# The Temporal-Residual-Mean Velocity. Part II: Isopycnal Interpretation and the Tracer and Momentum Equations

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ABSTRACT

Mesoscale eddies mix fluid parcels in a way that is highly constrained by the stratified nature of the fluid. The temporal-residual-mean (TRM) theory provides the link between the different views that are apparent from temporally averaging these turbulent flow fields in height coordinates and in density coordinates. Here the original TRM theory is modified so that it applies to unsteady flows. This requires a modification not only to the streamfunction (and hence the velocity vector) but also a specific interpretation of the density field; it is not the Eulerian-mean density. The TRM theory reduces the problem of parameterizing the eddy flux from three dimensions to two dimensions. The three-dimensional TRM velocity is shown to be the same as is obtained by averaging with respect to instantaneous density surfaces and the averaged conservation equations in height coordinates and in density coordinates are the same except for a nondivergent flux that is identified and explained. The TRM theory demonstrates that the tracers (such as salinity and potential temperature) that are carried by an eddyless ocean model must be interpreted as the thickness-weighted tracers that result from averaging in density coordinates.

The extra streamfunction of the temporal-residual-mean flow, termed the quasi-Stokes streamfunction, has a simple interpretation that proves valuable in developing plausible boundary conditions for this streamfunction: at any height *z*, the quasi-Stokes streamfunction is the contribution of temporal perturbations to the horizontal transport of water that is more dense than the density of the surface having time-mean height *z*. Importantly, the extra three-dimensional velocity derived from the quasi-Stokes streamfunction is not the bolus transport that arises when averaging in density coordinates. Therefore the Gent and McWilliams eddy parameterization scheme is not a parameterization of the bolus velocity but rather of the quasi-Stokes velocity of the temporal-residual-mean circulation. The physical interpretation of the quasi-Stokes streamfunction implies that it must be tapered smoothly to zero at the top and bottom of the ocean rather than having delta functions of velocity against these boundaries. The common assumption of downgradient flux of potential vorticity along isopycnals is discussed and it is shown that this does not sufficiently constrain the three-dimensional quasi-Stokes advection because only the vertical derivative of the quasi-Stokes streamfunction is specified. Near-boundary uncertainty in the potential vorticity fluxes translates into uncertainty in the depth-averaged heat flux. The horizontal TRM momentum equation is derived and leads to an alternative method for including the effects of eddies in eddyless models.

## 1. Introduction

The task of representing unresolved mesoscale eddy motions in a three-dimensional eddyless model is significantly more complicated than the corresponding problem under zonal averaging. There are two principal reasons for the added complexity. First, the mean flow field in a forward ocean model is three-dimensional and is of zeroth order in perturbation amplitude, in contrast to the two-dimensional-velocity field of the zonally averaged problem where the Eulerian-mean flow is of second order in perturbation amplitude. Second, one needs to account for unresolved fields in both space and time. The temporal-residual-mean (TRM) theory of the present paper addresses the first issue, while the second issue should be considered separately and will be addressed in a subsequent paper [see McDougall (1998) for an introduction to the problem of limited horizontal resolution].

In a previous paper (McDougall and McIntosh 1996, hereafter called TRM-I) we derived a temporally low-passed density conservation equation that had the property that, so long as the mean and the eddy density fields were steady, the mean advection velocity had a diapycnal component if and only if the instantaneous flow were diapycnal. The standard Eulerian-averaged equations do not have this property for either zonal or temporal averaging. Rather, the Eulerian-mean flow has a diapycnal component forced by the divergence of the eddy density flux even if the flow is instantaneously adiabatic at all times. This velocity of TRM-I was the sum of the Eulerian-mean velocity and a component that

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depended on the horizontal eddy density flux and the density variance. By analogy with the existing literature on the zonal-residual-mean flow, we named this mean velocity the temporal residual mean velocity since the averaging operator was temporal rather than zonal. Equally well, the acronym TRM could be interpreted as the ''three-dimensional residual mean'' because the TRM velocity is three-dimensional rather than two-dimensional in the case of zonal averaging.

