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The propagation of water waves across a laterally sheared current

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Abstract

This paper studies surface gravity waves propagating across a shearing current in water of constant depth according to linear theory. The approach adopted is to represent the current by a series of vortex sheets separating regions of constant velocity. In each such region the method of solution employed is one of expansion in terms of eigenfunctions. These are then matched at the boundaries between regions. This leads to a large set of linear algebraic equations to solve for the coefficients. The results are then compared with those obtained using simpler semi-analytic theories which neglect the evanescent modes and can be described as analogues of the mild-slope equation. The general conclusion is that these simple approaches are accurate when the wavelength of the incident waves is much less than the lateral length scale over which the current varies but become less and less accurate as the wavelength increases.

Keywords: Water waves; Shear current; Interaction

1. Introduction

This investigation is concerned with the problem of the propagation of water waves across a horizontally-sheared current. Even the full linear problem is intractable so various simplifications have been introduced over the years. At one extreme, current changes have been modelled as one or more vortex sheets as in Evans [1], McKee and Tesoriero [4], Kirby, Dalrymple and Seo [2] and two papers by Smith [8, 9]. At the other extreme, the currents have been assumed to be slowly-varying on the scale of a wavelength. This leads to WKB-type solutions or extensions thereof (see McKee [5] and Mei [7]). In particular McKee [5] derived an analogue in this context of the celebrated mild-slope equation used by coastal engineers to study the progagation of waves over slowly-varying bottom topography which he called the mild-shear equation. In a later paper [6] he sought to extend this to currents which are more rapidly varying by including some terms which depend explicitly on the first and second derivatives of the current velocity. Both these equations neglect the contributions of the evanescent modes. One of the aims of this paper is to test the validity of these simple semi-analytic approaches by means of an alternative method which approximates the shear current by a large number of vortex sheets separating regions of constant current. The contributions of the evanescent modes are explicitly included. Rather than following the approach used in the case of a single vortex sheet by Evans [1] and McKee and Tesoriero [4] of reducing the problem to one of solving two integral equations at each vortex sheet, a method involving multiplication by eigenfunctions will be used in which the pressure-continuity condition at each vortex sheet is multiplied by the eigenfunctions relevant to one side of the vortex sheet and then integrated from top to bottom whereas the kinematic condition is treated analogously using the eigenfunctions relevant to the other side of the vortex sheet.

This method of dealing with the two matching conditions at a discontinuity of depth and/or current was used by Kirby, Dalrymple and Seo [2] in studying the propagation of waves across a trench in the presence of piecewise-constant currents and seems to date back to Takano [10] who considered depth variations only and studied wave propagation across a ridge of rectangular cross-section in the sea floor. In each of [2] and [10] there were only two points of discontinuity of depth and/or current. Here, there are an arbitrary number of such points but, in order to concentrate on the effects of the current shear, it will be assumed that the depth is constant. The case of variable

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depth, but no current, was treated by Mattioli [3] using the same methodology.

2. The basic equations

As in [5,6] we consider wave motion in an inviscid fluid of constant density ρ. The x and z axes are taken horizontal with the y axis vertically down. The undisturbed free surface is at $y = 0$ and the bottom at $y = H$. In order to focus on the effects of the current, H will be assumed to be constant. The basic current is a shear flow $(0, 0, W(x))$. Wavy perturbations to this basic state are now considered in which all perturbation quantities are proportional to $\exp \mathrm{i}(nz - \omega t)$ where $\omega > 0$. If W varies with x on a length scale L, we scale x and z with L, y and H with g/ω^2, W with g/ω, n with ω^2/g and the perturbation pressure due to the waves with $\rho g a$ where a is a typical free-surface amplitude of the waves. This scaling uses velocity, wave number and depth scales appropriate to waves in deeper water. As shown in [5], the dimensionless pressure perturbation, p, on linear theory satisfies the equations

$$(p_x/\Omega^2)_x + \epsilon^2 p_{yy}/\Omega^2 - n^2\epsilon^2 p/\Omega^2 = 0, \tag{1}$$

