + S_{\rm dis} - \int_{-1}^{0}\epsilon D\,d\varsigma. \quad (14)$$

It will be noticed that I have excluded $(\overline{\langle w'u_i'^2\rangle} + \overline{\langle w'p'\rangle})_0$ obtainable from the last term on the left of (11). There could be direct turbulence diffusion from the atmosphere identifiable with this term, but here it will be subsumed in $S_{\rm dis}$. Craig and Banner (1994), Terray et al. (1996), and others (see also Mellor and Blumberg 2004) include the effect of wave breaking as surface turbulence diffusion. As used in (11), we regard this strategy as a mathematical approximation whereby

$$s_{\rm dis}(z) = S_{\rm Sdis}\delta(\hat{\eta} - z) + S_{\rm Bdis}\delta(h + z) \quad (15)$$

and where δ is the Dirac delta function. The surface and bottom dissipations are denoted by $S_{\rm Sdis}$ and $S_{\rm Bdis}$, respectively. Equivalently, the dissipation could enter into the turbulence energy equation as a diffusional boundary condition. The bottom turbulent boundary layer is known to be very thin (Grant and Madsen 1986; Mellor 2002) and the assumption would be that the breaking part of the active wave region is also thin. Thus, $S_{\rm dis} = S_{\rm Sdis} + S_{\rm Bdis}$, but a smaller amount of wave dissipation should also be included below the wave-breaking region and in the case of swell (Weber 1983; Jenkins 1987).

The impact of $\tau_{p\alpha}$ on the turbulence energy equation was not considered in M03. In fact, inclusion of the production term, $\tau_{p\alpha}\partial U_\alpha/\partial z$, in (11) is the result of the derivation of the corresponding sink terms in (12) in this paper; its inclusion is necessary to achieve energy balance. Thus, it will be seen that all of the source/sink terms cancel such that, for a closed system, all of the atmospheric work terms are converted into turbulence dissipation and thence to thermal energy. Figure 1 is an energy flow diagram as determined by (8), (11), and (13). Baroclinic energy exchanges between equations for mean potential energy, mean kinetic energy, and turbulence kinetic energy are not shown.

4. Boundary layers

The purpose of this section is to discuss outstanding issues that need further research and understanding against the backdrop of the equations cited above.

In surface boundary layer models, the term $\tau_{p\alpha}$ has, erroneously it now appears, generally been lumped in with the Reynolds flux and modeled as turbulence. Now there is need to model $\tau_{p\alpha}(0) = \overline{\tilde{p}_{w\eta}\partial\tilde{\eta}/\partial x_\alpha}$, after which the depth dependence given by (5b) should apply. Whether the previous modeling (e.g., Mellor and Yamada 1982) suffices for $\tau_{t\alpha}$ is a ripe subject for new observations and study.

In the absence of surface forcing, Longuet-Higgins (1953) pointed out the incompatibility of irrotationality and zero (constant viscosity) stress thus creating a thin surface boundary layer; this phenomenon is probably of minor importance (Phillips 1977).

The bottom boundary layer in shallow water as described by Longuet-Higgins (1953; see also Mei 1983, Phillips 1977, Huang 1970, and Liu and Davis 1977) is enigmatic. For a barotropic, progressive, horizontally homogeneous wave, zero Coriolis parameter, zero wind forcing and zero current, Longuet-Higgins provided a continuation of the Stokes drift to the bottom assuming