If the mean density or the density variance evolves slowly in time, then the TRM velocity of McDougall and McIntosh (1996) generally has a diabatic component related to this change. The present paper overcomes this disadvantage by deriving a modified density conservation equation that is written for a modified density variable. Rather than using the average density at constant height as in TRM-I, we ensure that the mean height of the density surface coincides with the nominal height at which averaging is being performed, as recommended by de Szoeke and Bennett (1993). We note that the concept of having a mean conservation equation in which both the advection velocity and the quantity being advected are different to the Eulerian mean values is not unusual. In the generalized Lagrangian-mean approach of Andrews and McIntyre (1978), not only is the velocity the Lagrangian-mean velocity but also the tracer is the Lagrangian-mean tracer value.

In the conservation equation for the modified density variable, the turbulent flux of density is not simply the Reynolds-averaged density flux, but is a modified version of it. The present paper utilizes the density variance equation to show that this modified density flux can be decomposed into an advection of modified density by the quasi-Stokes velocity plus a nondivergent density flux (or equivalently, a skew-diffusive flux of density and a different nondivergent density flux). The challenge for the future is to find a sufficiently accurate parameterization for the quasi-Stokes streamfunction for use in eddyless ocean models. We regard the Gent and McWilliams (1990) procedure as the first such parameterization for the temporal-residual-mean circulation and note that the Gent and McWilliams (1990) procedure has been shown by Danabasoglu and McWilliams (1995) and Hirst and McDougall (1996) to greatly improve our ability to model the deep ocean. Since this paper presents an improved definition of the TRM velocity, the introduction section of McDougall and McIntosh (1996) is also applicable to the present paper, and we will not repeat that material here. Rather, the reader is encouraged to read the present paper in conjunction with that paper. In the present paper, as in TRM-I, the averaging operator is a low-pass averaging operator in time. The temporal averaging aspect can be understood as accounting for unresolved processes such as mesoscale eddies that are not present in eddyless ocean models, or perhaps submesoscale processes that are not resolved in eddy-permitting models.

In section 2 below a Taylor series analysis is used to provide the following physical interpretation of the quasi-Stokes (or eddy-induced) streamfunction. This streamfunction is the contribution of eddies to the horizontal transport of water of a certain density class, and this physical interpretation assists in applying boundary conditions to the quasi-Stokes streamfunction. This leads immediately to the physical interpretation of the horizontal TRM velocity as the thickness-weighted velocity of density coordinates. Section 5 shows that the diapycnal component of the TRM velocity is the same as would be obtained by averaging the diapycnal velocity following a given density surface. These results are used to show that the TRM approach is the way of representing in height coordinates the velocity and the conservation equations that would apply if one averaged the instantaneous flow in density coordinates. There are other ways of averaging the conservation equations such as the Lagrangian-mean approach and the Effective approach, but these are less suitable for our purposes, as discussed briefly by McDougall (1998).

This paper concentrates on theoretical features of residual-mean theory. We argue that the Gent and McWilliams (1990) scheme for advecting tracers in models is not equivalent to downgradient diffusion of thickness, despite this being the original justification of the scheme. The construction of the vertical component of the eddy-induced velocity assumes that the eddy-induced velocity is three-dimensionally nondivergent, which ensures that the eddy-induced velocity is not the bolus velocity because the three-dimensional bolus velocity is divergent (see section 10 below). This also is apparent in the diapycnal nature of the resolved-scale velocity in such ocean models (see section 9 below). In this way, the Gent et al. (1995) scheme can be seen as a parameterization of the quasi-Stokes velocity of the TRM theory. This realization provides the direction for improved parameterizations, and it also has implications for how we must interpret the model variables in such coarse-resolution models.

The Gent et al. (1995) implementation of the TRM circulation adds either a skew flux or a quasi-Stokes advection of tracer to the tracer conservation equations but generally leaves the momentum equations unaltered. We show that with errors that are cubic in perturbation amplitude, there is an alternative way that these TRM concepts may be implemented in an ocean model, namely by adding a horizontal stress to the horizontal momentum equations. Then the continuity, tracer, and momentum equations carry only one velocity variable: the TRM velocity.

