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Key Points:
- Review of the two main theories of surface wave/ocean circulation
- Arguments are presented favoring the wave radiation stress theory
- Errors in the vortex force theory are explained

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On theories dealing with the interaction of surface waves and ocean circulation

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ABSTRACT The classic theory for the interaction of surface gravity waves and the general ocean circulation entails the so-called wave radiation stress terms in the phase-averaged momentum equation. The equations of motion are for the combined Eulerian current and Stokes drift. On the other hand, a more recent approach includes the so-called vortex force term in the momentum equation wherein the only wave property is Stokes drift. The equations of motion are for the Eulerian current. The idea has gained traction in the ocean science community, a fact that motivates this paper. A question is: can both theories be correct? This paper answers the question in the negative and presents arguments in favor of the wave radiation theory. The vortex force approach stems from an interesting mathematical construct, but it does stand up to physical or mathematical scrutiny as described in this paper. Although not the primary focus of the paper, some discussion of Langmuir circulation is included since the vortex force was first introduced as the basis of this oceanic cellular phenomenon. Finaly the paper explains the difference in the derivation of the radiation stress theory and the vortex force theory: the later theory entails errors related to its use of curl and reverse-curl [or uncurl] processes.

1. Introduction

To treat the interaction of surface waves with the underlying circulation, a theory was introduced by *Longuet-Higgins and Stewart* [1962, 1964, hereinafter L-HS] and extended by [*Phillips*, 1977, hereinafter Phillips]. The theory applies to vertically integrated, phase-averaged equations of motion. The prognostic dependent variable was the combined Eulerian current and Stokes drift which responded to surface forcing but also to a "wave radiation stress" term in the momentum equation. Papers by *Mellor* [2003, 2015, hereinafter Mellor] extended the L-HS theory to obtain vertically resolved equations. The combined Eulerian current and Stokes drift satisfied the nondivergent continuity equation; separately the current and Stokes drift are generally divergent.

In pursuit of a theory of Langmuir cells, observed on the ocean surface as long rows of convergent contaminants or as "windrows," *Craik and Leibovich* [1976, hereinafter CL] developed a theory where stream-wise vorticity was forced by the curl of a so-called "vortex force." The theory was extended by *McWilliams and Retrepo* [1999, hereinafter MR] wherein the vorticity, when "uncurled," devolved into a momentum equation for the Eulerian current alone and which was forced by the vortex force, an element of which was Stokes drift. The Eulerian current and Stokes drift velocities are separately nondivergent.

Whereas the motivation of CL was to offer a "rational model of Langmuir circulation," MR applied the theory to oceanic circulation and, in particular, to determine its effect on large scale circulation. A later paper, *McWilliams et al.* [2004], extended MR to account for infragravity waves whose scales are intermediate between surface wave and current scales; the paper is quite complicated. However, in a subsequent paper [*Uchiyama et al.*, 2010], the infragravity wave and current scales were combined therefore reverting to the two scale scheme of MR.

The vortex force term in the momentum equation is a major player in many papers that have appeared in the last decade or so. Some numerical ocean models incorporate the vortex force term in their algorithms [*Uchiyama et al.*, 2010, *Bennis et al.*, 2011, *Kumar et al.*, 2012]; it has also played a role in numerical turbulence simulations [*McWilliams et al.*, 1997, *Sullivan et al.*, 2004] and turbulence closure models [*Kantha and Clayson*, 2004, *Harcourt*, 2013].

If the "radiation stress" and "vortex force" theories are both correct, then one should be able to derive one from the other, but apparently that cannot be done according to *Mellor* [2015] who invoked the paper by *Smith* [2006a]. This paper is written in support of the L-HS and Mellor theory whereas we show where and how the theory of MR and papers dependent on the vortex force are incorrect.

1.1. Wave Relations

To represent waves, all of the references cited above use the linear wave relations either directly or indirectly such that wave elevation, velocities and pressure are

$$\tilde{\eta}=a\cos\psi \tag{1}$$

$$\tilde{u}_\alpha=k_\alpha ac\frac{\cosh k(z+h)}{\sinh kD}\cos\psi,\ \tilde{w}=kac\frac{\sinh k(z+h)}{\sinh kD}\sin\psi \tag{2a, 2b}$$

$$\tilde{p}=kac^2\frac{\cosh k(z+h)}{\sinh kD}\cos\psi \tag{2c}$$

where k_α is the wave number vector and $k=|k_\alpha|$; a is wave amplitude; $c=\sigma/k$ is phase speed; $\psi\equiv k_\beta x_\beta-\omega t$ and $\omega=\sigma-k_\beta u_{A\beta}$ where σ is intrinsic frequency and $u_{A\beta}$ is the Doppler velocity defined in *Mellor* [2003]. Greek subscripts, α or β, denote horizontal components so that $x_\alpha=(x,y)$ are horizontal coordinates and $\tilde{u}_\alpha=(\tilde{u}_x,\tilde{u}_y)$ are horizontal velocity components; the vertical, particle following coordinate is z and the vertical wave velocity is $\tilde{w}$. Repeated subscripts denote summation; e.g., $\partial u_\beta/\partial x_\beta=\partial u_x/\partial x+\partial u_y/\partial y$.

There will also be a need for vector nomenclature such that $\mathbf{x}=(x_\alpha,z)=(x,y,z)$ and $\tilde{\mathbf{u}}=(\tilde{u}_\alpha,\tilde{w})=(\tilde{u}_x,\tilde{u}_y,\tilde{w})$.

It is noteworthy that (2) applies to the entire water column, $\eta>z>-h$, where $\eta=\hat{\eta}+\tilde{\eta}$; the mean elevation is $\hat{\eta}$ and $\tilde{\eta}$ is the instantaneous wave elevation. The mean water column depth is $D=\hat{\eta}+h$. Wave amplitude, wave number and frequency are assumed to vary slowly, spatially relative to k^{-1} and temporally relative to σ^{-1} (Phillips). Waves are taken to be monochromatic from which, it is assumed, spectra can be formed.

It should be understood that the wave energy equation must be added to the mix of equations to supply wave energy, $E=ga^2/2$, and therefore wave amplitude.

1.2. Phase-Averaging

Throughout this paper, phase-averaging is denoted by an over-bar such that

$$\overline{(\quad)}=\frac{1}{2\pi n}\int_0^{2\pi n}(\quad)d\psi \tag{3a}$$

where n is an integer equal or greater than one. It can also be reckoned by holding *x, y* and $\bar{z}=$ const. (z denotes the vertical location of material surfaces subject to wave motion, $\bar{z}$ is the phase-average of z). Thus,

$$\overline{(\quad)}=\frac{1}{nT}\int_0^{nT}(\quad)dt, \tag{3b}$$

or holding t and $\bar{z}=$ const. so that

$$\overline{(\quad)}=\frac{1}{nL}\int_0^{nL}(\quad)dx \tag{3c}$$

where here x is taken in the direction of wave propagation. T and L are wave period and wave length respectively.

