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On the 'quasimomentum rule' for wave-induced mean forces on obstacles immersed in a material medium.

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Abstract

The time-averaged resultant force, or radiation recoil, due to waves scattered from, or generated by, an obstacle immersed in a material medium, is often — but not always — given by the 'quasimomentum rule', which states that, correct to $O(a^2)$ in wave amplitude a, the radiation recoil is the same as if (i) the medium were absent, and (ii) the waves had momentum equal to their quasimomentum. The quasimomentum, or pseudomomentum, is equal to the wavenumber vector times the wave action or 'number of quanta'. It is a wave property in the sense that it is calculable from linearized wave theory alone, permitting the radiation recoil to be obtained with minimal computational effort. There are, however, clear *exceptions* to the quasimomentum rule, even among cases involving nondissipative fluid media and having what might appear to be the appropriate translational invariance property. This paper derives a new extension of Noether's theorem that shows why such exceptions arise, as well as clarifying the general conditions under which the rule does hold.

The new theorem covers a very wide class of waves in nondissipative, nonrelativistic fluid media including cases with frozen fields like Alfvén, internal-gravity, and inertio-gravity waves, in addition to surface gravity and acoustic waves. Similar results are expected to hold for the electrodynamic, 'Abraham-Minkowski' case, but require a relativistic version of the theorem not yet available.

The theorem shows that if the waves propagate in a space of dimension $N = 2$ or 3 then the rule holds for a finite-sized, impermeable obstacle of the same dimensionality provided that

(a) the wave field near the obstacle is steady,

(b) the wave-induced $O(a^2)$ mean flow near the obstacle is steady, and

(c) the obstacle can be shrunk to dimensions $\leq (N-2)$, i.e. to a point if the space is two-dimensional, or to a point or a line if three-dimensional, in a kinematically consistent way that leaves the properties of the medium everywhere translationally invariant, in the appropriate direction or directions, at the *end* of the process.

'Kinematically consistent' means in such a way that the medium remains in contact with the boundary of the shrinking obstacle while undergoing a continuous, mass-conserving motion that also conserves any frozen fields, such as entropy stratification or magnetic flux, in the same way as in the actual nondissipative dynamics of the medium. Condition (c) is required for finite-sized obstacles because the proof of the theorem depends on a consideration of the singular, finite-amplitude Lagrangian displacement field that would result from intrusion of the obstacle into the medium. 'Near the obstacle' means a region within which this displacement field must be considered larger than $O(a)$. All the exceptional cases known to the authors violate condition (c).

1. Background

Physical systems defined by translationally invariant Lagrangian or Hamiltonian functionals conserve momentum **p**. The translational invariance giving rise to the conservation of, say, the x component of **p** involves translating the whole physical system in the x direction. If the physical system is a material medium through which waves can propagate, then a second kind of translational invariance can arise, namely translational invariance of the properties of the medium. Associated with this is a second conservable quantity **q**, distinct from **p** in every case except that of waves *in vacuo*. The quantity **q** has no universally accepted name, being known in various contexts as quasimomentum, pseudomomentum, wave-vector, impulse, Poynting's momentum, Minkowski momentum, radiation momentum, field momentum, tensor momentum, phonon momentum, acoustic momentum, crystal momentum, canonical momentum, wave momentum, or simply as momentum.*

For the sake of definiteness, we shall call **q** the quasimomentum, even though the equally well established term pseudomomentum has much to recommend it (e.g. Sturrock 1968b; Gordon 1973; Peierls 1976, 1979). Its rotational equivalent, important for instance in models of waves in rotationally symmetric media, such as models of stellar interiors or planetary atmospheres, may likewise be called the angular quasimomentum. The term quasimomentum is both notationally convenient for present purposes, and etymologically apt since in many circumstances **q** behaves *as if* it were momentum.

In a wide variety of problems concerning waves in material media the quasimomentum

* For historical insights into the confusion that this has caused, the reader may consult e.g. Euler (1746), Poynting (1905) [[Collected Papers p 336??]], Brillouin (1925), Post (1953, 1960), Bretherton (1969), Gordon (1973), Peierls (1976), Israe l (1977), Beyer (1978), Jones (1978), Gibson et al. 1980, McIntyre (1981), Wach et al. (1981), and Newcomb (1983[[?]]).

$\mathbf{q}$ is a more useful quantity than the momentum $\mathbf{p}$. The reason is that not only does $\mathbf{q}$ often behave as if it were momentum, but it is also far easier to calculate, in most problems. A celebrated example is the time-averaged recoil force due to wave generation or scattering by an obstacle immersed in a material medium. A calculation of this force from momentum conservation is straightforward in principle, but laborious and error-prone in practice because $\mathbf{p}$ and its flux have to be computed correct to second order in wave amplitude a, taking account of the $O(a^2)$ wave-induced mean motion of the medium and the $O(a^2)$ mean stresses supported by the medium. Both may be crucially affected, for instance, by the presence of boundaries, including boundaries at large distances from the obstacle, and so all boundary conditions must be carefully evaluated correct to $O(a^2)$. Related to this is the fact that if the distant boundaries are 'at infinity' then the integral $\iiint \mathbf{p}\, dxdydz$ giving the total momentum may be ill-defined in some cases (e.g. Batchelor 1967, §§6.4, 7.2, 7.3; Peierls 1983).