## 2. Isopycnal interpretations of $\boldsymbol{\Psi}$ and $\overline{\mathbf{V}}^{\#}$

### *a. Horizontal transport of water denser than $\tilde{\gamma}$*

Here we follow the Taylor series approach of McIntosh and McDougall (1996) and consider the horizontal volume transport of fluid that is denser than a certain fixed density. In that paper the averaging operation was a zonal

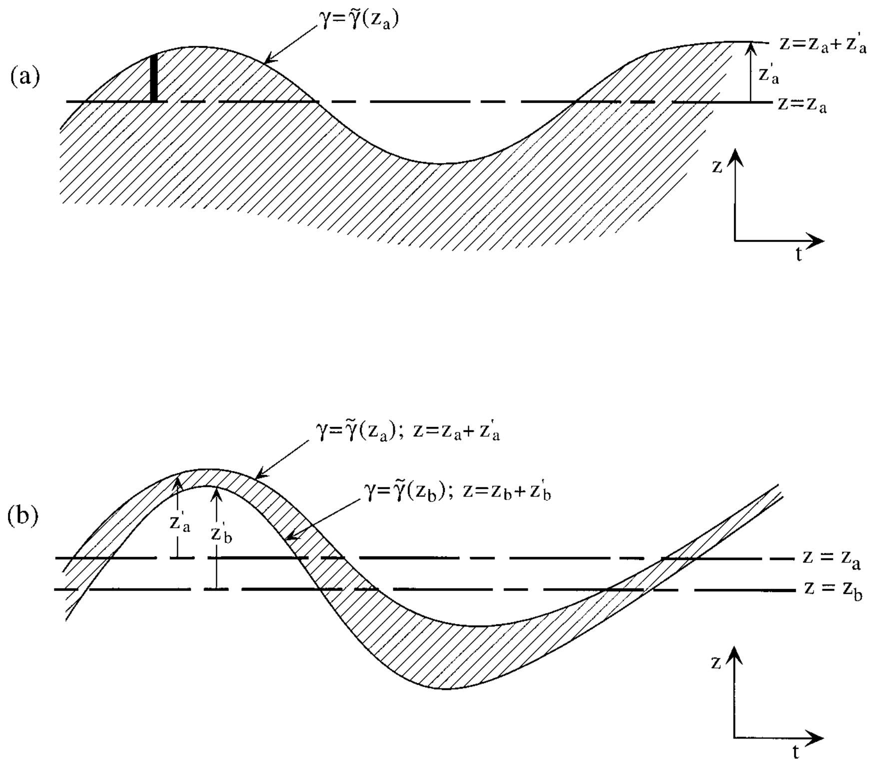


FIG. 1. (a) Sketch of the height of a $\tilde{\gamma}$ density surface as a function of time at a given latitude and longitude. The quasi-Stokes streamfunction $\boldsymbol{\Psi}$ is equal to the horizontal volume transport between the fixed height, $z = z_a$, and the instantaneous height of this density surface. This volume flux is equal to the temporal average of the perturbation height, $z'$, and the velocity appropriate to the column of fluid shown, viz. $(\mathbf{V} + \frac{1}{2}\mathbf{V}_z z')$, where both $\mathbf{V}$ and $\mathbf{V}_z$ are evaluated at the fixed height, $z = z_a$. It is this volume transport that must be added to the volume transport found by using the horizontal Eulerian-mean velocity in order to correctly estimate the transport of the shaded fluid that is more dense than $\tilde{\gamma}(z)$. (b) The thickness-weighted horizontal velocity of density coordinates, $\hat{\mathbf{V}}$, is defined to be the average horizontal velocity of the shaded fluid, and this is equal to the horizontal TRM velocity, $\overline{\mathbf{V}}^{\#} \equiv \overline{\mathbf{V}} + \boldsymbol{\Psi}_z$, with an error that is cubic in the amplitude of the temporal perturbations.

average, whereas here we are performing a temporal average at a fixed horizontal position.