1.3. Stokes Drift

Phillips and many others determine Stokes drift according to phase-averages of the Lagrangian wave velocity,

$$u_{S\alpha}=\overline{\tilde{u}_\alpha+\tilde{x}_\beta\frac{\partial\tilde{u}_\alpha}{\partial x_\beta}+\tilde{z}\frac{\partial\tilde{u}_\alpha}{\partial z}},\ w_S=\overline{\tilde{w}+\tilde{x}_\beta\frac{\partial\tilde{w}}{\partial x_\beta}+\tilde{z}\frac{\partial\tilde{w}}{\partial z}}, \tag{4a, 4b}$$

where $\tilde{x}_\beta\equiv\int_0^t\tilde{u}_\beta(t')dt'$ and $\tilde{z}\equiv\int_0^t\tilde{w}(t')dt'$. Without a change in nomenclature, let $z\rightarrow\bar{z}$ in $\tilde{u}_\alpha$ and $\tilde{w}$ of (2a, 2b) and then insert the result into (4a, 4b) yielding Stokes drift,

$$u_{S\alpha}=k_\alpha a(ka)c\,\frac{\cosh 2k(\bar{z}+h)}{2\sinh^2 kD}\ \text{and}\ w_S=0 \qquad (5a, 5b)$$

Thus, $\mathbf{u}_S=(u_{Sx},u_{Sy},0)$ is generally divergent as in Phillips.

1.4. This Paper

Section 2 is a review of the CL and MR theory. At first, I had difficulty grasping all the steps in the CL and MR analyses; however, their derivation is simplified in section 2 by omitting Coriolis and baroclinic terms; nevertheless the ingredients leading to the vortex force term in the momentum equation are retained. The review does not substantively change their analysis or final result. An alternative development is presented in section 3 using the same scaling as in section 2. Section 4 is a brief review of the L-HS theory and its vertically resolved counterpart as most recently set forth by *Mellor* [2015] which reviewed the L-HS and Phillips vertically integrated derivation and the Mellor vertically resolved derivation from the same integral continuity and momentum equations.

Reasons for favoring the LH-S and Mellor theory are set forth in section 5. *Langmuir* circulation is discussed in sections 6 and 7. It is included because it first motivated the concept of the vortex force. Finally, section 8 explains why the radiation stress theory differs from the vortex force theory. Section 9 is a summary of this paper's findings.

2. Review of the CL and MR Theory

To aid comprehension, this review adds some intervening steps to the analysis of CL and MR. Although the Coriolis and baroclinic terms are omitted, their inclusion in the phase-averaged momentum equation is straight forward and is the same in CL and MR and Mellor.

Adopting MR nomenclature, the continuity and momentum equations are

$$\nabla\cdot\mathbf{q}=0 \qquad (6)$$

and

$$\frac{\partial\mathbf{q}}{\partial t}+\mathbf{q}\cdot\nabla\mathbf{q}+\nabla p=-\mathbf{g}+\nu\nabla^2\mathbf{q} \qquad (7)$$

or using an established vector identity [*Hildebrand*, 1976]

$$\frac{\partial\mathbf{q}}{\partial t}=\mathbf{q}\times(\nabla\times\mathbf{q})-\nabla(p+\mathbf{q}\cdot\mathbf{q}/2)-\mathbf{g}+\nu\nabla^2\mathbf{q}. \qquad (8)$$

In the above, $\mathbf{q}$ is the velocity vector, p is kinematic pressure, $\mathbf{g}$ is the gravity vector and ν is an eddy viscosity.

CL and MR establish an ordering scheme such that

$$\mathbf{q}=\varepsilon\,\mathbf{u}^w+\varepsilon^2\mathbf{v},\quad \nu=\varepsilon^2\nu_0 \qquad (9a, 9b)$$

plus higher order terms. In (9), $\varepsilon\equiv$ order (ka) and ka is the wave slope. Note that $\mathbf{u}^w(\mathbf{x},t)$, representing wave motion, is taken to be a function of space and a fast time variable, t, such that a phase average $\overline{\mathbf{u}^w}=0$ whereas $\mathbf{v}(\mathbf{x},t,t_s)$ is additionally a function of the slow time variable, t_s. After inserting (9) into (8), taking the curl of the result and finally dividing by ε^2, CL and MR obtain

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\varepsilon^2\frac{\partial\boldsymbol{\omega}}{\partial t_s}=\nabla\times\left[(\varepsilon\,\mathbf{u}^w+\varepsilon^2\mathbf{v})\times\boldsymbol{\omega}\right]+\varepsilon^2\nu_0\nabla^2\boldsymbol{\omega} \qquad (10)$$

where $\nabla\cdot\mathbf{u}^w=\nabla\times\mathbf{u}^w=0$ and $\boldsymbol{\omega}\equiv\nabla\times\mathbf{v}$. In (10), note that $\mathbf{q}\times(\nabla\times\mathbf{u}^\mathbf{w})=0$ and has been deleted. [Initially, MR used the small parameters γ and δ to modify different parts of (10); subsequently they were related to ε which here are incorporated ab initio.] Next, expand

$$\mathbf{v}=\mathbf{v}_0+\varepsilon\mathbf{v}_1+\varepsilon^2\mathbf{v}_2+\ldots \qquad (11a)$$

$$\boldsymbol{\omega}=\boldsymbol{\omega}_0+\varepsilon\boldsymbol{\omega}_1+\varepsilon^2\boldsymbol{\omega}_2+\ldots \tag{11b}$$

so that, evidently, $\boldsymbol{\omega}_0=\nabla\times\mathbf{v}_0$ and $\boldsymbol{\omega}_1=\nabla\times\mathbf{v}_1$, etc.

From (10) and (11), an equation of order ε^0 is

$$\frac{\partial\boldsymbol{\omega}_0}{\partial t}=0 \tag{12}$$

and of order ε

$$\frac{\partial\boldsymbol{\omega}_1}{\partial t}=\nabla\times\left(\mathbf{u}^w\times\boldsymbol{\omega}_0\right) \tag{13}$$

and of order ε^2

$$\frac{\partial\boldsymbol{\omega}_2}{\partial t}+\frac{\partial\boldsymbol{\omega}_0}{\partial t_s}=\nabla\times(\mathbf{v}_0\times\boldsymbol{\omega}_0)+\nabla\times(\mathbf{u}^w\times\boldsymbol{\omega}_1)+\nu_0\nabla^2\boldsymbol{\omega}_0. \tag{14}$$

From (12), $\boldsymbol{\omega}_0=\boldsymbol{\omega}_0(\mathbf{x},t_s)$ and, also $\mathbf{v}_0=\mathbf{v}_0(\mathbf{x},t_s)$. A time integral of (13) is

$$\boldsymbol{\omega}_1=\nabla\times(\mathbf{U}\times\boldsymbol{\omega}_0) \tag{15}$$

where, $\mathbf{U}\equiv\int\mathbf{u}^w(t')dt'$ as obtained from (2a, 2b). ($\mathbf{U}$ is used here and in Appendix B for different variables. The context in which they appear should make the distinction clear.) A phase average of (14) is

$$\frac{\partial\boldsymbol{\omega}_0}{\partial t_s}=\nabla\times(\mathbf{v}_0\times\boldsymbol{\omega}_0)+\nabla\times(\overline{\mathbf{u}^w\times\boldsymbol{\omega}_1})+\nu_0\nabla^2\boldsymbol{\omega}_0. \tag{16}$$