By contrast, the quasimomentum and its flux are relatively simple to calculate since they are $O(a^2)$ *wave properties*, by which we mean that they can be computed correct to $O(a^2)$ from a knowledge of the linear, $O(a)$ wave solution alone. Moreover in many cases of interest the time-averaged recoil force, on an immersed obstacle scattering or generating waves, is given simply by the rate of change of quasimomentum of the wave field, or alternatively by the net flux of quasimomentum into a control volume that surrounds the obstacle and within which a steady state has been reached. In other words the time-averaged recoil force can be calculated, by analogy, *if* the problem were simply one of scattering *in vacuo*; or, stated more explicitly, the force is the same, in many cases, *as if*

(i) the medium were absent, and

(ii) the waves had momentum equal to their quasimomentum.

We shall call this statement the 'quasimomentum rule'. The words 'as if', and the proviso about the absence of the medium, are crucial. They emphasise the nontrivial nature of the assertion that for the purpose at hand one can ignore the $O(a^2)$ mean stresses and momentum changes that exist in the material of the medium and that actually contribute to the resultant force on the obstacle.

Although a correct and completely general proof of the quasimomentum rule has never, to our knowledge, been given for finite-sized obstacles, (except in the vacuum case, for which $\mathbf{p} = \mathbf{q}$ and the rule holds trivially), a large number of corroborative examples are known. The earliest for which a correct theoretical justification was given may have been the celebrated unpublished proof by Langevin for the case of acoustic waves (Bicquard 1936). Some of the examples have been verified not only theoretically, for the particular cases concerned (e.g. King 1934?; Westervelt 195?; Gordon 1973; Longuet-Higgins 1977), but also experimentally (e.g. Fox 1940; Jones and Richards 1954; Rooney 1973; Hasegawa and Yosioka 1975; Rudnick 1977; Jones and Leslie 1978; Longuet-Higgins, op.cit.; Gibson et al. 1980), in some cases with impressive accuracy. For instance the rule is today used routinely in ultrasonic engineering as a means of making absolute determinations, with an accuracy of a percent or so, of the intensity of a beam of sound waves (e.g. Rooney 1973; Hasegawa and Yosioka, op.cit., 1979). In this context the mean force on the obstacle is often called the 'Langevin radiation pressure'. For light waves reflected from an obstacle suspended in liquids of different refractive indices n, a recent high-precision experiment by Jones and Leslie (op.cit.) has verified that the rule holds to better than a fraction of a percent of the refractive index difference $n - 1$.

The quasimomentum rule has, indeed, been found to be true in so many cases that it has sometimes been thought to be a universal principle of physics. That, however, is known not to be the case (e.g. Gordon 1973; McIntyre 1972, 1973, 1981; Newcomb [[??]]).

Moreover, very careful theoretical calculations, which include all the relevant $O(a^2)$ effects and which have been cross-checked in various ways, have shown that exceptions to the rule do exist, even in cases of nondissipative fluid media which are homogeneous in the appropriate direction, i.e. which appear at first sight to have the appropriate translational invariance property. Some of the known exceptions will be described briefly in §3 of this paper.

The question thus arises, what then are the conditions for the rule to hold? The purpose of this paper is to answer this question in a much more general way than appears to have been done in the past, by deriving an extension of Noether's theorem that covers a wide variety of cases and that accounts for all the exceptions cited. The essential idea behind the theoretical analysis is explained in the next section.

2. THE TRANSLATIONAL SYMMETRY OPERATION

The starting point is to analyze what 'translational invariance of the properties of the medium' actually means, when applying Noether's theorem to derive conservation of $\mathbf{q}$. The relevant symmetry operation can be thought of as a spatial shift of the disturbance pattern, keeping the undisturbed positions of material elements of the medium fixed. This differs, as already mentioned, from the symmetry operation corresponding to conservation of momentum $\mathbf{p}$. The latter operation consists of shifting both the wave pattern *and* the undisturbed positions of elements of the medium by the same amount.

To make these ideas both precise and general we need to be able to use the machinery of analytical dynamics. This involves specifying the dynamical system by a Lagrangian or Hamiltonian functional having the required symmetry properties, which in turn involves introducing generalized coordinates. In the presence of a material medium this requires us, *inter alia*, to keep track of the positions of each material element, if complete generality is to be retained.* In particular, the definition of 'disturbance pattern' involves specifying the wave-induced displacement $\xi(\mathbf{x}, t)$ of each material element from a suitably defined reference position $\mathbf{x}$ representing its undisturbed position at a given time t. Translational invariance 'of the medium' can then be defined as translational invariance of an undisturbed reference state in which $\xi(\mathbf{x},\ t) = 0$ everywhere. Such invariance implies that, if $\xi(\mathbf{x},\ t)$ describes a dynamically possible wave pattern, then so also does the shifted wave pattern $\xi(\mathbf{x} - \epsilon,\ t)$, $\phi(\mathbf{x} - \epsilon t)$, where ϵ is a constant vector pointing in the direction of translational symmetry.

