The Effects of a Jet-Like Current on Gravity Waves in Shallow Water

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ABSTRACT

The effects of a strong current with horizontal shear on shallow water waves is studied. For jet-like currents, the existence of trapped waves and the reciprocity of scattering coefficients are pointed out. Detailed consequences of current dimensions and intensity on waves are analyzed for a top-hat current.

1. Introduction

The dynamical effects of a strong current on surface waves are of interest in the refraction of swell by major oceanic currents such as the Gulf Stream or the Antarctic Circumpolar Current (Kenyon, 1971). In these cases the wavelength is usually very short compared to the sea depth and to the length scale of currents; the WKB or geometrical optics approximation for slowly varying media is applicable and has produced many useful results. Comprehensive reviews of theories of this type and their applications have been made by Peregrine (1976) and more recently by Peregrine and Jonsson (1983). In the case of an abruptly varying current and arbitrary water depth, theoretical solutions are much more difficult. Peregrine and Smith (1975) have considered the three-dimensional problem of time-independent waves in a jet-like current and gave explicit results for a top-hat current bounded at the sides and below by vortex sheets. Evans (1975) has studied the scattering of deep water waves across two regions of constant but different current velocities separated by a vortex sheet. In principle, existing numerical methods via integral equations or finite elements, for analyzing the scattering of gravity waves by abrupt changes of depth, can be adapted for the scattering by a jet-like stream or a large eddy whose horizontal dimension is comparable to the wavelength. However, we are unaware of such explicit computations in the published literature.

As in the similar case of scattering by topography, much physical insight can be gained from the mathematically simpler problem of long waves in shallow water. While the analysis is straightforward and is well-known in related theories for varying depth, the physics are quite different and do not seem to have been sufficiently explored. In reality, strong currents can also occur in shallow coastal regions where river discharges or tidal flows are strong. For example, the mean discharge velocity of the Connecticut River at ebb tide can be 0.5 m s^{-1} at the mouth where the depth is on the order of 2 m. The resulting jet is roughly 100 m wide and 20 km long (Garvine, 1974). At the mouth of Ishikari River in Hokaido, Japan, the depth is roughly 5 m and width is 500 m; the river jet has a mean discharge velocity as high as 1 m s^{-1}, extending several kilometers offshore (Kashiwamura and Yoshida, 1978). In these cases, the river water is fresh and stratification is also an important feature.

In this note we concentrate on the detailed effects of jet-like currents on the propagation of shallow water waves in a homogeneous sea. The current will be assumed to be steady and parallel and its velocity, to vary only transversely. The governing equation for the wave motion is ultimately reduced to a second-order ordinary differential equation with variable coefficients. For a top-hat current and constant depth, quantitative effects of current width and strength on trapping and scattering coefficients will be examined. For general current profiles and colinear depth variation, we employ known arguments in quantum mechanics and deduce some reciprocity relations and the necessary condition for wave trapping.

2. Linearized governing equations

We assume that the horizontal length scales of the waves, the current and the depth variations are much greater than the water depth. The classical equations of Airy then apply

$$\left.\begin{aligned} \frac{\partial \mathbf{Q}}{\partial t} + \mathbf{Q}\cdot\nabla\mathbf{Q} &= -g\nabla Z \\ \frac{\partial Z}{\partial t} + \nabla\cdot[\mathbf{Q}(Z+h)] &= 0 \end{aligned}\right\}, \tag{2.1}$$

where $\mathbf{Q}$ refers to the horizontal components of the depth-averaged velocity, Z the free surface displacement, $\mathbf{x} = (x, y)$ the horizontal coordinates and t, time.

Consider the special case where the depth variation is one-dimensional [$h = h(x)$] and the current is steady and jet-like [$U = 0$, $V = V(x)$]. Then, from the time-independent version of (2.1), the corresponding Z must be constant everywhere and may be taken to be zero.

Let $\mathbf{u}(\mathbf{x}, t)$ and $\zeta(\mathbf{x}, t)$ be the velocity and free-surface height respectively of a superimposed wave field, i.e.,

$$\mathbf{Q} = [0, V(x)] + \mathbf{u}, \quad z = \zeta. \tag{2.2}$$

Assuming small amplitudes, the linearized equations governing the disturbance must be

$$\frac{\partial \mathbf{u}}{\partial t} + V\frac{\partial \mathbf{u}}{\partial y} + u\frac{\partial V}{\partial x}\mathbf{e}_y = -g\nabla\zeta, \tag{2.3a}$$

$$\frac{\partial \zeta}{\partial t} + V\frac{\partial \zeta}{\partial y} + \frac{\partial uh}{\partial x} + h\frac{\partial v}{\partial y} = 0. \tag{2.3b}$$