CL and MR engage in complicated tensor algebra (their Appendix A) to evaluate $\boldsymbol{\omega}_1$ from (15) and obtain

$$\overline{\mathbf{u}^w\times\boldsymbol{\omega}_1}=\mathbf{u}^S\times\boldsymbol{\omega}_0 \tag{17}$$

the now well-known vortex force where $\mathbf{u}^S$ is the Stokes drift. A review of the derivation of (15) and (17) in this paper's Appendix B indicates that (17) is missing a vertical component. Nevertheless, continuing with the MR analysis, (16) can be written

$$\frac{\partial\boldsymbol{\omega}_0}{\partial t_s}=\nabla\times(\mathbf{v}_0\times\boldsymbol{\omega}_0)+\nabla\times\left(\mathbf{u}^S\times\boldsymbol{\omega}_0\right)+\nu_0\nabla^2\boldsymbol{\omega}_0. \tag{18}$$

Now "uncurl" (18) to obtain

$$\frac{\partial\mathbf{v}_0}{\partial t_s}-(\mathbf{v}_0\times\boldsymbol{\omega}_0)=\mathbf{u}^S\times\boldsymbol{\omega}_0-\nabla\Phi+\mathbf{g}+\nu_0\nabla^2\mathbf{v}_0 \tag{19}$$

where $\mathbf{g}$ is any constant vector and Φ is any scalar function. However, to conform to the conventional momentum equation, let $\mathbf{g}=(0,0,-g)$ be the gravity constant and now define $\Phi=p_0+(\mathbf{v}_0\cdot\mathbf{v}_0)/2$ where p_0 is phase-averaged pressure. Because $\mathbf{v}_0\times\boldsymbol{\omega}_0=-\mathbf{v}_0\cdot\nabla\mathbf{v}_0+\nabla(\mathbf{v}_0\cdot\mathbf{v}_0)/2$ [*Hildebrand*, 1976], (19) can be written

$$\frac{\partial\mathbf{v}_0}{\partial t_s}+\mathbf{v}_0\cdot\nabla\mathbf{v}_0=-\nabla p_0+\mathbf{g}+\mathbf{u}^S\times\boldsymbol{\omega}_0+\nu_0\nabla^2\mathbf{v}_0 \tag{20}$$

This is equation (19) in MR.

A necessary addition to (20) is the continuity equation in the form

$$\nabla\cdot\mathbf{v}_0=0 \tag{21}$$

which is obtained from (6), (9a) and (11a) to lowest order.

Henceforth, the subscripts, 0, will be removed.

2.1. The Boundary Layer Approximation

Applying the boundary layer approximation (also known as the hydrostatic approximation) to (20) yields

$$\frac{\partial v_\alpha}{\partial t_s}+v_\beta\frac{\partial v_\alpha}{\partial x_\beta}+v_z\frac{\partial v_\alpha}{\partial z}=-\frac{\partial p}{\partial x_\alpha}+\omega_z(u_y^S,-u_x^S)+\nu\nabla^2 v_\alpha. \tag{22a}$$

Since the vertical component of the vortex force is small relative to g,

$$\frac{\partial p}{\partial z}=-g \tag{22b}$$

so that $p=p_{atm}+g(\hat{\eta}-z)$ and $\partial p/\partial x_\alpha=\partial(p_{atm}+g\hat{\eta})/\partial x_\alpha$ for insertion into (22a); p_{atm} is the atmospheric pressure and $\hat{\eta}$ is the mean surface elevation. Finally, the continuity equation from (21) is

$$\frac{\partial v_\beta}{\partial x_\beta}+\frac{\partial v_z}{\partial z}=0. \tag{23}$$

Multiply both terms in (23) by v_α and add to (22a) to obtain the flux form of the momentum equation,

$$\frac{\partial v_\alpha}{\partial t_s}+\frac{\partial(v_\beta v_\alpha)}{\partial x_\beta}+\frac{\partial(v_z v_\alpha)}{\partial z}=-\frac{\partial p}{\partial x_\alpha}+\omega_z(u_y^S,-u_x^S)+\nu\nabla^2 v_\alpha. \tag{24}$$

The development in this section is an impressive display of mathematical acumen and manipulative skill. However, I do not think the development is correct. The reasons follow in sections 3, 5 and 7.

3. Alternate Derivation

The ordering scheme adopted by CL and MR is used here except that it is applied directly to the primitive equations of motion. Now, write again (6)

$$\nabla\cdot\mathbf{q}=0, \tag{25}$$

For later convenience, add $\mathbf{q}\nabla\cdot\mathbf{q}$ to (7) to obtain the flux form of the momentum equation and obtain

$$\frac{\partial\mathbf{q}}{\partial t}+\nabla\cdot(\mathbf{qq})+\nabla p=\mathbf{g}+\nu\nabla^2\mathbf{q}. \tag{26}$$

Boundary conditions at $z=\eta$ are $q_z=q_\beta\partial\eta/\partial x_\beta+\partial\eta/\partial t$ and at $z=-h$, $q_z=-q_\beta\partial h/\partial x_\beta$.

Adding pressure, the ordering scheme as in (9) is repeated

$$\mathbf{q}=\varepsilon\mathbf{u}^w+\varepsilon^2\mathbf{v},\quad \nu=\varepsilon^2\nu_o,\quad p=\varepsilon p^w+\varepsilon^2 p^v,\ \eta=\varepsilon\eta^w+\varepsilon^2\eta^v. \tag{27a, 27b, 27c, 27d}$$

Inserting (27) in (25) and (26) and collecting terms of order ε,

$$\nabla\cdot\mathbf{u}^w=0 \tag{28}$$

$$\frac{\partial\mathbf{u}^w}{\partial t}+\nabla p^w=\mathbf{g} \tag{29}$$

where $\mathbf{g}=(0,0,-g)$. Boundary conditions at $z=\eta$, are $w^w=\partial\eta^w/\partial t$ and at $z=-h$, $w^w=-u_\beta^w\partial h/\partial x_\beta$.

The order ε^2 equations are

$$\nabla\cdot\mathbf{v}=0 \tag{30}$$

$$\frac{\partial\mathbf{v}}{\partial t}+\nabla\cdot\left(\mathbf{u}^w\mathbf{u}^w\right)+\nabla p^v=\nu\nabla^2\mathbf{v}. \tag{31}$$

At $z=\eta$, $v_z=u_\beta^w\partial\eta^w/\partial x_\beta+\partial\eta^v$ and at $z=-h$, $v_z=-v_\beta\partial h/\partial x_\beta$. The ordering scheme has the nice feature that (28), (29) – which vanished in section 2 - and boundary conditions immediately deliver the linearly wave solutions, (1) and (2).