$$p_y + \Omega^2 p = 0 \quad \text{at } y = 0, \tag{2}$$

$$p_y = 0 \quad \text{at } y = H, \tag{3}$$

where all variables are now dimensionless,

$$\Omega(x) = 1 - nW(x) \quad \text{and} \quad \epsilon = \omega^2 L/g.$$

The parameter ϵ is a measure of how rapidly the current changes on the scale of the waves. Smaller values of ϵ signify a more rapidly-changing current. As discussed in [5], values of ϵ of order 100 or more are to be expected in most offshore oceanographical situations because of the large lateral length scales over which most ocean currents vary. In such situations, the following analogue of the mild-slope equation, called the mild-shear equation and hereinafter referred to as the MSE, was derived in McKee [5]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\Gamma(x)\frac{\mathrm{d}\eta}{\mathrm{d}x}\right) + \epsilon^2 (k^2(x) - n^2)\Gamma(x)\eta = 0, \tag{4}$$

where $k(x)$ is the local total wavenumber and is the unique positive root of the local dispersion relation

$$\Omega^2(x) = k(x)\tanh(k(x)H). \tag{5}$$

In deriving this equation, it is assumed that

$$p = \eta(x)\,\mathcal{Z}(y; x), \tag{6}$$

where

$$\mathcal{Z}(y; x) = \frac{\cosh(k(x)(y - H))}{\cosh(k(x)H)} \tag{7}$$

is the local surface wave eigenfunction and

$$\Gamma(x) = \Omega^{-2}(x)\int_0^H \mathcal{Z}^2(y; x)\,\mathrm{d}y.$$

The quantity η can be interpreted as the dimensionless free surface deformation due to the waves. This approach

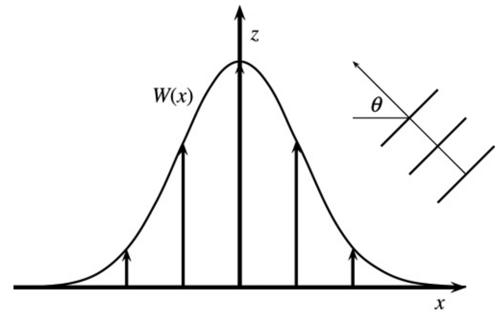

Fig. 1. Definition diagram for waves incident from still water at $x = \infty$ with angle of incidence θ upon a laterally-sheared current $W(x)$.

completely neglects the contributions of the evanescent modes. In principle, (6) can be extended to include the evanescent modes, but the resulting system of ordinary differential equations for the amplitudes is stiff and the more evanescent modes are included the stiffer it becomes.

The MSE was later generalised by McKee [6] to include some extra terms on the right of (4) which depend upon the first and second derivatives of W. This equation, hereinafter called the extended mild-shear equation (EMSE), would be expected to be more accurate than the MSE since it does not assume that ϵ is large, yet it still makes the assumption (6) and so neglects the evanescent modes.

In the case of rip currents or tidal flows between closely-spaced islands, values of ϵ of $O(1)$ or smaller might be expected. The aim of the present work is to develop an essentially numerical method of solution which, although computationally more intensive than solving the MSE or EMSE numerically, can be used to investigate the circumstances under which those two methods give accurate results as well as being a numerical method of general applicability. We will investigate the situation where waves are incident with angle of incidence θ from still water at $x = \infty$ towards a current. Thus $W(\infty)$ will be assumed to be zero. The two canonical current profiles of interest are a jet-type current which will here generally be modelled by a Gaussian current profile $W = \beta\exp(-x^2)$ and a shear layer which will generally be modelled by $W = \frac{1}{2}\beta(1 - \tanh x)$. Fig. 1 depicts the situation for a jet-type current. Without loss of generality, it may be assumed that $0 \le \theta \le \pi/2$. If $\beta > 0$ the waves are entering a following current. If $\beta < 0$ they are entering an opposing or adverse current.

3. The numerical method

If we assume that the current velocity W is piecewise constant, (1)–(3) can be solved by the usual separation of variables technique and the solutions matched at the vortex sheets separating the different regions by enforcing the physically correct matching conditions at the discontinuities. This approach is analogous to that of Mattioli [3] for water wave propagation over a series of steps in the bottom.