We further assume simple harmonic dependence on y and t, i.e.,

$$(u, v, \zeta) \rightarrow [u(x), v(x), \zeta(x)] \exp(-i\beta y - i\omega t). \tag{2.4}$$

It then follows from the momentum equation that

$$\left.\begin{aligned} u &= -\frac{ig\zeta'}{(\omega + \beta V)} \\ v &= -\left[\frac{\beta g \zeta}{\omega + \beta V} + \frac{g\zeta' V'}{(\omega + \beta V)^2}\right] \end{aligned}\right\}. \tag{2.5}$$

Substituting these into the continuity equation (2.3b), we get

$$\zeta'' + \left[\frac{h'}{h} - \frac{2\beta V'}{\omega + \beta V}\right]\zeta' + \left[\frac{(\omega + \beta V)^2}{gh} - \beta^2\right]\zeta = 0. \tag{2.6}$$

Since (2.6) may be reduced to the one-dimensional Schrödinger equation, knowledge in quantum mechanics can, therefore, be transferred to the problem at hand.

We now treat a special case of a shear current with a top-hat profile. Since the depth effect is relatively well-known, we shall take h = constant. Let the current distribution be

$$V(x) = \begin{cases} V = \text{const} > 0, & |x| < a \\ 0, & |x| > a, \end{cases} \tag{2.7}$$

then (2.6) becomes

$$\left.\begin{aligned} &\zeta'' + \left(\frac{\omega^2}{gh} - \beta^2\right)\zeta = 0, \quad |x| > a \\ &\zeta'' + \left[\frac{(\omega + \beta V)^2}{gh} - \beta^2\right]\zeta = 0, \quad |x| < a \end{aligned}\right\}. \tag{2.8}$$

We always assume the current to be in the positive y-direction, $V > 0$. Depending now on the magnitude of ω and β, a variety of physical situations arise.

3. Trapped modes in a top-hat current

If

$$\frac{(\omega + \beta V)^2}{gh} > \beta^2 > \frac{\omega^2}{gh}, \tag{3.1}$$

the free surface is exponential outside but wavelike within the current:

$$\left.\begin{aligned} \zeta &= Ae^{\gamma_1(x+a)}, & x &< -a \\ &= Be^{i\alpha_2 x} + Ce^{-i\alpha_2 x}, & |x| &< a \\ &= De^{-\gamma_1(x-a)}, & x &> a \end{aligned}\right\} \tag{3.2}$$

with

$$\gamma_1{}^2 = \beta^2 - \frac{\omega^2}{gh}, \quad \alpha_2{}^2 = \frac{(\omega + \beta V)^2}{gh} - \beta^2.$$

Eq. (3.2) clearly represents a wave trapped within the current. The subscripts 1 and 2 designate regions outside and within the current, respectively. By requiring continuity of ζ and $\partial\zeta/\partial x$ at $x = -a$, we get

$$A = Be^{-i\alpha_2 a} + Ce^{i\alpha_2 a}, \tag{3.3}$$

$$\gamma_1 A = i\alpha_2(Be^{-i\alpha_2 a} - Ce^{i\alpha_2 a}), \tag{3.4}$$

which may be combined, giving

$$0 = (\gamma_1 - i\alpha_2)Be^{-i\alpha_2 a} + (\gamma_1 + i\alpha_2)Ce^{i\alpha_2 a}. \tag{3.5}$$

Similar requirements at $x = a$ lead to

$$D = Be^{i\alpha_2 a} + Ce^{-i\alpha_2 a}, \tag{3.6}$$

$$-\gamma_1 D = i\alpha_2[Be^{i\alpha_2 a} - Ce^{-i\alpha_2 a}], \tag{3.7}$$

$$0 = (\gamma_1 + i\alpha_2)Be^{i\alpha_2 a} + (\gamma_1 - i\alpha_2)Ce^{-i\alpha_2 a}. \tag{3.8}$$

From (3.5) and (3.8), an eigenvalue condition for ω is obtained

$$\tan 2\alpha_2 a = \frac{2\gamma_1\alpha_2}{\alpha_2{}^2 - \gamma_1{}^2}. \tag{3.9}$$

Let

$$\frac{\gamma_1}{\alpha_2} = \tan\delta. \tag{3.10}$$

From (3.9) we get

$$\tan 2\alpha_2 a = \frac{2\tan\delta}{1 - \tan^2\delta} = \tan 2\delta.$$

Hence,

$$\delta = n\pi/2 + \alpha_2 a. \tag{3.11}$$

Taking the tangent of both sides of (3.11) we get

$$\left.\begin{aligned} \tan\alpha_2 a \\ -\cot\alpha_2 a \end{aligned}\right\} = \frac{\gamma_1}{\alpha_2}\begin{cases} n = \text{even} \\ n = \text{odd}. \end{cases} \tag{3.12}$$