Now reassemble the equations by combining the product of ε and equation (28) and the product of ε^2 and equation (30) to give

$$\nabla\cdot(\tilde{\mathbf{u}}+\hat{\mathbf{u}})=0 \tag{32}$$

where we define $\tilde{\mathbf{u}}=\varepsilon\,\mathbf{u}^w$ and $\hat{\mathbf{u}}\equiv\varepsilon^2\mathbf{v}$. Similarly add (29) and (31) to give

$$\frac{\partial}{\partial t}(\hat{\mathbf{u}}+\tilde{\mathbf{u}})+\nabla\cdot[(\hat{\mathbf{u}}+\tilde{\mathbf{u}})(\hat{\mathbf{u}}+\tilde{\mathbf{u}})]+\nabla(\hat{p}+\tilde{p})=\mathbf{g}+\nu\nabla^2\hat{\mathbf{u}} \tag{33}$$

A slight disadvantage of the ordering scheme in (27) is that, in the absence of waves, the conventional advective terms in (33) are not recovered. Therefore, consistent with (27), we have simply added higher order terms to (33) by replacing $\tilde{\mathbf{u}}$ with $\hat{\mathbf{u}}+\tilde{\mathbf{u}}$ in the terms in square brackets. After further development below and phase-averaging, this will add an error of order ε^4 relative to retained terms of order ε^2. By similar addition, the boundary conditions become at $z=\eta$, $\hat{w}+\tilde{w}=(\hat{u}_\beta+\tilde{u}_\beta)\partial\eta/\partial x_\beta+\partial\eta/\partial t$ and at $z=-h$, $\hat{w}+\tilde{w}=-(\hat{u}_\beta+\tilde{u}_\beta)\partial h/\partial x_\beta$.

Note that the horizontal and vertical components of $\mathbf{u}^w\mathbf{u}^w$ or $\tilde{\mathbf{u}}\tilde{\mathbf{u}}$ after phase averaging become part of the radiation stress term in the following equation.

4. The L-HS and Mellor Theories

Although representation of the viscous term differs, (32) and (33) are the initial equations of L-HS and *Mellor* followed by further manipulation and phase-averaging. The L-HS theory, as recorded by Phillips, is obtained by vertical integration of (32) and (33) while incorporating boundary conditions after which the integrals are divided into wave portions (mean surface level to crest or trough levels) and the remainder portion and then phase-averaged. It is a mostly physical argument and, thus differs from the predominantly mathematical argument of CL and MR. The methodology is reviewed in *Mellor* [2015] and shown to be related to the vertically resolved equations of *Mellor* [2003].

The equations in *Mellor* [2003] were written in sigma coordinates, but, here, the equations have been transformed to Cartesian coordinates in order to compare with CL and MR; the transformation process is detailed in *Mellor* [2005].

In Mellor, properties were expanded about mean surfaces, $\bar{z}$ which denotes phase-averaged (resting) material surfaces. Then instantaneous material surfaces are denoted by $z=\bar{z}+\tilde{s}$ where the vertical deviation, $\tilde{s}=a[\sinh k(\bar{z}+h)/\sinh kD]\cos\psi$; at $\bar{z}=\hat{\eta}$, the material surface elevation is $\tilde{s}=\tilde{\eta}=a\cos\psi$. Next define $\tilde{u}_{D\alpha}\equiv\tilde{u}_\alpha(z=\bar{z})$, Now see that, to lowest order, the $\tilde{u}_\alpha$ in (2a) can be written as $\tilde{u}(z,t)=\tilde{u}_{D\alpha}+(\partial\tilde{u}_{D\alpha}/\partial\bar{z})\tilde{s}$ (The subscript, D, is suggested by the definition, $D\equiv\hat{\eta}+h$, to denote the fact that $\hat{\eta}>\bar{z}>-h$.) Similar expansions apply to the vertical wave velocity and pressure

The phase-averaged continuity equation is

$$\frac{\partial U_\beta}{\partial x_\beta}+\frac{\partial W}{\partial z}=0 \tag{34}$$

In Appendix A and as an example of the derivation process, the continuity equation is derived in a way somewhat different from Mellor but nevertheless consistent with that paper.

In Mellor (2a, 2b, 2c) and in Appendix A, the term $u_{S\alpha}=\partial(\overline{\tilde{u}_{D\alpha}\tilde{s}})/\partial\bar{z}$ emerges which yields the same result for $u_{S\alpha}$ as in (5a). Furthermore, $u_{S\alpha}$ appears as an addition to the Eulerian current, $\hat{u}_\alpha$, so that $U_\alpha\equiv\hat{u}_\alpha+u_{S\alpha}$ and $W\equiv\hat{w}$ since by the same reckoning $w_S=\overline{\tilde{s}\partial\tilde{w}_D/\partial\bar{z}}=0$.

The phase-averaged momentum equation is

$$\frac{\partial U_\alpha}{\partial t}+\frac{\partial U_\alpha U_\beta}{\partial x_\beta}+\frac{\partial WU_\alpha}{\partial z}+\frac{\partial S_{\alpha\beta}}{\partial x_\beta}+F_{S\alpha}=-\frac{\partial}{\partial x_\alpha}(g\hat{\eta}+p_{atm})+\frac{\partial}{\partial z}(\tau_{T\alpha}+\tau_{P\alpha}) \tag{35a}$$

wherein the radiation stress term is

$$S_{\alpha\beta}=\overline{\tilde{u}_{D\alpha}\tilde{u}_{D\beta}}-\delta_{\alpha\beta}\overline{\tilde{w}_D^2}+\delta_{\alpha\beta}\left[\frac{\partial}{\partial\bar{z}}\left(\overline{\tilde{p}_D\tilde{s}}-g\overline{\tilde{s}^2}/2\right)\right]. \tag{35b}$$

The Kronecker delta $\delta_{\alpha\beta}=1$ if $\alpha=\beta$ and $=0$ otherwise. The system is closed after substituting (37c) (wherein $z\rightarrow\bar{z}$) into (35b).

Note that $F_{S\alpha}\equiv S_{\alpha\beta}D^{-1}(\partial D/\partial x_\beta)+\left[\partial\hat{\eta}/\partial x_\beta+D^{-1}(z-\hat{\eta})\partial D/\partial x_\beta\right]\partial S_{\alpha\beta}/\partial z$ whereas in deep water, $F_{S\alpha}\cong 0$. Albeit complicated, $F_{S\alpha}$ is necessary to obtain the simple integral equation of L-HS because

$\int_{-h}^{\hat{\eta}}(\partial S_{\alpha\beta}/\partial x_\beta)dz+F_{S\alpha}=\partial\left(\int_{-h}^{\hat{\eta}}S_{\alpha\beta}dz\right)/\partial x_\beta$ and also to convert back to sigma coordinates where the term is also simple.

The boundary conditions are

$$W=\frac{\partial\hat{\eta}}{\partial t}+U_\beta\frac{\partial\hat{\eta}}{\partial x_\beta}\text{ at }z=\hat{\eta}\text{ and }W=-U_\beta\frac{\partial h}{\partial x_\beta}\text{ at }z=-h \tag{36a, 36b}$$

Thus, equations (34) and (35a) are much like those without waves except for the addition of the radiation stress term and the fact that Stokes drift is included in the definition of U_α. In *Mellor* [2015], it is established that, whereas, $S_{\alpha\beta}=c^2O(ka)^2$ is retained, terms of order $(ka)^4$ are neglected.