Consider the density surface whose average height at a given horizontal location is $z_a$. This density surface we will label as $\tilde{\gamma}(z_a)$ and the instantaneous horizontal transport of water (per unit horizontal distance) that is denser than $\tilde{\gamma}(z_a)$ is given by (Fig. 1a):

$$\int_{-H}^{z_a+z_a'} \mathbf{V}(x, y, z, t)\, dz$$
$$= \int_{-H}^{z_a} \mathbf{V}(x, y, z, t)\, dz + \int_{z_a}^{z_a+z_a'} \mathbf{V}(x, y, z, t)\, dz, \quad (1)$$

where the bottom of the ocean is at $z = -H$ and $z_a'$ is the instantaneous displacement of the density surface from its mean height $z_a$. We assume that the water column is stably stratified. The right-hand side here is simply a convenient rearrangement of the integral into two parts. The horizontal velocity vector, $\mathbf{V} \equiv u\mathbf{i} + \upsilon\mathbf{j}$, is a function of space and time, as emphasized by the notation in (1). The water that is more dense than $\tilde{\gamma}(z_a)$ we call ''marked fluid'' as though it were colored differently to the water that is less dense than $\tilde{\gamma}(z_a)$.

Expanding the horizontal velocity in a vertical Taylor series about the fixed height, $z = z_a$, the last term in (1) may be written:

$$\int_{z_a}^{z_a+z_a'} \mathbf{V}\, dz = \int_{z_a}^{z_a+z_a'} \{\mathbf{V}(x, y, z_a, t) + \mathbf{V}_z(x, y, z_a, t)[z - z_a] + \cdots\}\, dz$$
$$= \mathbf{V}(x, y, z_a, t)z_a' + \frac{1}{2}\mathbf{V}_z(x, y, z_a, t)[z_a']^2 + O(\alpha^3), \quad (2)$$

where terms up to second order in perturbation quantities have been retained. The terminology $O(\alpha^3)$ indicates additional terms that are of cubic or higher order in perturbation quantities, and $\alpha$ is a general measure of perturbation amplitude. That is, the remaining terms in (2) each contain the product of at least three primed quantities. Forming the temporal average of (2) and Reynolds decomposing the horizontal velocity into mean and perturbation parts, we obtain two equivalent expressions:

$$\overline{\int_{z_a}^{z_a+z_a'} \mathbf{V}\,dz} = \overline{\mathbf{V}'z_a'} + \frac{1}{2}\overline{\mathbf{V}}_z\overline{(z_a')^2} + O(\alpha^3)$$

$$= \overline{\left(\mathbf{V} + \frac{1}{2}\mathbf{V}_z z_a'\right)z_a'} + O(\alpha^3). \quad (3)$$

It is emphasized that the instantaneous velocity $\mathbf{V}$, the instantaneous velocity shear $\mathbf{V}_z$, the perturbation velocity $\mathbf{V}'$, and the mean shear $\overline{\mathbf{V}}_z$ on the right of (3) are all evaluated at the constant height $z_a$.

We define the perturbation volume transport of (3) to be a two-dimensional streamfunction, which we call the quasi-Stokes streamfunction, that is,

$$\boldsymbol{\Psi}(z_a) \equiv \overline{\int_{z_a}^{z_a+z_a'} \mathbf{V}\,dz}. \quad (4a)$$

The perturbation height of a $\tilde{\gamma}$ surface, $z'$, is related to the density perturbation at a fixed height, $\gamma'$, by $z' = -\gamma'|_z/\overline{\gamma}_z + O(\alpha^2)$ [see (10) below]. Hence (3) can be written

$$\boldsymbol{\Psi} = -\frac{\overline{\mathbf{V}'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{\mathbf{V}}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right) + O(\alpha^3), \quad (4b)$$

where $\overline{\phi} \equiv \frac{1}{2}\overline{(\gamma')^2}$ is half the density variance at height $z$. This equation shows that the quasi-Stokes streamfunction $\boldsymbol{\Psi}$ can be expressed in terms of the temporal correlations $\overline{\mathbf{V}'\gamma'}$ and $\overline{\phi}$ that are evaluated at constant height, together with an error term that is cubic in perturbation amplitude. We will later show that the Taylor series expansion beaks down near the top and bottom of the ocean where the quasi-Stokes streamfunction (4a) is actually zero even though the approximate expression (4b) would indicate otherwise. The average transport of fluid that is denser than $\tilde{\gamma}(z_a)$ is obtained by taking the temporal average of (1), giving the exact result

$$\overline{\int_{-H}^{z_a+z_a'} \mathbf{V}\,dz} = \int_{-H}^{z_a} \overline{\mathbf{V}}\,dz + \boldsymbol{\Psi}(z_a). \quad (5)$$