To attack the problem using this formulation, let us suppose that the discontinuities in W occur at $x = \xi_j$ for $j = 1, \ldots, L$. In (ξ_j, ξ_{j+1}), let $W = W_j$ and $\Omega = \Omega_j = 1 - nW_j$. In

$(-\infty, \xi_1)$, let $W = W_0$ and $\Omega = \Omega_0 = 1 - nW_0$ and in (ξ_L, ∞), let $W = W_{L+1}$ and $\Omega = \Omega_{L+1} = 1 - nW_{L+1}$ where we may take $W_{L+1} = 0$ since the current velocity is assumed to tend to zero as $x \to \infty$. Then the solution in (ξ_j, ξ_{j+1}) is

$$p_j(x, y) = \sum_{m=0}^{\infty} \Phi_m^{(j)}(x)\,\Psi_m^{(j)}(y) \qquad (8)$$

where

$$\Psi_0^{(j)}(y) = \frac{\cosh(K_j(y-H))}{\cosh(K_j H)},$$

$$\Psi_m^{(j)}(y) = \cos(\kappa_{jm}(y-H)),$$

$$\Phi_0^{(j)} = F_0^{(j)} \exp(-il_j(x-\xi_{j+1})) + B_0^{(j)} \exp(il_j(x-\xi_j)),$$

$$\Phi_m^{(j)} = F_m^{(j)} \exp(\sigma_{jm}(x-\xi_{j+1})) + B_m^{(j)} \exp(-\sigma_{jm}(x-\xi_j)),$$

$$l_j = \epsilon\sqrt{K_j^2 - n^2}, \quad \text{and} \quad \sigma_{jm} = \epsilon\sqrt{\kappa_{jm}^2 + n^2}.$$

In these equations, K_j is the unique positive root of

$$\Omega_j^2 = K_j \tanh(K_j H)$$

and κ_{jm} is the mth positive root of

$$\Omega_j^2 = -\kappa_{jm} \tan(\kappa_{jm} H).$$

The use of $x - \xi_{j+1}$ and $x - \xi_j$ in the exponential terms rather than just x lessens the possible consequences of loss of significant figures which can occur when very large and very small exponentials appear in an expression. If we introduce fictitious points $\xi_0 = \xi_1$ and $\xi_{L+1} = \xi_L$ then the above formulation can be said to hold in all intervals if we take $B_m^{(0)} = 0$ for all m with $F_m^{(L)} = 0$ for all $m > 0$ and $F_0^{(L)} = 1$ being the amplitude of the wave incident from $+\infty$. In terms of the angle of incidence θ, $n = K_{L+1}\sin\theta$. The infinite sum in (8) is truncated at $m = M$. At a discontinuity of W, the correct matching conditions are that p and $\Omega^{-2}\partial p/\partial x$ are continuous. The first of these is necessary to avoid infinite accelerations of the fluid particles on the vortex sheet and the second follows from integrating (1) across the vortex sheet. Evans [1] gave equivalent conditions in terms of the velocity potential not pressure. These conditions are applied at a typical point of discontinuity ξ_j as follows. First, the expressions for p_j and p_{j-1} are equated at $x = \xi_j$. This expression is then multiplied by $\Psi_m^{(j-1)}(y)$ and integrated from $y = 0$ to $y = H$. This is done for $m = 0, \ldots, M$ to give $M + 1$ simultaneous linear equations linking the unknown amplitudes. Some of the coefficients are zero because of the orthogonality of the eigenfunctions. Next, the expressions for $\Omega_j^{-2}\partial p_j/\partial x$ and $\Omega_{j-1}^{-2}\partial p_{j-1}/\partial x$ are equated at $x = \xi_j$ and treated analogously, except that we multiply by the eigenfunctions $\Psi_m^{(j)}(y)$ this time to give a further $M + 1$ simultaneous linear equations. Doing this at each of the ξ_j for $j = 1, \ldots, L$ gives a total of $2L(M + 1)$ simultaneous linear equations for the $2L(M + 1)$ coefficients. All the integrals are found analytically.