To examine the eigenvalues, it is convenient to adopt the following dimensionless parameters

$$K = \frac{\omega}{\beta\sqrt{gh}} = \frac{k}{\beta}, \quad F^2 = \frac{V^2}{gh}, \tag{3.13}$$

where $k = \omega/\sqrt{gh}$, then (3.12) can be expressed as

$$\left.\begin{matrix}\tan\{[(K+F)^2-1]^{1/2}\beta a\} \\ -\cot\{[(K+F)^2-1]^{1/2}\beta a\}\end{matrix}\right\} = \frac{(1-K^2)^{1/2}}{[(K+F)^2-1]^{1/2}}. \quad (3.14)$$

In view of (3.1), K must fall within the range

$$1 - F < K < 1, \quad (3.15)$$

as shown in Fig. 1. Introducing

$$\xi = [(K+F)^2 - 1]^{1/2},$$

i.e.,

$$K = (\xi^2 + 1)^{1/2} - F, \quad (3.16)$$

then (3.15) becomes

$$0 < \xi < (F^2 + 2F)^{1/2}, \quad (3.17)$$

while (3.14) becomes

$$\left.\begin{matrix}\tan\xi\beta a \\ -\cot\xi\beta a\end{matrix}\right\} = \xi^{-1}\{1 - [(\xi^2+1)^{1/2} - F]^2\}^{1/2}. \quad (3.18)$$

The roots may be examined graphically by plotting both sides of the preceding equation as illustrated in Fig. 2 for $F = 0.5$ and $\beta a = 2\pi$. Since as ξ varies from 0 to $(F^2 + 2F)^{1/2}$, the right-hand side varies from $+\infty$ to 0 monotonically, the first even mode, to be labeled by $n = 0$, exists for any $F > 0$. As $\beta a(F^2 + 2F)^{1/2}$ passes the threshold $m\pi$ ($m = 1, 2, 3, \cdots$) there are $m + 1$ even modes. These modes will be labeled in ascending order as $\xi_0, \xi_2, \ldots, \xi_{2m}$; the corresponding eigenvalues are of $k_0, k_2, \ldots, k_{2m}$. Similarly, as $\beta a(F^2 + 2F)^{1/2}$ passes the threshold $(m - 1/2)\pi$, there are m odd modes $\xi_1, \xi_3 \cdots \xi_{2m-1}$ with the corresponding eigenvalues $k_1, k_3 \cdots k_{2m-1}$. The number of modes increases as βa or F increases. The eigenvalues are listed in Table 1 for $\beta a = \pi$ and 2π.

To have some idea of the physical values, we take $h = 5$ m, $g = 9.82$ m s^{-2}, then $\sqrt{gh} = 7.0$ m s^{-1}. Consider the lowest mode $n = 0$ trapped in a current of half-width $a = 100$ m, then the period $T = 15.6$ s with $kh = 0.288$ if $F = 0.1$ and $T = 19.8$ s with $kh = 0.226$ if $F = 0.3$, both for $\beta a = 2\pi$.

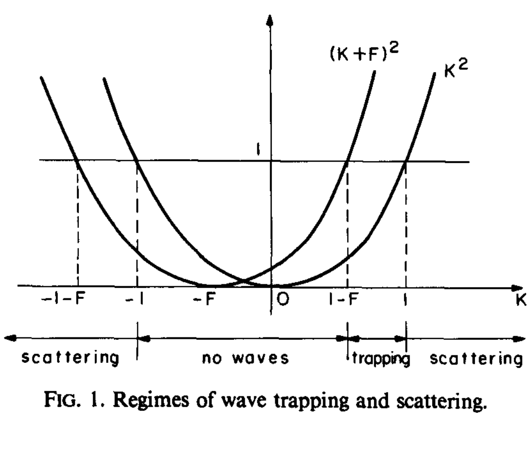

FIG. 1. Regimes of wave trapping and scattering.

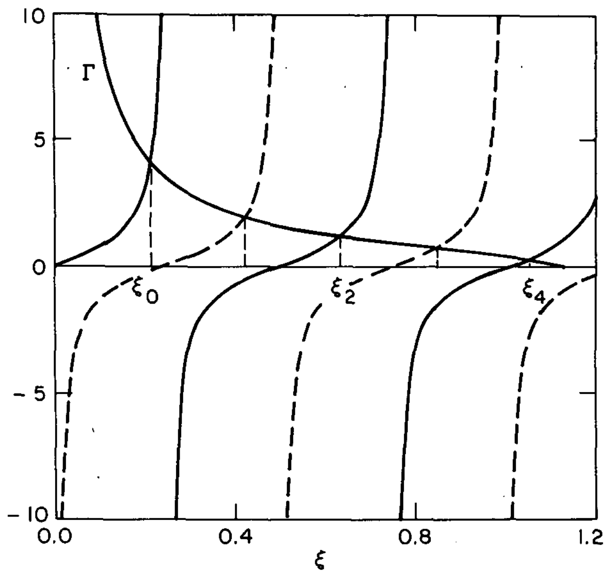

FIG. 2. Graphic solution of eigenvalues ξ_n for $F = 0.5$ and $\beta a = 2\pi$. The right-hand side of (3.18) is represented by curve Γ, while $\tan\xi\beta a$ and $-\cot\xi\beta a$ are represented respectively by the solid and dashed lines.