Now in order to deduce the quasimomentum rule for an immersed obstacle, it is necessary to consider a symmetry operation in which the obstacle is moved through the medium by the same amount ϵ as the wave pattern. It is easy enough to apply this idea to cases where the obstacle is infinitesimally small and does not rupture the medium. Such a particle-like obstacle may be represented mathematically by a weak, localized potential field centred on a given position $\mathbf{x}_0$, and there is no difficulty in imagining it being translated through the medium along with the wave pattern, i.e. $\mathbf{x}_0 \rightarrow \mathbf{x}_0 + \epsilon$. This case has practical importance for understanding the results of experiments on the interactions of particles and quasiparticles in solid and liquid state physics (e.g. Gibson et al. 1980). Proofs of the quasimomentum rule for such particle-like obstacles have been given for a problem in one-dimensional acoustics by Brenig (1955) and by Gilbert and Mollow (1968); and the extension to two and three dimensions, and to dynamical systems permitting other wave types, is straightforward. However, none of this can explain the exceptions to the

* The general theory of phase-space reduction under the lagrangian relabelling group has in recent years given us a beautiful noncanonical Eulerian theory of Hamiltonian structure; but it is not sufficiently general to handle all the cases of interest here, owing to the technical problem of "too few Casimirs" that arises in, for example, the acoustic case.

rule, which, as we shall see in the next section, all involve finite obstacles that rupture the medium.

Here the key point is to recognize that, for the purposes of applying Noether's theorem, the 'disturbance' to the medium must include not only the displacement field attributable to the wave motion *per se*, but also the large-amplitude displacement field $\xi^B(\mathbf{x})$, say, attributable to the introduction of the obstacle into the medium. The translationally-invariant reference state that is relevant to Noether's theorem is the state of the medium before either the waves or the obstacle is introduced. One way to visualize this (and to model it mathematically) is via a thought-experiment in which a very strong repulsive field representing the obstacle is imagined to grow from nothing, displacing the medium by the necessary amount. Because the obstacle-induced displacement field $\xi^B(\mathbf{x})$ is generally finite in amplitude, theories assuming a small-amplitude disturbance are inapplicable.

The function $\xi^B(\mathbf{x})$ must also, quite evidently, be singular. There must be at least one reference position from the neighbourhood of which material particles move apart from one another by a finite amount, as the medium is displaced. Because of the singular behaviour of the function $\xi^B(\mathbf{x})$, the applicability of Noether's theorem is by no means self-evident without further analysis. The standard form of the theorem assumes that the translational symmetry operation is applied to continuous, indeed differentiable, fields, since the operator $\epsilon \cdot \nabla$ is applied to those fields in the course of the analysis.

We shall see that this is by no means an academic question. Rather, it is the heart of the matter. Noether's theorem is, indeed, applicable in some cases, but inapplicable in others, precisely because of the singular behaviour of $\xi^B(\mathbf{x})$; and this does, indeed, account for the known exceptions to the quasimomentum rule.

Whether or not the singularity is fatal to the applicability of Noether's theorem in a given case turns out to depend, as we shall see, on the dimensionality of the region from which the obstacle may be considered to expand and displace the medium (in a manner consistent with the translational symmetry of the reference state). A careful discussion of what this implies will be given in §4.

Before proceeding to the details of the general theory, we briefly describe some of the known exceptions.

3. EXCEPTIONS TO THE QUASIMOMENTUM RULE

The known exceptions are of two quite different kinds. The first is the general problem for a finite, impermeable obstacle in one dimension. The rule then fails in general. (It holds fortuitously in a tiny set of special cases, the best known of which is the artificial case of sound waves in a fictitious gas whose sound speed does not change with density. In other acoustics problems the thermodynamic derivative $\partial(\ln c)/\partial(\ln \rho)$ enters and contributes a term unrelated to $\mathbf{q}$.) [[refs?]]

The second kind of exception is more subtle and interesting, and occurs in two and three dimensions. The cases in question all involve frozen fields which, together with the translational invariance requirement, constrain the function $\xi^B(\mathbf{x})$ to be more singular than it would otherwise be, causing Noether's theorem fail. We describe two examples in this category. The first involves an entropy field with a continuous gradient, frozen into an 'adiabatic' medium in which no heat transport is allowed, so that the material derivative DS/Dt of the specific entropy S is zero. In the presence of gravity such a medium supports internal gravity waves. There are cases for which the q.r. holds, and also exceptions. The second involves a magnetic field frozen into a perfectly-conducting medium. Such a medium supports various kinds of magnetohydrodynamic waves. Again, the rule holds in some cases but not in others.

[[Bottom para of p 2 of my letter to Peierls?]]

[[We should consider whether §3 should mention *inter alia* the parametric acoustic array (the only direct experimental evidence of physical reality of $\partial(lnc/\partial ln\rho)$, J. Fluid Mech. **106**, bot.p.343), since all the other evidence is theoretical.]]