It was found that, irrespective of the order of truncation, the method conserved wave action flux exactly to within the round-off error. The method can handle critical layers (where $\Omega = 0$) and caustics provided the nodes ξ_j are chosen so that none of them coincides with the location of the actual critical layer. A slight modification is required for a shear-layer profile if $\Omega(-\infty) < 0$. As pointed out in [4], group velocity considerations show that we must then take $F_0^{(0)} = 0$ instead of $B_0^{(0)} = 0$ in order to satisfy the radiation condition of no incoming energy at $x = -\infty$. In such a situation there is a critical layer and possible over-reflection of the incident waves. Such a regime would be unlikely to occur in oceanographical applications.

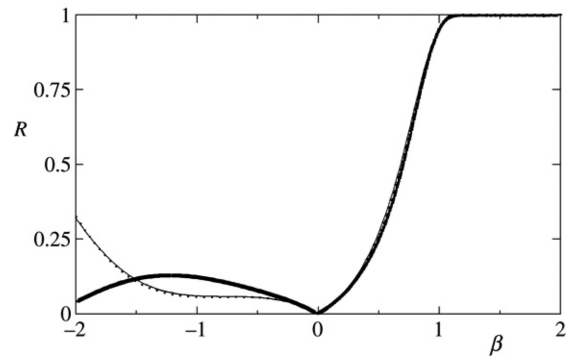

Fig. 2. The amplitude reflection coefficient as a function of the dimensionless maximum current strength β for a jet current $W = \beta\exp(-x^2)$. The dimensionless water depth is $H = 1$ and the angle of incidence is $\theta = 45°$. The parameter ϵ has the value 0.25. The three lines (thick, thin and dotted) are for calculations using zero, two and four evanescent modes respectively. The convergence is demonstrated by the fact that the latter two are virtually indistinguishable.

Although the method conserves wave action flux, the results of course depend upon the order of truncation used. As a general rule, it was found that the method converged quite quickly and it was only necessary to use a few evanescent modes in the expansions. In all the figures presented here sufficient modes and nodes were used to ensure convergence to the accuracy required for graphical presentation. Except for the case of waves encountering opposing currents so strong as to be beyond the range of physical relevance, the method did not suffer greatly from the problem of ill-conditioning.

An example of the convergence of the method is shown in Fig. 2 for a jet current $W = \beta\exp(-x^2)$ when $H = 1$, $\theta = 45°$ and $\epsilon = 0.25$. In calculating these results, the current was assumed to be zero outside $[-B, B]$ and this interval divided up into a large number (typically 99) of subintervals of equal length. In each such subinterval the value of W was taken as that of $\beta\exp(-x^2)$ at the midpoint of the subinterval. A value of $B = 3$ proved sufficient and using more than four evanescent modes or more than 99 nodes gave results graphically indistinguishable from those shown. As a general rule, the smaller the value of ϵ, the more evanescent modes were required. This is as expected since smaller values of ϵ imply a more rapidly-varying current. It is worth observing here that only relatively small values of β are likely to be of importance in applications since the velocity scale adopted is $g\tau/2\pi$ where τ is the wave period in seconds. Thus for $\tau = 10$ and a strong current of 2 m/s, the value of β is only about 0.13.

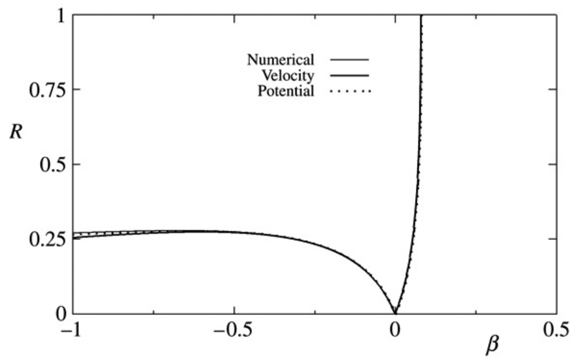

Fig. 3. The amplitude reflection coefficient for infinite depth for a single vortex sheet of strength β when $\theta = 60°$. The potential and velocity approximations of Evans [1] are compared with a numerical solution for very large H which is shown by the thin sold line.