To examine the free surface of these eigenmodes, we first get from (3.5), after using (3.13) and (3.14)

$$\frac{B}{C} = e^{-in\pi}.$$

It then follows that for n even, $B = C$ and $A = D$, so that

$$\zeta = 2B\cos\alpha_2 x = 2B\cos\{[(K+F)^2 - 1]^{1/2}\beta x\}$$
$$= 2B\cos\xi_n\beta x. \quad (3.19)$$

Thus, the free surface is even in x. Since $0 < \xi_n\beta a < \pi/2$, the eigenmode $n = 0$ has no node within the width of the current. If $\pi < \xi_n\beta a < 3\pi/2$ the eigenmode $n = 2$ has two nodes symmetrically located within $|x| < a$, etc. Similarly for n = odd, $B = -C$ and $A = -D$; the free surface is odd in x

TABLE 1. Eigenvalues ξ_n of the trapped modes.

	F					
	$\beta a = \pi$			$\beta a = 2\pi$		
n	0.5	0.3	0.1	0.5	0.3	0.1
0	0.2113	0.2044	0.1820	0.3668	0.3451	0.2817
1	0.4234	0.4080	0.3551	0.7355	0.6768	
2	0.6358	0.6083		1.0738		
3	0.8464	0.7927				
4	1.0470					

$$\zeta = 2iB \sin\alpha_2 x = 2iB \sin\{[(K+F)^2 - 1]^{1/2}\beta x\}$$
$$= 2i\beta \sin\xi_n \beta x. \tag{3.20}$$

If $\pi/2 < \xi_n\beta a < \pi$, $n = 1$, there is one node at the center $x = 0$. If $3\pi/2 < \xi_n\beta a < 2\pi$, $n = 3$, there are 3 nodes within $|x| < a$, etc. Thus, for the nth mode there are n nodes. These results can also be established qualitatively for a continuous $V(x)$ by invoking the oscillation theorem in ordinary differential equations.

4. Scattering of waves by a top-hat current

As is evident from (2.8), wave solutions are possible outside the current if $K^2 > 1$ as marked in Fig. 1. If $\beta > 0$, the incidence angle is $0 < \theta < 90°$ where

$$\theta = \sin^{-1}(\beta/k) = \sin^{-1}(1/K), \tag{4.1}$$

i.e., waves are propagating against the current. Then $(K + F)^2 - 1 > 0$ as well, so that the free surface within the current is also wavelike. However, if $\beta < 0$, $(K < -1)$, the incidence angle is $-90° < \theta < 0$, i.e., the waves propagate with the current. The free surface within $|x| < a$ is then wavelike if $K < -1 - F$ and monotonic if $-1 - F < K < -1$ (see Fig. 1). In both cases, the free surface is wavelike outside $|x| > a$.

a. Wavelike response everywhere ($K > 1$ and $K < -1 - F$)

In these regimes of K, we have, from definitions,

$$\alpha_1^2 = \frac{\omega^2}{gh} - \beta^2 > 0,$$
$$\alpha_2^2 = \frac{(\omega + \beta V)^2}{gh} - \beta^2 > 0. \tag{4.2}$$

The solution with an incident wave from the left can be expressed as

$$\left.\begin{aligned}\zeta &= e^{i\alpha_1(x+a)} + Re^{-i\alpha_1(x+a)}, && x < -a\\ &= Ae^{i\alpha_2 x} + Be^{-i\alpha_2 x}, && |x| < a\\ &= Te^{i\alpha_1(x-a)}, && x > a\end{aligned}\right\}, \tag{4.3}$$

where R and T are the reflection and transmission coefficients respectively. Matching at $x = \pm a$ then yields four equations for the unknowns T, R, A and B. We give only the results

$$\left.\begin{aligned}T &= \frac{4b}{(1+b)^2 e^{-2i\alpha_2 a} - (1-b)^2 e^{2i\alpha_2 a}}\\ R &= \frac{-(1-b^2)[e^{-2i\alpha_2 a} - e^{2i\alpha_2 a}]}{(1+b)^2 e^{-2i\alpha_2 a} - (1-b)^2 e^{2i\alpha_2 a}}\end{aligned}\right\}, \tag{4.4}$$

where

$$b = \alpha_1/\alpha_2. \tag{4.5}$$

The squared magnitudes are

$$\begin{pmatrix}|T|^2\\ |R|^2\end{pmatrix} = \begin{pmatrix}4b^2\\ (1-b^2)^2 \sin^2 2\alpha_2 a\end{pmatrix} \times [4b^2 + (1-b^2)^2 \sin^2 2\alpha_2 a]^{-1}. \tag{4.6}$$