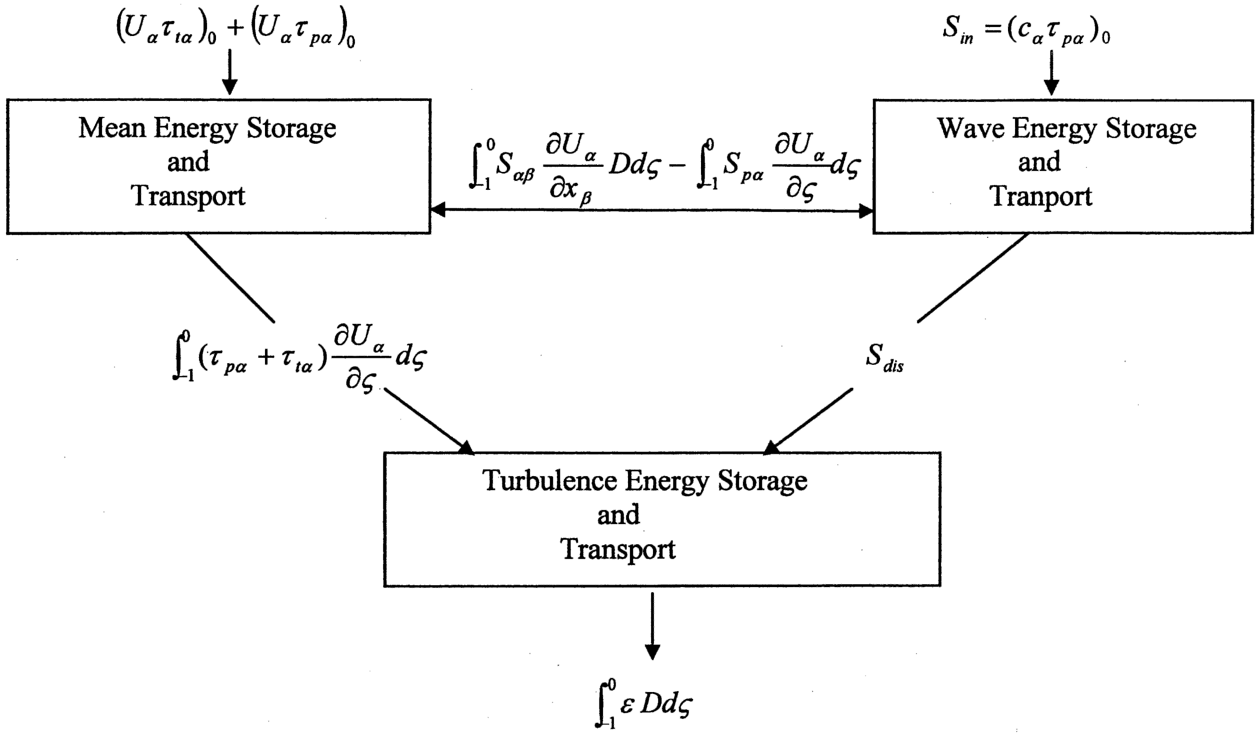

FIG. 1. The energy flow diagram for a barotropic system. The final dissipation, $\int_{-1}^{0} \varepsilon D \, d\varsigma$, is a source term in the thermal energy equation (not shown).

constant viscosity. When a turbulence model is used in conjunction with (5a) and then (2a), it is probable that the part of the bottom boundary layer described by (16) will be adequately addressed together with enhanced turbulence production as in Mellor (2002).

However, for constant viscosity, Longuet-Higgins found that the orbital velocities in the bottom layer produced in-phase orbital velocities that combine for a mean wave stress such that

$$\frac{\partial \overline{\tilde{u}_{b\alpha}\tilde{w}_b}}{\partial z'} = \frac{E}{g} \frac{o^2 k_\alpha}{\sinh^2 kD} e^{-\beta z'} \times [e^{-\beta z'} - \cos\beta z' + \beta z'(\cos\beta z' - \sin\beta z')], \tag{16}$$

where $z' \equiv D(1 + \varsigma)$ is measured from the bottom and $\beta = \sqrt{\sigma/(2\nu)}$. For $\nu \partial^2 U_\alpha / \partial z'^2 = \partial \overline{\tilde{u}_{b\alpha}\tilde{w}_b}/\partial z'$, one obtains a velocity solution that, at the outer edge of the boundary layer (but where $\varsigma \cong -1$), is independent of viscosity and is 3/2 times the Stokes drift, a counterintuitive but now accepted result.

It is herewith proposed that $-\partial \overline{\tilde{u}_{b\alpha}\tilde{w}_b}/\partial z$ be added to the right side of (2a), in which case, near the bottom, $-\overline{w' u'_\alpha} + \nu \partial U_\alpha / \partial z' - \overline{\tilde{w}\tilde{u}_\alpha} = \tau_{t\alpha}(0) + (\partial p / \partial x_\alpha)_0 z' + \cdots$ and $\tau_{t\alpha}(0) = \nu(\partial U_\alpha / \partial z')_0$ in the case of a smooth bottom.