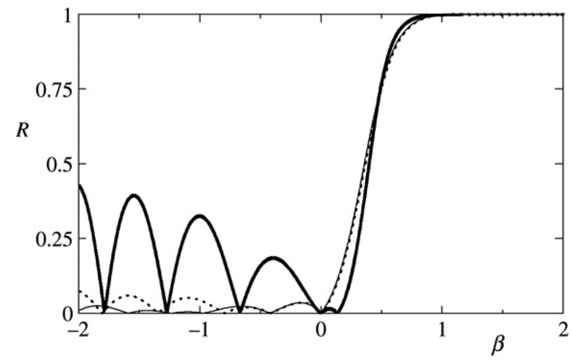

Fig. 4. The reflection coefficient as a function of the dimensionless maximum current strength β for a jet current $W = \beta \exp(-x^2)$ with $H = 1$, $\theta = 45°$ and $\epsilon = 1$. The results for the mild-shear equation are shown by a thick solid line, those for the extended method by a thin solid line and those for the purely numerical solution by a dotted line. The latter two are virtually indistinguishable except for $\beta < 0.5$ approximately.

The method is not strictly applicable when the depth is infinite since the sum over the evanescent modes is then replaced by an integral over a continuous spectrum. Nevertheless, results for increasingly large values of the non-dimensional depth H tend to settle down relatively quickly to values which can be considered to be representative of infinite depth. The normalisation adopted here of the surface wave eigenfunction by dividing by $\cosh(K_j H)$ ensured that no numerical problems such as ill-conditioning arose for large values of H.

The present method can also be used to deal with the problem of a single vortex sheet. Evans [1] treated this problem in the case of infinite depth by formulating it as one requiring the solution of two singular integral equations of the first kind. He then sought approximate solutions of these in terms of two-term Galerkin expansions using the surface wave eigenfunctions on the two sides of the vortex sheet. This technique was extended to the case of finite depth by McKee and Tesoriero [4] who also included some evanescent modes in the Galerkin expansions. The present method is numerically superior to this method in that it is far less prone to ill-conditioning. Smith [8,9] discusses other possible approximate methods of solution for this problem. Evans in fact gave two formulations of the problem: one in which the unknown in the integral equations is essentially the potential on the vortex sheet and the other in which it is the horizontal velocity. Fig. 3 compares the results of both these with the numerical solution using a large value of H for an angle of incidence of $\theta = 60°$. Both the potential and velocity approximations used by Evans are seen to be really very good indeed.

4. Some numerical results

One aim of this work is to use the numerical method outlined above to investigate the circumstances under which the MSE and the EMSE will give accurate predictions of the reflection and transmission of waves crossing a shearing current. In all cases we will consider a wave of unit amplitude incident from $x = +\infty$ with angle of incidence θ. The other parameters of interest are the dimensionless water depth H and the current-scale parameter ϵ as well as the maximum current speed for which we will use the symbol β. Fig. 4 shows the amplitude reflection coefficient as a function of β for a jet with velocity $W = \beta \exp(-x^2)$ when $H = 1$, $\theta = 45°$ and $\epsilon = 1$. As can be seen, the EMSE is in excellent agreement with the numerical solution for small values of β (which are the only ones of physical relevance) whereas the MSE is not nearly so accurate.

Many such figures have been produced. The general picture which emerges is that the larger the value of ϵ the better the agreement between both the MSE and EMSE with the EMSE generally being more accurate. This is just what one would expect.

With so many parameters in the problem, it is difficult to know the best way to present the results. The basic aim of this work is to investigate the circumstances under which the MSE and EMSE are accurate. One would expect that the MSE would be accurate for large ϵ since the current would then be slowly-varying on the scale of the waves and less accurate for smaller values of ϵ. The derivation of the EMSE, on the other hand, made no explicit assumption as to the size of ϵ yet it also neglected the evanescent modes. Hence it would seem sensible to present results as functions of ϵ for representative values of the other parameters.