It is easy to check that $|T|^2 + |R|^2 = 1$, in accordance with a general result (5.11) to be deduced later. In terms of the normalized variables we have

$$b^2 = \frac{K^2 - 1}{(K+F)^2 - 1}, \tag{4.7a}$$

$$\alpha_2 a = [(K+F)^2 - 1]^{1/2} \frac{ka}{K}. \tag{4.7b}$$

For a given incidence angle (i.e., K) and F, the coefficients may be calculated as functions of ka. Sample results for $\theta > 0$ are shown in Figs. 3a–c for $F = 0.1$, 0.3 and 0.5. As is intuitively expected, stronger current

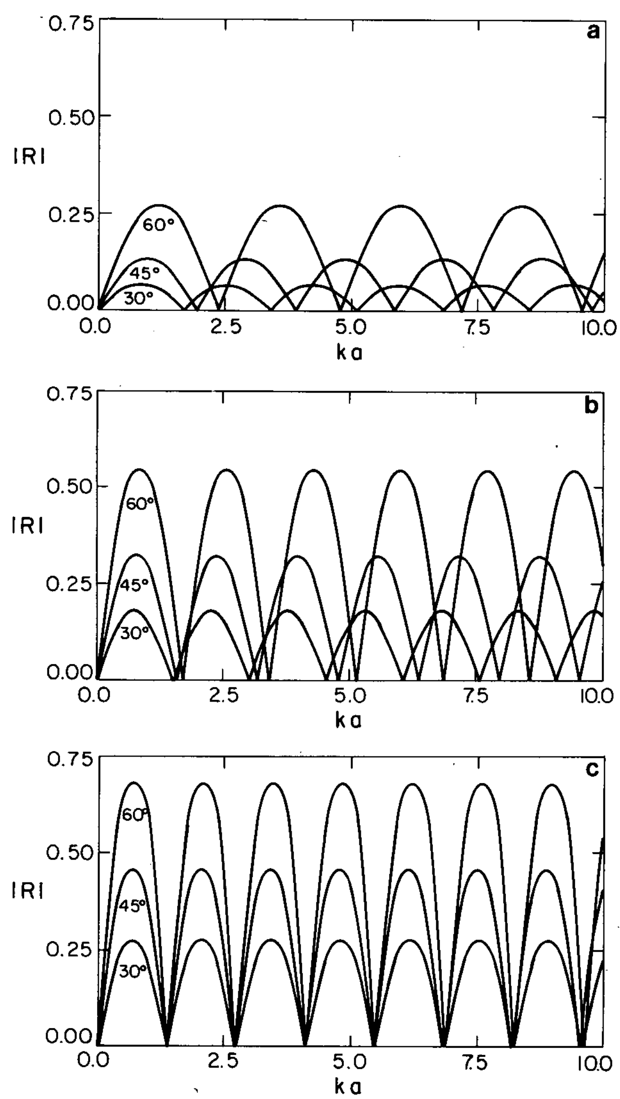

FIG. 3. Reflection coefficient $|R|$ as a function of ka for various angles of incidence $\theta > 0°$. The Froude number F is (a) 0.1, (b) 0.3 and (c) 0.5.