The first term on the right of (5) is the straightforward transport estimate based on the Eulerian-mean velocity, while the quasi-Stokes streamfunction $\boldsymbol{\Psi}(z_a)$ accounts for the balance of the total horizontal transport of marked fluid. That is, $\boldsymbol{\Psi}$ is the contribution of perturbations (or eddies) to the horizontal flux of fluid that is denser than $\tilde{\gamma}$. This interpretation of $\boldsymbol{\Psi}$ is also evident from the second line of (3). There the quasi-Stokes streamfunction $\boldsymbol{\Psi}$ is seen to be the average of the instantaneous velocity at the midheight $z = z_a + \frac{1}{2}z_a'$, namely $(\mathbf{V} + \frac{1}{2}\mathbf{V}_z z_a')$, multiplied by $z_a'$, the height of the small water column between the fixed height $z_a$ and the instantaneous height of the density surface. The product, $(\mathbf{V} + \frac{1}{2}\mathbf{V}_z z_a')z_a'$, is then the instantaneous value of the volume transport between the fixed height and the height of the $\tilde{\gamma}(z_a)$ surface. The time average of this volume transport must be added to the Eulerian-mean transport in order to obtain the total transport of marked fluid.

### *b. The mean velocity between pairs of $\tilde{\gamma}$ surfaces*

In the isopycnal reference frame one is interested in the transport between a pair of isopycnals. This transport is obtained by taking the difference between two expressions of the form (5) (see Fig. 1b):

$$\overline{\int_{z_b+z_b'}^{z_a+z_a'} \mathbf{V}\,dz} = \int_{z_b}^{z_a} \overline{\mathbf{V}}\,dz + \boldsymbol{\Psi}(z_a) - \boldsymbol{\Psi}(z_b). \quad (6)$$

The left-hand side of this equation is the thickness-weighted mean horizontal transport between the density surfaces $\tilde{\gamma}(z_b)$ and $\tilde{\gamma}(z_a)$. Dividing (6) by $(z_a - z_b)$ and taking the limit $(z_a - z_b) \to 0$ gives

$$\hat{\mathbf{V}} \equiv \tilde{\mathbf{V}} + \overline{\mathbf{V}'|_{\tilde{\gamma}}(z_a')_z} = \overline{\mathbf{V}} + \boldsymbol{\Psi}_z \equiv \overline{\mathbf{V}}^{\#}. \quad (7)$$

Equation (7) provides a link between time-mean horizontal velocities in the density and height reference frames. On the left-hand side of (7) the thickness-weighted mean horizontal velocity $\hat{\mathbf{V}}$ has been expressed as a function of density with $\tilde{\mathbf{V}} \equiv \overline{\mathbf{V}|_{\tilde{\gamma}}}$ and $\mathbf{V}'|_{\tilde{\gamma}}$ being the mean and perturbation velocity components evaluated following the $\tilde{\gamma}$ density surface. (We use the tilde as a general notation for time-mean at constant density.) The term $\mathbf{V}^B \equiv \overline{\mathbf{V}'|_{\tilde{\gamma}}(z_a')_z}$ is the correlation between the perturbation velocity along density surfaces and the relative perturbation thickness between adjacent density surfaces; it was coined the "bolus" velocity by Rhines (1982).

The right-hand side of (7) is written in terms of quantities evaluated in height coordinates, namely, $\overline{\mathbf{V}}$ and $\boldsymbol{\Psi}$. This equation shows that the horizontal component of the TRM velocity, $\overline{\mathbf{V}}^{\#} \equiv \overline{\mathbf{V}} + \boldsymbol{\Psi}_z$, is equivalent to the thickness-weighted mean velocity in density coordinates, $\hat{\mathbf{V}}$. This result is exact when the definition (4a) is used for the quasi-Stokes streamfunction but, when the approximate expression (4b) is used, there is an error that is cubic in perturbation quantities. Note that the expression for the streamfunction, (4b), is not the same as that found in McDougall and McIntosh (1996) [their Eq. (11)]. The reason for this difference is due to the different density surfaces that are being considered. In McDougall and McIntosh (1996) we considered the Eulerian-averaged density surface, $\overline{\gamma}$, that is, the density surface whose perturbation density (at fixed height) was zero. By contrast, here we have chosen the density surface whose perturbation height (at constant density) is zero. To develop the TRM theory for this new density variable we first need to derive an expression for $\tilde{\gamma}$ in terms of quantities evaluated at fixed height.