In (35a), the terms, $\tau_{T\alpha}$ and $\tau_{P\alpha}$, represent vertical momentum transfer due to turbulence and pressure respectively; their relative roles are examined in *Mellor* [2005].

After U_α is determined and after wave properties are known, the Eulerian velocity may be obtained by simply subtracting Stokes drift obtained from (5a).

5. Critique

Waves at the surface are indeed wavy and material surfaces in the interior are also wavy such that $z=\bar{z}+\tilde{s}$, a seemingly trivial statement were it not for the fact that it is intrinsic to the analysis of Mellor but does not play a role in the analysis of section 2.

5.1. Fundamentals

Reverting to fundamentals, the Eulerian velocity is

$$\hat{\mathbf{u}}\equiv\frac{\hat{\mathbf{x}}(t+T_T)-\hat{\mathbf{x}}(t)}{T_T} \tag{37a}$$

whereas Stokes drift is

$$\mathbf{u}_S\equiv\frac{\tilde{\mathbf{x}}(t+T_P)-\tilde{\mathbf{x}}(t)}{T_P} \tag{37b}$$

where T_T is a time scale long relative to the characteristic turbulence scale and T_P is one or more wave periods; both $\hat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ are particle locations, but to describe the motion of *a particle of fixed identity*, we need the sum of $\hat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$. Therefore, set $T=T_T=T_p$ and $\mathbf{x}_p=\hat{\mathbf{x}}+\tilde{\mathbf{x}}$ so that

$$\hat{\mathbf{u}}+\mathbf{u}_S\equiv\frac{\mathbf{x}_p(t+T)-\mathbf{x}_p(t)}{T}. \tag{37c}$$

Equation is simple evidence that the Eulerian current and Stokes drift should be dynamically combined as in (34) and (35a). The equation will be useful in the *boundary condition* subsection below.

5.2. Continuity

In their Appendix A, MR derive their result that $\nabla\cdot\mathbf{u}^S=0$. I have tried to repeat the MR derivation in Appendix B, but do not obtain their nondivergent result. The determination in (4) is that $w_S=0$ so that, in general, $\partial u_{S\alpha}/\partial x_\alpha\neq0$ and $\partial\hat{u}_\alpha/\partial x_\alpha+\partial\hat{w}/\partial z\neq0$; nevertheless, $\partial(\hat{u}_\beta+u_{S\beta})/\partial x_\beta+\partial\hat{w}/\partial z=0$. Furthermore, a mean horizontal drift for a horizontally propagating wave is conceptually acceptable; a mean vertical drift is not acceptable. *McWilliams et al*. [2004] do recognize that $w_S=0$, but allow for a vertical "pseudo-Stokes drift" here denoted by w^S and defined by integrating $\partial w^S/\partial z=-\partial u_\beta^S/\partial x_\beta$. In view of (5b), I can assign no physical meaning to w^S.

In Phillips and Mellor, it is the *combination* of Eulerian current and Stokes drift that is nondivergent as in (34) and as derived in Appendix A.

5.3. Momentum

Temporarily neglecting the expansion of $\tilde{u}_\alpha$ about a mean state as in section 4, the horizontal component of the momentum flux in (33) can, after phase-averaging, be written $\overline{(\hat{u}_\alpha+\tilde{u}_\alpha)(\hat{u}_\beta+\tilde{u}_\beta)}=\hat{u}_\alpha\hat{u}_\beta+\overline{\tilde{u}_\alpha\tilde{u}_\beta}$ wherein $\overline{\tilde{u}_\alpha\tilde{u}_\beta}=\overline{\tilde{u}_{D\alpha}\tilde{u}_{D\beta}}$ [+ terms of order $(ka)^4$] is the first term on the right in (35b). It is similar to a turbulent fluctuation term. It is hard to see how a theory of wave-current interaction can avoid this term's inclusion in the momentum equation as in (24).

In Appendix B, the vertical component of the vector equality in (17) is shown to contain an error such that the vertical component of the vortex force in (19) to (21) also contains an error. However, due to the boundary layer approximation, the vertical component is negligibly small in (24).

5.4. Pressure

L-HS, Phillips and Mellor find that waves introduce a significant term in the pressure relation; such that $\bar{p}=p_{atm}+g(\hat{\eta}-\bar{z})-\overline{\tilde{w}_D^2}$ and $\partial\bar{p}/\partial x_\alpha=\partial(p_{atm}+g\hat{\eta})/\partial x_\alpha-\partial\overline{\tilde{w}_D^2}/\partial x_\alpha$. The addition of $\overline{\tilde{w}_D^2}$ is not subtle; it is found in either of two ways: integrate the vertical component of the differential momentum equation from the free surface to an underlying surface [*Mellor*, 2015] or apply the integral momentum equation to a control volume that envelopes the free surface and an underlying surface [*Mellor*, 2011]. Thus, $\overline{\tilde{w}_D^2}$ joins $\overline{\tilde{u}_{D\alpha}\tilde{u}_{D\beta}}$ and is the second term on the right in (35b). As shown in section 8, the ratio of the term, $\omega_z(u_{Sy},-u_{Sx})$ in (24) to $\partial\left(\overline{\tilde{u}_{D\alpha}\tilde{u}_{D\beta}}-\delta_{\alpha\beta}\overline{\tilde{w}_D^2}\right)/\partial x_\alpha$ in (35b) is of order $(ka)^2$ and can be neglected.

Pressure also contributes the last term in (35b) as derived in Mellor.

5.5. Boundary conditions

If is integrated from $z=-h$ to $z=\hat{\eta}$, one obtains

$$\frac{\partial}{\partial x_\beta}\int_{-h}^{\hat{\eta}} v_\beta dz-v_\beta(\hat{\eta})\frac{\partial\hat{\eta}}{\partial x_\beta}-v_\beta(-h)\frac{\partial h}{\partial x_\beta}+w(\hat{\eta})-w(-h)=0. \tag{38}$$

The boundary conditions cited in MR are

$$\text{at } z=\hat{\eta},\quad w=\nabla\cdot\mathbf{M}^S \text{ and at } \quad z=-h,\quad w=-\mathbf{v}\cdot\nabla h \tag{39a, 39b}$$

where $M_\alpha^S=\int_{-h}^{\hat{\eta}} u_\alpha^S dz$. Equations (39a) and (39b) are incorrect as indicated below. Inserting into (38) gives

$$\frac{\partial}{\partial x_\beta}\int_{-h}^{\hat{\eta}} v_\beta dz+\frac{\partial M_\beta^S}{\partial x_\beta}-v_\beta\frac{\partial\hat{\eta}}{\partial x_\beta}=0\,.$$

Of course, this result is incorrect. However, in a later applications paper by *Uchiyama et al*. [2010], (39a) is amended such that

$$w=\nabla\cdot\mathbf{M}^S+\frac{\partial\eta}{\partial t}+\mathbf{v}\cdot\nabla\eta \quad \text{at } z=\hat{\eta} \tag{39a, 39b}$$

which, to this writer, seems hard to justify. However, then the integrated continuity equation becomes

$$\frac{\partial}{\partial x_\beta}\int_{-h}^{\hat{\eta}}\left(v_\beta+u_\beta^S\right)dz+\frac{\partial\hat{\eta}}{\partial t}=0 \tag{40}$$

With a change in nomenclature and using (36), (40) may be properly obtained from (34) after vertical integration. Whereas (40) is correct (39a') is, as shown next, incorrect.