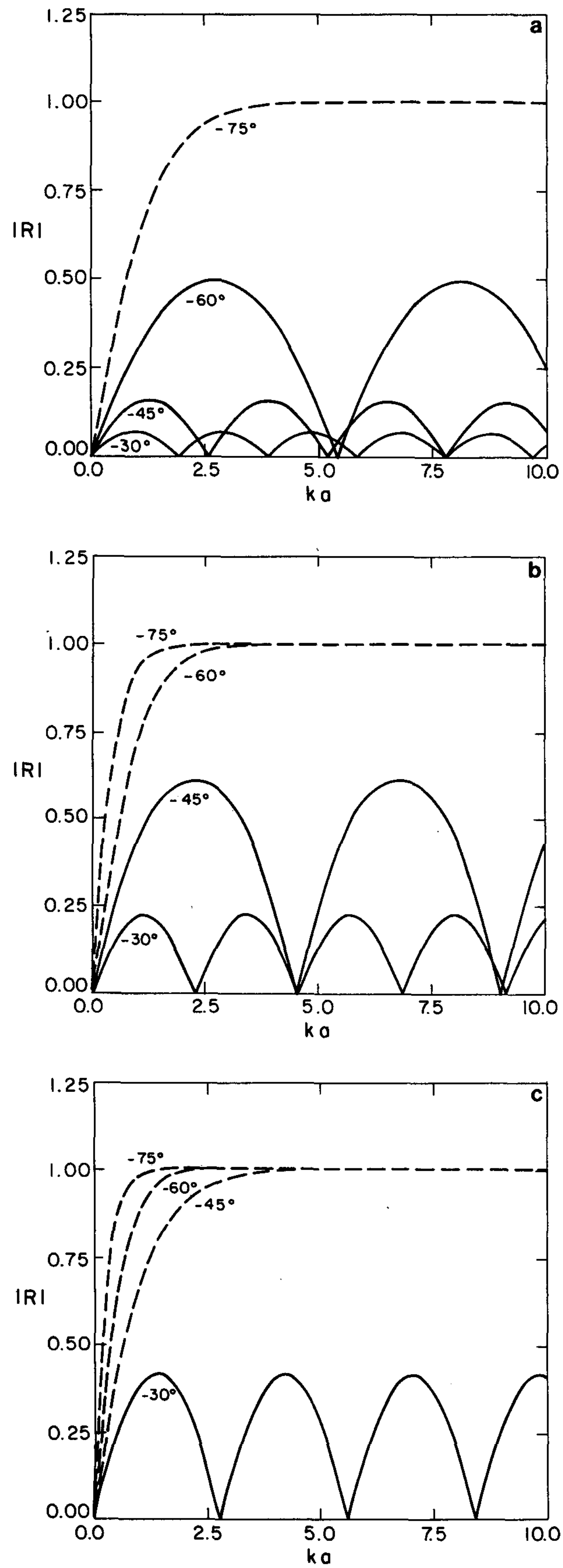

FIG. 4. Reflection coefficient $|R|$ as a function of ka for various angle of incidence $\theta < 0°$. The Froude number F is (a) 0.1, (b) 0.3 and (c) 0.5.

(larger F) causes greater reflection and less transmission.

Note that for near glancing incidence $\theta \uparrow 90°$, $K \downarrow 1$, then $b^2 \downarrow 0$, $|T| \downarrow 0$ and $|R|^2 \uparrow 1$; the reflection is complete. For nearly normal incidence $\theta \downarrow 0$, $K \gg 1$ then $b \uparrow 1$, $|T| \uparrow 1$ and $|R| \downarrow 0$; the transmission is complete.

In Figs. 4a–c, sample results for $\theta < 0$ and $K < -1 - F$ are presented for $F = 0.1$, 0.3 and 0.5. To satisfy (4.2) we must have

$$|\theta| < \theta^* \equiv \sin^{-1}\left(\frac{1}{1+F}\right). \tag{4.8}$$

The relevant curves are shown in solid lines. Note that for $\theta \downarrow 0$, $K \ll -1 - F$, $b^2 \uparrow 1$, hence, $|T| \uparrow 1$ and $|R| \downarrow 0$. A little algebra shows that as $|\theta| \uparrow \theta^*$, $K + 1 + F \uparrow 0$ and $|T|^2$ attains the finite limit

$$|T|^2 \to \left[1 - \frac{F^2 + 2F}{(1+F)^2}(ka)^2\right]^{-1}. \tag{4.9}$$

As F or ka increases, the limiting $|T|^2$ decreases. The corresponding $|R|^2$ may be inferred from energy conservation.

The dependence of $|T|^2$ and $|R|^2$ on the current width ka is oscillatory through $\sin^2\alpha_2 a$. For $2\alpha_2 a = n\pi$, i.e., $n = 0, 1, 2, \ldots$, $|R|^2 = 0$ and $|T|^2 = 1$ one gets maximum transmission. For $2\alpha_2 a = (n + 1/2)\pi$, minimum transmission and maximum reflection are obtained, i.e.,

$$\left.\begin{aligned} \min|T|^2 &= 4b^2/(1+b^2)^2 \\ \max|R|^2 &= (1-b^2)^2/(1+b^2)^2 \end{aligned}\right\}. \tag{4.10}$$

For $\theta > 0$, $K > 1$, b increases with K from 0 to 1; hence, the minimum transmission increases from 0 to 1. For $\theta < 0°$ and $0° < |\theta| < \theta^*$, $K < -1 - F$, b decreases with increasing $|K|$ from ∞ to 1; the minimum transmission coefficient also increases from 0 to 1.

b. Monotonic response in an opposing current

We now turn to the next case $-90° < \theta < -\theta^*$, i.e., $-1 - F < K < -1$,

$$\frac{(\omega + \beta V)^2}{gh} < \beta^2 < \frac{\omega^2}{gh}. \tag{4.11}$$

Let

$$\alpha_1{}^2 = \frac{\omega^2}{gh} - \beta^2, \quad \gamma_2{}^2 = \beta^2 - \frac{(\omega+\beta V)^2}{gh}. \tag{4.12}$$