Fig. 5 shows the amplitude reflection coefficient as a function of ϵ for a jet with velocity $W = 0.2 \exp(-x^2)$ when $H = 1$ and $\theta = 45°$. Both the MSE and EMSE are in excellent agreement with the numerical solution for large values of ϵ but as ϵ decreases the MSE loses accuracy. The EMSE is quite accurate except for very small values of ϵ where it erroneously predicts that $R \to 1$ as $\epsilon \to 0$.

Similar behaviour is shown in Fig. 6 in which the velocity profile is a shear layer $W = 0.1(1 - \tanh x)$. In this case the EMSE tracks the exact numerical solution almost perfectly until ϵ reaches a value of about 0.1 below which it again erroneously predicts that $R \to 1$ as $\epsilon \to 0$. The general behaviours exhibited in Figs. 5 and 6 have been found in all examples investigated.

We conclude with some contour plots (Figs. 7–9) of the amplitude reflection coefficient as a function of β and ϵ when

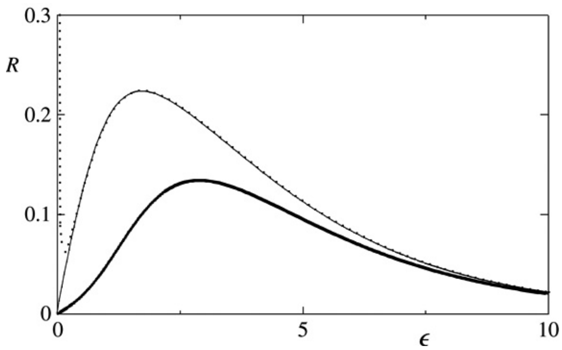

Fig. 5. The amplitude reflection coefficient as a function of the scale parameter ϵ for a jet current $W = 0.2\ \exp(-x^2)$. The dimensionless water depth is $H = 1$ and the angle of incidence is $\theta = 45°$. The MSE is shown by a thick solid line, the EMSE by a dotted line and the numerical results by a thin solid line. The latter two are virtually indistinguishable except for very small values of ϵ.

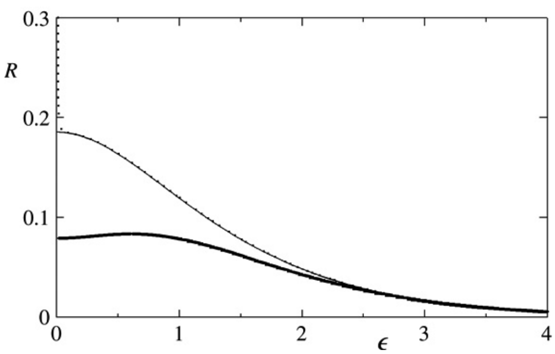

Fig. 6. The amplitude reflection coefficient as a function of the scale parameter ϵ for a shear-layer current $W = 0.1\,(1 - \tanh x)$. The dimensionless water depth is $H = 1$ and the angle of incidence is $\theta = 45°$. The MSE is shown by a thick solid line, the EMSE by a dotted line and the numerical results by a thin solid line. The latter two are virtually indistinguishable except for very small values of ϵ.

$H = 10$ and $\theta = 45°$ for three different current profiles. In Fig. 8 we have the standard Gaussian jet profile $W = \beta \exp(-x^2)$ whereas the current in Fig. 7 is $\beta \cos^2(x)$ for $|x| \leq \pi/2$ and zero otherwise. In Fig. 9 the profile is $W = \beta \exp(-|x|x^2)$. These three figures are generally quite similar, except where the reflection is weak, which indicates that the maximum current strength is more important than the precise shape of the current in determining the amount of energy reflected.

5. Discussion

This work has developed a general method of finding solutions, albeit numerical ones, to the linear problem of water wave propagation across a laterally-sheared current in water of constant depth. The method could be applied to any current profile, including one observed in field or laboratory experiments. The main thrust of this investigation, however, is to investigate the circumstances under which the simple semi-analytic approaches of [5] (which leads to the MSE) and [6] (which leads to the EMSE) are accurate. The general conclusion reached is that both are accurate for large values of ϵ. In