If the bottom flow is characterized by a constant molecular or eddy viscosity and the value is known, then $-\partial \overline{\tilde{u}_{b\alpha}\tilde{w}_b}/\partial z$ from (16) could simply be added to the right side of (2a). It is noted that (16) does not involve viscosity except in the scale factor β. For turbulent flow, the scale is $\beta \propto \sigma/u_*$, where u_* is the bottom friction velocity and where we speculate that the constant of proportionality is of order unity.

With the help of phase-resolved numerical modeling (Mellor 2002), details of an oscillating turbulence bottom boundary layer have been obtained including enhanced bottom friction and dissipation; the eddy viscosity was not constant spatially or temporally. However, the oscillating flow was a kind of slug flow such that $k = 0$, whence $\overline{\tilde{u}_{b\alpha}\tilde{w}_b} = 0$. It is hoped that that line of research can be extended for $k > 0$. The work of Groeneweg and Klopman (1998) would seem to bear on this problem but as yet does not supply $-\partial \overline{\tilde{u}_{b\alpha}\tilde{w}_b}/\partial z$ in the needed parametric form.

5. Summary

The precursor paper, M03, was replete with a detailed and somewhat complicated derivation of the three-dimensional phase-averaged, continuity, momentum, and wave energy equations. It is the purpose of this paper to contribute to an understanding of the equations by simplification of nomenclature, through the development of energy pathways, by presenting the Cartesian version of the equations in appendix A, and by contrasting with another set of current–wave interaction equations in appendix B.

It will be recognized that some terms that must be modeled based on observations and laboratory data do double duty. Thus, the term $\tau_{p\alpha}(0) = \overline{\tilde{p}_{w\eta}\partial\tilde{\eta}/\partial x_\alpha}$ is important to the wave energy equation and the momentum equation. The source term s_{dis} in the turbulence kinetic energy equation, when vertically integrated, is a sink term in the wave energy equation.

The next goal is to develop a properly coupled wave, circulation model; many of the pieces exist, but some issues require further research, particularly those mentioned in section 4.

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APPENDIX A

Conversion to Cartesian Coordinates

The use of wave-following sigma coordinates was a useful strategy to derive (1) and (2). However, after phase averaging, the equations appear as "ordinary" sigma equations but with additional terms and with the knowledge that U_α includes both the current and the Stokes drift. For some ocean circulation models, the equations of motion in sigma coordinates are preferred; otherwise, Cartesian coordinates are the usual way of describing these equations. Following the reasoning in M03, they can be obtained by a reverse transformation so that, letting $\phi(x_\alpha, \varsigma, t) = \phi^*(x_\alpha^*, z, t^*)$, we have

$$\frac{\partial\phi}{\partial x_\alpha} = \frac{\partial\phi^*}{\partial x_\alpha^*} + \frac{\partial\phi^*}{\partial z}\left(\frac{\partial\hat{\eta}}{\partial x_\alpha} + \varsigma\frac{\partial D}{\partial x_\alpha}\right), \tag{A1a}$$

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi^*}{\partial t^*} + \frac{\partial\phi^*}{\partial z}\left(\frac{\partial\hat{\eta}}{\partial t} + \varsigma\frac{\partial D}{\partial t}\right), \quad \text{and} \tag{A1b}$$

$$\frac{\partial\phi}{\partial\varsigma} = D\frac{\partial\phi^*}{\partial z}, \tag{A1c}$$

where $\varsigma \equiv (z - \hat{\eta})/D$. Using these transformations on (1) and (2) together with $\Omega \equiv W - U_\alpha(\hat{\eta}_\alpha + \varsigma D_\alpha) - \hat{\eta}_t - \varsigma D_t$ [notice how Cartesian surface and bottom boundary conditions are satisfied by virtue of the fact that $\Omega(\varsigma = 0) = \Omega(\varsigma = -1) = 0$], the following are obtained, after some algebra and after deleting the asterisks,

$$\frac{\partial U_\alpha}{\partial x_\alpha} + \frac{\partial W}{\partial z} = 0, \tag{A2}$$