The free surface is

$$\left.\begin{aligned} \zeta &= e^{i\alpha_1(x+a)} + Re^{-i\alpha_1(x+a)}, & x < -a \\ &= Ae^{\gamma_2 x} + Be^{-\gamma_2 x}, & |x| < a \\ &= Te^{i\alpha_1(x-a)}, & x > a \end{aligned}\right\}, \tag{4.13}$$

where T and R can be obtained from (4.4) by replacing $i\alpha_2$ by γ_2; thus,

$$\begin{pmatrix}|T|^2\\|R|^2\end{pmatrix} = \begin{pmatrix}4b^2\\(1+b^2)^2\sinh^2 2\gamma_2 a\end{pmatrix} \times [4b^2 + (1+b^2)^2 \sinh^2 2\gamma_2 a]^{-1}, \quad (4.14)$$

where

$$b = \frac{\alpha_1}{\gamma_2}. \quad (4.15)$$

In addition, we have

$$\begin{pmatrix}A\\B\end{pmatrix} = \frac{T}{2} e^{\mp\gamma_2 a}\left(1 \pm \frac{i\alpha_1}{\gamma_2}\right). \quad (4.16)$$

In normalized variables, we have

$$\gamma_2 a = \frac{ka}{K}[1 - (K+F)^2]^{1/2},$$

$$b^2 = \frac{(K^2-1)^{1/2}}{[1-(K+F)^2]^{1/2}}. \quad (4.17)$$

Sample variations of $|R|$ and $|R|^2$ are given as dashed curves in Figs. 4a–c for $F = 0.1$, 0.3 and 0.5. In all cases, reflection is large except for small ka. As $\theta \rightarrow -90°$, $K \uparrow -1$, $b^2 \downarrow 0$, $|T| \downarrow 0$ and $|R| \uparrow 1$. Near $K \rightarrow -1 - F$, (4.9) is recovered.

The free surface displacement above the current is monotonic in x according to

$$|\zeta|^2 = \frac{4b^2[\cosh^2\gamma_2(x-a) + b^2\sinh^2\gamma_2(x-a)]}{4b^2 + (1+b^2)^2\sinh^2 2\gamma_2 a}. \quad (4.18)$$

At the limit $K = -1 - F$ it can be shown that the free surface over the current is linear in x giving

$$\zeta = \frac{i\alpha_1 x}{(1 - i\alpha_1 a)} + 1, \quad (4.19)$$

so that

$$\left.\begin{aligned}\max|\zeta| &= |\zeta(-a)| = \left|\frac{-i\alpha_1 a}{1-i\alpha_1 a} + 1\right|\\ \min|\zeta| &= |\zeta(a)| = \left|\frac{1}{1-i\alpha_1 a}\right|\end{aligned}\right\}. \quad (4.20)$$

Finally, there are no waves for all x when $-1 < K < 1 - F$.

5. Waves in a continuous shear current along a ridge or trough

Let us first rewrite (2.6) as follows

$$(a\zeta')' + (1 - \beta\alpha^2)\zeta = 0, \quad (5.1)$$

where

$$a = \frac{gh}{(\omega + \beta V)^2} > 0. \quad (5.2)$$

We now assume that $V \rightarrow 0$ and $h \rightarrow h_\infty$ as $|x| \rightarrow \infty$ and follow the argument of Roseau (1976) for a one-dimensional Schrodinger equation [see also Mei (1983) for shallow water waves]. First, consider a general *scattering problem* with ζ being the solution which has the following asymptotic limits

$$\zeta \sim \begin{Bmatrix}A_- e^{i\alpha x} + B_- e^{-i\alpha x}, & x \sim -\infty\\ A_+ e^{i\alpha x} + B_+ e^{-i\alpha x}, & x \sim \infty\end{Bmatrix}. \quad (5.3)$$

Define Jost functions f_- and f_+, which are solutions due to unit transmitted waves from $x \sim -\infty$ and $+\infty$ respectively, as

$$f_-(x;\alpha) \sim \begin{cases}\frac{1}{T_-}e^{i\alpha x} + \frac{R}{T_-}e^{-i\alpha x}, & x \sim -\infty\\ e^{i\alpha x}, & x \sim \infty,\end{cases} \quad (5.4)$$

$$f_+(x;\alpha) \sim \begin{cases}e^{-i\alpha x}, & x \sim -\infty\\ \frac{1}{T_+}e^{-i\alpha x} + \frac{R}{T_+}e^{i\alpha x}, & x \sim \infty.\end{cases} \quad (5.5)$$

Here, $R_-(R_+)$ and $T_-(T_+)$ are the reflection and transmission coefficients for incidence from the left (right). Since f_- and f_+ are linearly independent, ζ can be expressed as a linear combination of them. As a consequence, we can write

$$\begin{Bmatrix}A_+\\B_-\end{Bmatrix} = [\mathbf{S}]\begin{Bmatrix}A_-\\B_+\end{Bmatrix}, \quad (5.6)$$

where the braces denote a column vector, brackets, a matrix and

$$[\mathbf{S}] = \begin{bmatrix}T_- & R_+\\ R_- & T_+\end{bmatrix} \quad (5.7)$$

is the scattering matrix.