Refer again to (37c). Thus, at the bottom, $z=-h$, $\mathbf{x}_p(t+T)=\mathbf{x}_p(t)+\mathbf{i}\Delta x_p+\mathbf{j}\Delta y_p-\mathbf{m}\Delta h_p$ where **i**, **j**, **m** are unit vectors in the *x, y, z* directions. From (37c), $\hat{u}_\alpha+u_{S\alpha}=\Delta x_{p\alpha}/T$ and $\hat{w}=-\Delta h_p/T$ (since we insist that $w_S=0$). Therefore, at $z=-h$, $\hat{w}=-(\hat{u}_\alpha+u_{S\alpha})\partial h/\partial x_\alpha$ after letting $\Delta h_p/\Delta x_{p\alpha}\rightarrow\partial h/\partial x_\alpha$ as in (36b); thus, (39b) cannot be correct. The same reasoning can be applied at the surface to show that $\hat{w}-\partial\hat{\eta}/\partial t=(u_\alpha+u_{S\alpha})\partial\hat{\eta}/\partial x_\alpha$ at $z=\hat{\eta}$ so that (39a) or (39a') is incorrect.

5.6. Energy

Starting from (33) the wave energy equation can be derived as in Phillips where velocities are assumed to be vertically constant and *Mellor* [2003] for vertically resolved velocities. It is

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x_\alpha}\left[(c_{g\alpha}+\hat{u}_{A\alpha})E\right]+\int_{-h}^{\hat{\eta}} S_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}dz=S_{in}-S_{dis} \tag{41}$$

where $c_{g\alpha}$ is the group velocity, $\hat{u}_{A\alpha}$ is an advective velocity and S_{in} and S_{dis} are wind source and wave dissipation terms. I see no way that the energy equation can be obtained from the vortex force formulation of section 2. Furthermore, in *Mellor* [2005] the mean energy ($U_\alpha^2/2$) equation is derived from (35a) wherein the term $\int_{-h}^{\hat{\eta}} S_{\alpha\beta}(\partial U_\alpha/\partial x_\beta)dz$ also appears as a source/sink term and is transported to the same term in (41) as a sink/source term. This portion of a complete energy balance [*Mellor*, 2005] is missing in the vortex force formulation rendering an energy balance unachievable.

6. Langmuir Circulation

CL seek a stationary solution to (18) which produces vorticity in the stream direction and the cellular characteristic of Langmuir circulation (henceforth LC). Recall that the subscripts, 0 have been deleted; then let

$$\nabla\times(\mathbf{u}^S\times\boldsymbol{\omega})=\boldsymbol{\omega}\cdot\nabla\mathbf{u}^S-\mathbf{u}^S\cdot\nabla\boldsymbol{\omega}$$

and for the curl of the vortex force

$$\nabla\times(\mathbf{u}^S\times\boldsymbol{\omega})=\boldsymbol{\omega}\cdot\nabla\mathbf{u}^S-\mathbf{u}^S\cdot\nabla\boldsymbol{\omega}$$

which, however, assumes the contentious $\nabla\cdot\mathbf{u}^S=0$! Nevertheless, inserting these expressions into (18) yields

$$\nu\nabla^2\boldsymbol{\omega}=(\mathbf{v}+\mathbf{u}^S)\cdot\nabla\boldsymbol{\omega}-\boldsymbol{\omega}\cdot\nabla(\mathbf{v}+\mathbf{u}^S) \tag{42}$$

which is then written in a coordinate system (*x,y,z*) where *x* is aligned with $\mathbf{u}_S=(u_{Sx},\ 0,\ 0)$ and, since a characteristic of LC's is that the cells are long in one direction, all property variations in the *x*-direction are assumed to be nil. Thus,

$$\nu\nabla^2\omega_x+\omega_y\frac{\partial u_x^S}{\partial y}+\omega_z\frac{\partial u_x^S}{\partial z}=v_y\frac{\partial\omega_{0x}}{\partial y}+v_z\frac{\partial\omega_{0x}}{\partial z} \tag{43}$$

The system (43) is forced by u_x^S, but, in order to obtain cell structure with periodicity in the *y*-direction, CL posit two intersecting waves with equal amplitude where one wave progressives in the direction (k_x, k_y) and the other in the direction $(k_x, -k_y)$ where here k_y is a positive number. The potential function of such a wave field is

$$\begin{aligned}\varphi=\varphi_++\varphi_-&=ace^{kz}\left[\cos(k_xx+k_yy-\sigma t)+\cos(k_xx-k_yy-\sigma t)\right]\\&=2ace^{kz}\left[\cos(\psi_x)\cos(k_yy)\right],\quad \psi\equiv k_xx-\sigma t\end{aligned} \tag{44}$$

from which

$$[\tilde{u},\tilde{v},\tilde{w}]=2ace^{kz}\left[-k_x\sin\psi_x\cos(k_y\tilde{y}),\ -k_y\cos\psi_x\sin(k_y\tilde{y}),\ k\cos\psi_x\cos(k_y\tilde{y})\right]. \tag{45}$$

The associated Stokes drift from (4a) using (45) and requiring considerable manipulation is

$$u_x^S=2(ka)^2ce^{2kz}\left[1+(k_x/k)^2\cos(2k_yy)\right] \tag{46}$$

thereby obtaining the requisite periodicity in the *y*-direction. After extensive analysis, CL do indeed obtain ω_x from (43) with *y*-periodicity. Thereafter, $\omega_x=\nabla^2\psi(y,z)$ and with proper boundary conditions on $\psi(y,z)$, cell structure normal to the *x*-direction is obtained.

A simpler formulation is obtained by assuming that perturbations of v_y and v_z are small relative to the *y*-average of v_x, call it $U(z)$, so that $\omega_y=\partial U/\partial z$ and (37c) becomes $\nu\nabla^4\psi=-(\partial U/\partial z)(\partial u_x^S/\partial y)$ and the right side is known.

7. Critique

The above LC theory in (42)and (43) depends on separate continuity equations, $\nabla\cdot\mathbf{u}^S=0$ and $\nabla\cdot\mathbf{v}=0$ and the vortex force term in the vorticity or momentum equation. Therefore, for this reason (37c) and reasons stated in section 5, this LC theory is deemed to be incorrect.