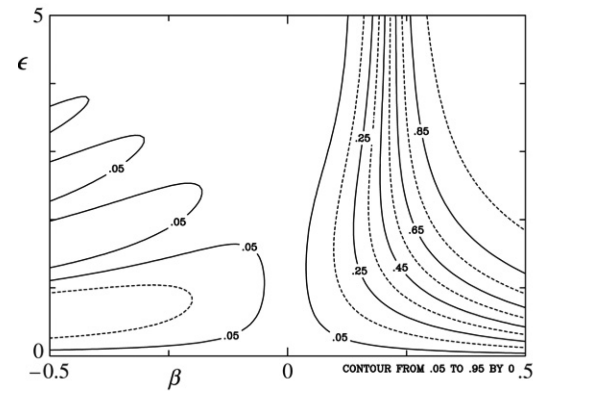

Fig. 7. The amplitude reflection coefficient as a function of the scale parameter ϵ and maximum current β for a jet current in which $W = \beta \cos^2(x)$ for $|x| \leq \pi/2$ and zero otherwise. The dimensionless water depth is $H = 10$ and the angle of incidence is $\theta = 45°$.

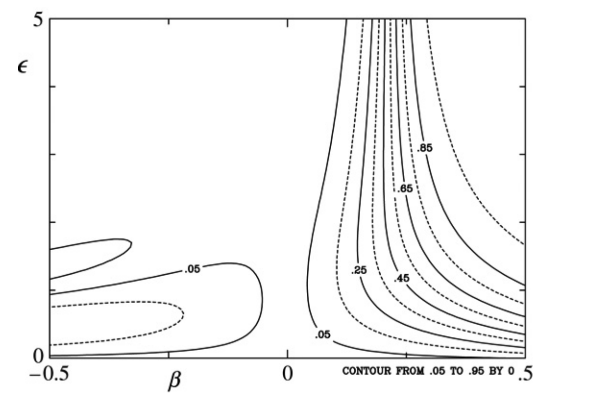

Fig. 8. As for Fig. 7 except that the current profile is $W = \beta\ \exp(-x^2)$.

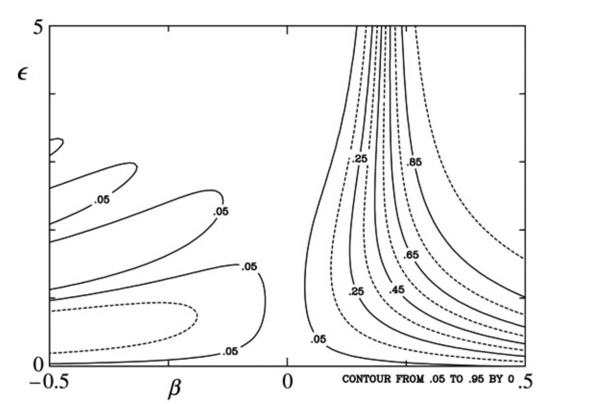

Fig. 9. As for Fig. 7 except that the current profile is $W = \beta\ \exp(-x^2|x|)$.

such situations, the waves are short when measured against the lateral length scale over which the current is changing. As ϵ decreases the MSE, unlike the EMSE, gradually loses its accuracy. For very small values of ϵ, the MSE is still inaccurate but the EMSE experiences a catastrophic failure in that it predicts complete reflection of the incident waves. Attempts to establish this behaviour rigorously using perturbation methods for $\epsilon \to 0$ have not been fruitful. From a practical point of view, this behaviour of the EMSE for very small ϵ is probably not a major concern since the EMSE requires a knowledge of the first and second derivatives of the current profile $W(x)$. These, particularly W'', would be very difficult, if not impossible, to estimate accurately from laboratory or field data. Thus the EMSE, though of theoretical interest and value, is probably not useful in laboratory or fieldwork situations. The MSE, on the other hand, requires only a knowledge of the current velocity, but not any of its derivatives, and so would be less prone to errors introduced by the uncertainties in measurements.

The method could be readily extended using the method of Kirby, Dalrymple and Seo [2] to deal with the case where the water depth H also varies with x. Another direction into which the present work could be extended would be to consider depth-dependent currents. Except in a few simple cases, such as currents varying linearly with depth, the eigenfunctions used in the expansion (8) would have to be determined numerically.

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