By linear independence, the Wronskian of f_- and f_+ does not vanish and is equal to a constant. Using (5.2) and (5.3) it can be shown that

$$T_- = T_+, \quad (5.8)$$

which is a reciprocity relation.

Applying Green's formula to ζ and its conjugate ζ^* and using (5.3), it can be shown that

$$|A_+|^2 + |B_-|^2 = |A_-|^2 + |B_+|^2, \quad (5.9)$$

which states energy conservation. From (5.6) it follows that $[\mathbf{S}]$ is unitary, i.e.,

$$[\mathbf{S}^T][\mathbf{S}^*] = [\mathbf{I}], \quad (5.10)$$

where $\mathbf{T}$ denotes the transpose and

$$[\mathbf{I}] = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$$

Using (5.7) we further obtain

$$|T_\pm|^2 + |R_\pm|^2 = 1, \quad (5.11)$$

$$T_- R_+^* + R_- T_+^* = 0. \quad (5.12)$$

Combination of (5.12) and (5.8) yields

$$|R_-| = |R_+|. \quad (5.13)$$

Now we examine the *trapped waves* which are similar to the *bound states* in quantum mechanics (Sitenko, 1971; see also Mei, 1983). It can be shown that $f_-(x; \alpha)$ and $f_-(x; -\alpha)$ are also independent; hence, $f_+(x; \alpha)$ can be equated to a linear combination of them. Differentiating this equation with respect to x to get a second equation, we then find from the two equations,

$$T_+ = -2i\alpha/W[f_-(x; \alpha), f_+(x; \alpha)]$$

$$R_+ = W[f_-(x; -\alpha), f_+(x; \alpha)]/W[f_-(x; \alpha), f_+(x; \alpha)], \quad (5.14)$$

where $W(f_-, f_+) = f_-f'_+ - f_+f'_-$ = Wronskian. In the complex α-plane, the poles of T_+ coincide with the zeros of $W[f_-(x; \alpha), f_+(x; \alpha)]$; hence, they are also the poles of R_+ and of **[S]**. Let the poles be denoted by α_n, $n = 1, 2, 3, \ldots$, then

$$f_+ \sim \begin{cases} e^{-i\alpha_n x}, & x \sim -\infty \\ \dfrac{R_+}{T_+} e^{i\alpha_n x}, & x \sim \infty. \end{cases} \quad (5.15)$$

Applying Green's formula to f_+ and f_+^* we get

$$\mathrm{Im}\alpha_n{}^2 \int_{-\infty}^{\infty} |f_+|^2 dx = 2(\mathrm{Re}\alpha_n)(\mathrm{Im}\alpha_n) \int_{-\infty}^{\infty} |f_+|^2 \alpha x = 0. \quad (5.16)$$

Assuming that $\mathrm{Im}\alpha_n > 0$, we get from (5.16) that $\mathrm{Re}\alpha_n = 0$, so that the poles lie on the positive imaginary axis. The corresponding f_+ are clearly trapped modes, being exponentially vanishing as $|x| \to \infty$.

Since for a trapped mode α_n is pure imaginary, we can define the corresponding eigenmode ζ, i.e., $f_+(x; \alpha_n)$ to be a real function. Multiplying (5.1) by ζ and integrating, we get

$$\int_{-\infty}^{\infty} a\zeta'^2 dx + \int_{-\infty}^{\infty} (1 - \beta^2 a)\zeta^2 dx = 0. \quad (5.17)$$

This equality is possible if and only if $1 - \beta^2 a = 1 - \beta^2 gh/(\omega + \beta V)^2$ changes sign at certain finite x. As $|x| \to \infty$, $h \to h_\infty$ and $V \to 0$; we must have

$$1 - \beta^2 a = 1 - \beta^2 gh_\infty/\omega^2 < 0 \quad (5.18)$$

for exponential decay. Therefore,

$$1 - \frac{\beta^2 gh}{(\omega + \beta V)^2} > 0 \quad (5.19)$$

for some range of x. In other words, the inequality

$$\frac{(\omega + \beta V)^2}{gh} > \beta^2 > \frac{\omega^2}{gh_\infty} \quad (5.20)$$

is necessary for the existence of trapped waves.

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