Another direct approach is to refer back to (8) wherein one can obtain

$$\nabla\times(\mathbf{q}\times\boldsymbol{\omega})+\nu\nabla^2\boldsymbol{\omega}=0$$

or

$$\boldsymbol{\omega}\cdot\nabla\mathbf{q}-\mathbf{q}\cdot\nabla\boldsymbol{\omega}+\nu\nabla^2\boldsymbol{\omega}=0$$

where $\boldsymbol{\omega}\equiv\nabla\times\mathbf{q}$. For the x-component of vorticity and assuming all properties do not vary in the x-direction as in section 6, one finds that the first term on the left disappears so that

$$\mathbf{q}\cdot\nabla\omega_x-\nu\nabla^2\omega_x=0$$

In other words, the system only decays and cannot produce cells. If $\mathbf{q}=\mathbf{i}q_x$, the solution is trivial. Cells can be produced by adding buoyancy gradients or curvature terms [*Taylor*, 1923; *Gortler*, 1954] or other body force terms.

Since the discovery of cellular phenomena in the ocean by *Langmuir* [1938], various attempts have been made to provide an underlying theory. For example, *Garrett* [1976] described a feedback mechanism whereby a disturbance or nascent jet in the x-component of velocity is amplified. Wave theory does dictate that waves encountering the jet will amplify and therefore might break and create additional Eulerian current. However, *Craik* [1977] states that "several aspects of his analysis are unacceptable" and proceeds to alternative analyses based on the vortex force as does *Leibovich* [1980].

I offer a suggestion: theory and experiment show that flow on concave walls can form cells in laminar flow and in turbulent flow [*So and Mellor*, 1976]. Waves have curvature, but the curvature alternates between convex stabilizing flow and concave destabilizing cellular flow. Does the concave portion dominate? Until more experimental or theoretical foundation is provided, the suggestion remains only one of many.

8. Why do the Two Formulations Differ?

The derivation of CL and MR in section 2 begins with the curl of (8) whereby the gradient of pressure and all of (29) drop out of contention. Upon uncurling, $\nabla\Phi$ where Φ is any scalar, appears but the portion of $\nabla\Phi=\nabla(\mathbf{u}^w\cdot\mathbf{u}^w/2)=\nabla\cdot(\mathbf{u}^w\mathbf{u}^w)$, as in the wave portion of $\nabla(\mathbf{q}\cdot\mathbf{q}/2)$ in (8), is not reinstated. Adding (29) and $\nabla\cdot(\mathbf{u}^w\mathbf{u}^w)$ to (19) comes close to (33) in section 3. However, the vortex force term does not appear in (33). The vortex term is somehow buried in the complexity of section 2 and its dependent appendix B. Contrast this with the relative simplicity of section 3.

It is concluded that the vortex term does not exist. However, if one assumes that it does exist, it is possible to gage the contribution of $\mathbf{u^S}\times\boldsymbol{\omega}_0$ or $\omega_z(u_{Sy},-u_{Sx})$. Suppose the term were simply added to (35a). The vortex force term is of the order $(\hat{u}/L)u_S$ where L is the length scale of mean spatial variations. Next $\hat{u}$ is of order $u_S=$ order $c(ka)^2$ [in fact, in some cases there is evidence that they very nearly cancel [*Ursell*, 1950, *Monismith et al*., 2007, *Smith*, 2006b] so that the vortex force term is of order $c^2(ka)^4/L$. However, quadratic terms in (35a, b) are order $c^2(ka)^2/L$. Thus, the ratio of the vortex force to the wave radiation stress term is of order $(ka)^2$ ($\approx 10^{-2}$) so that the vortex force term can be discarded.

9. Summary Discussion

This paper contrasts the vertically resolved equations of CL and MR with that of Mellor which, upon vertical integration, agree with the equations of L-HS, *Phillips* and *Smith* [2006a].

The continuity equation derived by MR involving only the Eulerian current disagrees with the results of L-HS, Phillips, Mellor and the simple derivation in Appendix A where the combined current and Stokes drift is required for volume conservation (or mass conservation after insertion of constant density).

The surface boundary conditions of MR and amended in *Uchiyama et al*. [2010] includes a strange term in (39a′) but otherwise involves only the current as does the bottom boundary condition. Instead, direct recourse to equation requires the combination of current and Stokes drift. In this paper and those of L-HS, Phillips and Mellor, it is recognized that the vertical component of Stokes drift is nil and avoids introduction of a vertical "pseudo Stokes drift."

The MR version of the momentum equation is a prognostic equation for the current alone. It also substitutes the vortex force term in place of the wave radiation stress term and therefore neglects quadratic wave momentum; the latter is elementary according to the reasoning of L-HS, Phillips and Mellor.

Although the reasoning is slightly more complex, pressure is maltreated by MR according to this paper.

It is shown that the vortex force formulation is incompatible with the established wave energy equation.

The papers by L-HS, Phillips and Mellor consider the flow field as a simple combination of currents and waves. The difference in the CL and MR results is due to the fact that terms drop out after taking the curl of the basic primitive equations; the terms are not restored when the "uncurl" process is undertaken as discussed in section 8. Alternately, the scaling of CL and MR in (9) or (27) also reproduces the initial equations of L-HS, Phillips and Mellor when applied directly to the primitive equations of motion in which case the vortex force term does not appear. Even if were assumed that the vortex force term did exist, the quadratic terms are considerably larger than the vortex force term which is of an order that is generally neglected in most wave theories.

Appendix A: The Continuity Equation

The continuity equation can be derived by considering the flow on the faces of an elemental control volume $\Delta x \Delta y \Delta z$ where $\Delta z = \Delta \bar{z}(1+\partial\tilde{s}/\partial\bar{z})$ such that

$$\delta_x\left[\overline{\left(\hat{u}_x+\tilde{u}_{Dx}+\frac{\partial \tilde{u}_{Dx}}{\partial \bar{z}}\tilde{s}\right)\left(1+\frac{\partial \tilde{s}}{\partial \bar{z}}\right)\Delta\bar{z}}\right]\Delta y+\delta_y\left[\overline{\left(\hat{u}_y+\tilde{u}_{Dy}+\frac{\partial \tilde{u}_{Dy}}{\partial \bar{z}}\tilde{s}\right)\left(1+\frac{\partial \tilde{s}}{\partial \bar{z}}\right)\Delta\bar{z}}\right]\Delta x$$

$$+\delta_z\left[\overline{\left(\hat{w}+\tilde{w}_D+\frac{\partial \tilde{w}_D}{\partial \bar{z}}\tilde{s}\right)}\right]\Delta x\Delta y=0$$

where $\delta_x f = f(x+\Delta x)-f(x)$ and similar expressions apply to δ_y and δ_z.Working out the phase-averaging and noting that $\overline{\tilde{w}_D+(\partial\tilde{w}_D/\partial\bar{z})\tilde{s}}=0$, one obtains

$$\delta_x\left[\hat{u}_x+\frac{\partial(\overline{u_{Dx}\tilde{s}})}{\partial\bar{z}}\right]\Delta\bar{z}\Delta y+\delta_y\left[\hat{u}_y+\frac{\partial(\overline{u_{Dy}\tilde{s}})}{\partial\bar{z}}\right]\Delta\bar{z}\Delta x+\delta_z(\hat{w})\Delta x\Delta y=0$$

Now divide by $\Delta x\Delta y\Delta\bar{z}$ and let $\delta_x(\quad)/\Delta x \rightarrow \partial(\quad)/\partial x$, etc. and obtain

$$\frac{\partial}{\partial x_\beta}\left(\hat{u}_\beta+\frac{\partial(\overline{u_{D\beta}\tilde{s}})}{\partial\bar{z}}\right)+\frac{\partial\hat{w}}{\partial\bar{z}}=0$$

Using (2), it can be shown that $u_{S\beta}=\partial(\overline{\tilde{u}_{D\beta}\tilde{s}})/\partial\bar{z}$ so that (34) is obtained after recalling the definitions, $U_\beta \equiv \hat{u}_\beta+u_{S\beta}$ and $W\equiv\hat{w}$.