$$\frac{\partial U_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}(U_\beta U_\alpha) + \frac{\partial}{\partial z}(WU_\alpha) + \frac{\partial\hat{p}}{\partial x_\alpha} + \epsilon_{\alpha\beta z}f_z U_\beta = -\frac{1}{D}\frac{\partial DS_{\alpha\beta}}{\partial x_\beta} - \left(\frac{\partial\hat{\eta}}{\partial x_\beta} + \frac{z-\hat{\eta}}{D}\frac{\partial D}{\partial x_\beta}\right)\frac{\partial S_{\alpha\beta}}{\partial z} + \frac{\partial S_{p\alpha}}{\partial z} + \frac{\partial\tau_{t\alpha}}{\partial z} + \frac{\partial\tau_{p\alpha}}{\partial z}, \quad \text{and} \tag{A3a}$$

$$\frac{\partial\hat{p}}{\partial z} = -g\frac{\hat{\rho}}{\rho_o}. \tag{A3b}$$

Whereas the pressure term is simpler relative to the sigma version, the terms involving $S_{\alpha\beta}$ on the right of (A3a) are complicated. However, when they are integrated from $z = -h$ to $z = \hat{\eta}$, the result, after manipulation including use of Leibnitz's rule, is simply

$$-\partial\left(\int_{-h}^{\hat{\eta}} S_{\alpha\beta}\,dz\right)\bigg/\partial x_\beta,$$

where

$$\int_{-h}^{\hat{\eta}} S_{\alpha\beta}\,dz = E(k_\alpha k_\beta/k^2)(c_g/c) + \delta_{\alpha\beta}E(c_g/c - 1/2)$$

in accordance with Longuet-Higgins and Stewart (1960) and Phillips (1977).

APPENDIX B

Other Wave–Current Interaction Equations

The equations of Craik and Leibovich (1976) and the extensions of McWilliams and Restrepo (1999) are widely cited wave–current interaction equations and have recently been used to model the effects of Langmuir circulation (Kantha and Clayson 2004) on mixing. The questions are to what extent do they differ from the equations of this paper and why? When vertically integrated, they do differ from the equations found in Phillips (1977).

a. The equations of this paper

For convenience of comparison, the present momentum equation is written in vector form without making the boundary layer or hydrostatic approximation. Thus in Cartesian coordinates and from (A2) and (A3a) we have

$$\frac{\partial\mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U} + \mathbf{f}\times\mathbf{U} + \nabla\hat{p} - \mathbf{b} = \mathbf{G}, \tag{B1}$$

where $\mathbf{G} = \partial(\boldsymbol{\tau}_p + \boldsymbol{\tau}_t)/\partial z + D^{-1}\boldsymbol{\nabla}_h(D\mathbf{S}) + [\boldsymbol{\nabla}_h\hat{\eta} + D^{-1}(z - \hat{\eta})\,\boldsymbol{\nabla}_h D] \cdot \partial\mathbf{S}/\partial z - \partial\mathbf{S}_p/\partial z$ and $\mathbf{b} = \rho\mathbf{g}/\rho_o$ is the buoyancy wherein $\mathbf{g} = (0, 0, -g)$. The dyadic $\mathbf{S} = S_{\alpha\beta}$ from (3a), the vector $\mathbf{S}_p = S_{p\alpha}$ from (3b), and $\boldsymbol{\nabla}_h$ is the horizontal gradient operator. The several terms involving $\mathbf{S}$ are complicated, but, as mentioned above, when they are all integrated from $z = -h$ to $z = \hat{\eta}$ the result is simply $\boldsymbol{\nabla}_h \cdot \int_{-h}^{\hat{\eta}} \mathbf{S}\, dz$ in accord with the corresponding term in Phillips (1977).

Recall that $\mathbf{U} = \hat{\mathbf{u}} + \mathbf{u}_s$ = the current + Stokes drift, and thus, considering (A2) or $\boldsymbol{\nabla} \cdot \mathbf{U} = 0$ and only the left side of (B1), the current and the Stokes drift are subject to the same tendency, advective, and Coriolis processes.