Derivation of the momentum equation is similar but more complicated for which reference is made to *Mellor* [2015].

Appendix B: Review and Critique of a Nondivergence Stokes Drift

The following discussion will be restricted to deep water. In their Appendix A, MR purport to show that $\nabla\cdot\mathbf{u}^S=0$, or

$$\frac{\partial}{\partial x_m}\overline{\left(U_j\frac{\partial u^w_m}{\partial x_j}\right)}=0 \qquad \text{(B1)}$$

where we used indices, *i*, *j*, *m* (reserving *k* for wave number). Recalling that $U_i=\int u^w_i dt$, MR write

$$\overline{\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x_m}\left(U_j\frac{\partial U_m}{\partial x_j}\right)\right]}=\overline{\frac{\partial}{\partial x_m}\left(u^w_j\frac{\partial U_m}{\partial x_j}+U_j\frac{\partial u^w_m}{\partial x_j}\right)}=\frac{\partial}{\partial x_m}\left(\overline{u^w_j\frac{\partial U_m}{\partial x_j}}+\overline{U_j\frac{\partial u^w_m}{\partial x_j}}\right). \qquad \text{(B2)}$$

The left side is nil [which, in MR, appears to have an indexing error but is corrected here as suggested by (B1) and as necessary to match the right side of (B2)]. Since

$$u^w_i=ace^{kz}(k_x\cos\psi, k_y\cos\psi, k\sin\psi)$$

one obtains

$$U_i=ak^{-1}e^{kz}(-k_x\sin\psi, -k_y\sin\psi, k\cos\psi).$$

Working out the algebra, it is then found that

$$\overline{u_j^w \frac{\partial U_m}{\partial x_j}} = ka^2 c e^{2kz}(-k_x, -k_y, 0) \text{ and } U_j \frac{\overline{\partial u_m^w}}{\partial x_j} = ka^2 c e^{2kz}(+k_x, +k_y, 0). \quad \text{(B3)}$$

Inserting into (B2), one obtains 0 = 0, a sensible result but a result that does not support the MR assertion of (B1).

However, as noted in section 4, a later paper by *McWilliams et al*. [2004] agrees that $w_S=0$ but introduces a vertical "pseudo-Stokes drift" which is here labeled w^S and obtained from $w^S = -\partial\left(\int_{-h}^{z} u_\beta^S dz'\right)/\partial x_\beta$.

B1. Derivation of Equation (17)

A key step in section 3 is equation (17). First, to determine $\boldsymbol{\omega}_1$ appeal to a standard identity [*Hildebrand*, 1976] and write (15) as

$$\boldsymbol{\omega}_1 = \nabla \times (\mathbf{U} \times \boldsymbol{\omega}_0) = \boldsymbol{\omega}_0 \cdot \nabla \mathbf{U} - \mathbf{U} \cdot \nabla \boldsymbol{\omega}_0 + \mathbf{U}(\nabla \cdot \boldsymbol{\omega}_0) - \boldsymbol{\omega}_0(\nabla \cdot \mathbf{U}).$$

Now the divergences of $\mathbf{U}$ and $\boldsymbol{\omega}_0$ are nil. Furthermore, the ratio of the second term on the right relative to the first term is of order $(kL)^{-1}$ where L is the spatial scale of $\boldsymbol{\omega}_0$ and so the second term can be neglected. Thus,

$$\boldsymbol{\omega}_1 = \boldsymbol{\omega}_0 \cdot \nabla \mathbf{U}. \quad \text{(B4)}$$

Working out the vector algebra, it is found that

$$\boldsymbol{\omega}_1 = ak^{-1}e^{kz} \left\{ \begin{array}{l} \mathbf{i}\left[\cos\psi\left(-k_x^2\omega_{0x} - k_x k_y \omega_{0y}\right) - \sin\psi(kk_x\omega_{0z})\right] \\ +\mathbf{j}\left[\cos\psi\left(-k_y^2\omega_{0y} - k_x k_y \omega_{0x}\right) - \sin\psi(kk_y\omega_{0z})\right] \\ +\mathbf{m}\left[\cos\psi\left(k^2\omega_{0z}\right) - \sin\psi\left(kk_x\omega_{0x} + kk_y\omega_{0y}\right)\right] \end{array} \right\} \quad \text{(B5)}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{m}$ are unit vectors in the *x, y, z* directions. Since

$$\mathbf{u}^w = ace^{kz}\left(\mathbf{i}k_x \cos\psi + \mathbf{j}k_y \cos\psi + \mathbf{m}k \sin\psi\right). \quad \text{(B6)}$$

one obtains

$$\overline{\mathbf{u}^w \times \boldsymbol{\omega}_1} = ka^2 c e^{2kz}\left[\mathbf{i}(k_y\omega_{0z}) - \mathbf{j}(k_x\omega_{0z})\right] \quad \text{(B7)}$$

where many terms are nil since $\overline{\sin\psi\cos\psi} = 0$. It is noteworthy that the vertical component is nil. On the other hand

$$\mathbf{u}^S = ka^2 c e^{2kz}\left(\mathbf{i}k_x + \mathbf{j}k_y\right) + \mathbf{m}w^S.$$

However, as defined above, w^S is order $(kL)^{-1}$ relative to the other stokes drift components and can be neglected. Since $\boldsymbol{\omega}_0 = \mathbf{i}\omega_{0x} + \mathbf{j}\omega_{0y} + \mathbf{m}\omega_{0z}$ one obtains

$$\mathbf{u}^S \times \boldsymbol{\omega}_0 = ka^2 c e^{2kz}\left[\mathbf{i}(k_y\omega_{0z}) - \mathbf{j}(k_x\omega_{0z}) + \mathbf{m}(k_x\omega_{0y} - k_y\omega_{0x})\right]. \quad \text{(B8)}$$

Comparing (B7) and (B8), we find that

$$\overline{\mathbf{u}^w \times \boldsymbol{\omega}_1} = \mathbf{u}_S \times \boldsymbol{\omega}_0 - \mathbf{m}ka^2 c e^{2kz}(k_x\omega_{0y} - k_y\omega_{0x}) \quad \text{(B9)}$$

Agreement is obtained with (17) only if the vertical component of (B8) or (B9) is ignored.

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