Using a well-known vector identity, (B1) is

$$\frac{\partial \mathbf{U}}{\partial t} - \mathbf{U} \times (\mathbf{f} + \boldsymbol{\omega}) + \boldsymbol{\nabla}\left(p + \frac{\mathbf{U}\cdot\mathbf{U}}{2}\right) - \mathbf{b} = \mathbf{G}, \quad \text{(B2)}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{U} = \hat{\boldsymbol{\omega}} + \boldsymbol{\omega}_S$, with $\hat{\boldsymbol{\omega}} = \boldsymbol{\nabla} \times \hat{\mathbf{u}}$ and $\boldsymbol{\omega}_S = \boldsymbol{\nabla} \times \mathbf{u}_S$. The curl of (B2) is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times \mathbf{U} \times (\mathbf{f} + \boldsymbol{\omega}) - \boldsymbol{\nabla} \times \mathbf{b} = \boldsymbol{\nabla} \times \mathbf{G}. \quad \text{(B3)}$$

b. The equations of McWilliams and Restrepo

The equation derived by Craik and Leibovich (1976) and Leibovich (1980) and extended to include tendency and Coriolis terms by McWilliams and Restrepo (1999) is

$$\frac{\partial \hat{\boldsymbol{\omega}}}{\partial t} - \boldsymbol{\nabla} \times \mathbf{U} \times (\mathbf{f} + \hat{\boldsymbol{\omega}}) - \boldsymbol{\nabla} \times \mathbf{b} = \nu\boldsymbol{\nabla}^2\hat{\boldsymbol{\omega}}, \quad \text{(B4)}$$

where ν is the kinematic viscosity. It follows that $\partial\hat{\mathbf{u}}/\partial t - \mathbf{U} \times (\mathbf{f} + \hat{\boldsymbol{\omega}}) + \boldsymbol{\nabla}\Phi - \mathbf{b} = \nu\boldsymbol{\nabla}^2\hat{\mathbf{u}}$, where Φ is an arbitrary scalar. If one chooses $\Phi = \hat{p} + \hat{\mathbf{u}}\cdot\hat{\mathbf{u}}/2$, then

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} - \mathbf{U} \times (\mathbf{f} + \hat{\boldsymbol{\omega}}) + \boldsymbol{\nabla}\left(\hat{p} + \frac{\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}}{2}\right) - \mathbf{b} - \nu\boldsymbol{\nabla}^2\hat{\mathbf{u}}. \quad \text{(B5)}$$

Because $-\hat{\mathbf{u}} \times \hat{\boldsymbol{\omega}} + \boldsymbol{\nabla}(\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}/2) = \hat{\mathbf{u}} \cdot \boldsymbol{\nabla}\hat{\mathbf{u}}$, one obtains

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + \hat{\mathbf{u}}\cdot\boldsymbol{\nabla}\hat{\mathbf{u}} - \hat{\mathbf{u}} \times \mathbf{f} + \boldsymbol{\nabla}p - \mathbf{b}$$

$$= \nu\boldsymbol{\nabla}^2\hat{\mathbf{u}} + \mathbf{u}_S \times \mathbf{f} + \mathbf{u}_S + \hat{\boldsymbol{\omega}}, \quad \text{(B6)}$$

where $\mathbf{u}_S \times \mathbf{f}$ and $\mathbf{u}_S \times \hat{\boldsymbol{\omega}}$ are said to be the Stokes drift Coriolis force and the Stokes drift vortex force. McWilliams and Restrepo chose another scalar identity for Φ, and Kantha and Clayson chose yet another Φ. The choice here is made so that (B5) or (B6) are correct in the absence of waves ($\mathbf{u}_S = 0$) as is (B1) or (B2). (However, any other addition to Φ containing $\mathbf{u}_S$ would also be correct in the absence of waves.)

Putting aside this algebraic problem, (B6), (B5), and (B4) can be compared—in reverse order—with (B1), (B2), and (B3). It will be seen that the so-called Stokes drift Coriolis and Stokes drift vortex forces—the last terms in (B6)—are contained in (B1) but so are other terms such that the Stokes drift and the current are not uniquely different. More simply, compare (B3) and (B4). Assuming $\nu\boldsymbol{\nabla}^2\hat{\mathbf{u}}$ in (B6) is a model for $\mathbf{G} = \partial\boldsymbol{\tau}_t/\partial z$ in (B1), this would leave out $\partial\boldsymbol{\tau}_p/\partial z$, which, it is claimed here, is the important wave-induced source term for the Stokes drift portion in (B1). Wave radiation terms are also missing. A reason for the discrepancies, I believe, is that, in the derivation of (B4), a wavy free surface and underlying wavy material surfaces were not factored into the derivation and do not account for some second-order terms retained in (B3). What this does for the commonly accepted explanation for Langmuir circulation is beyond the scope of this paper (as is often stated in the absence of requisite wisdom).

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