## 3. The modified density, $\tilde{\gamma}$

The modified density surface, $\gamma = \tilde{\gamma}(z_a)$, appropriate to height $z = z_a$ is defined to have an instantaneous height of $z = z_a + z'_a$ such that the time-mean height is $z_a$ and the temporal average of the perturbation height is zero; that is, $\overline{z'_a} \equiv 0$ (see Fig. 1a). The density on this surface is obtained in terms of quantities evaluated at $z = z_a$ using the vertical Taylor series:

$$\gamma(z_a + z'_a) = \tilde{\gamma}(z_a) = \gamma(z_a) + \gamma_z z'_a + \frac{1}{2}\gamma_{zz}(z'_a)^2 + O(\alpha^3). \quad (8)$$

Taking a time average of this equation yields (using $\overline{z'_a} \equiv 0$)

$$\tilde{\gamma}(z_a) = \overline{\gamma}(z_a) + \overline{\gamma'_z z'_a} + \frac{1}{2}\overline{\gamma}_{zz}\overline{(z'_a)^2} + O(\alpha^3), \quad (9)$$

while the perturbation part of (8), applied to any height now (hence we drop the subscript $a$) shows that

$$z' = -\gamma'|_z/\overline{\gamma}_z + O(\alpha^2). \quad (10)$$

Substituting (10) into (9) gives an expression for the modified density variable, $\tilde{\gamma}$, in terms of quantities evaluated at fixed height:

$$\tilde{\gamma} = \overline{\gamma} - \left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)_z + O(\alpha^3), \quad (11)$$

where $\overline{\phi} \equiv \frac{1}{2}\overline{(\gamma')^2}$ is half the density variance at height $z$. This result is equivalent to realizing that the average height of the Eulerian-mean density surface, $\overline{\gamma}(z_a)$, is

$$z = z_a + (1/\overline{\gamma}_z)(\overline{\phi}/\overline{\gamma}_z)_z + O(\alpha^3).$$

As an example of the difference between these two versions of an average density, consider the situation where the vertical heaving of mesoscale eddies is as large as 150 m such as occurs in the mesoscale frequency band in the Southern Ocean (H. Phillips 2000, personal communication) in a typical vertical density gradient of $10^{-3}$ kg m$^{-4}$. This gives a value of half the density variance, $\overline{\phi}$, of $10^{-2}$ kg$^2$ m$^{-6}$ and, if $\overline{\phi}/\overline{\gamma}_z$, is assumed to vary in the vertical by its own magnitude in 1000 m, the density difference, $\tilde{\gamma} - \overline{\gamma}$, is 0.01 kg m$^{-3}$ and the associated height difference between these surfaces is 10 m. The use of the modified density rather than the Eulerian-mean density redresses the criticism of Smith (1999) of level coordinate models that they are "fundamentally unable to . . . conserve mass between ensemble-mean isopycnals."

## 4. Derivation of the TRM density equation

Since the modified density $\tilde{\gamma}(z)$ has the desirable property that its perturbation height is on average zero, we explore the consequences of writing the mean density conservation equation for $\tilde{\gamma}$ rather than for the Eulerian-mean density $\overline{\gamma}$. It will be found that in so doing, the temporal-residual-mean circulation of McDougall and McIntosh (1996) must be modified in such a way that the quasi-Stokes streamfunction becomes (4) above.

In developing the theory it is convenient to first deal with the density conservation equation and to initially ignore any nonlinearity in the equation of state. In practice ocean models do not carry a conservation equation for density but rather for conservative tracers such as salinity and potential temperature. Diapycnal mixing in the ocean is relatively weak, so it proves very useful to develop the residual-mean theory for the density conservation equation and to treat density as a conservative variable for this purpose. The instantaneous density conservation equation is

$$D_t\gamma \equiv \gamma_t + \mathbf{U}\cdot\nabla\gamma = Q, \quad (12)$$

where $\mathbf{U}$ is the instantaneous three-dimensional velocity, $\mathbf{U} = \mathbf{V} + \mathbf{k}w$, and $Q$ represents a source term (such as solar radiation), as well as the flux divergence of unresolvable diapycnal processes. The velocity is assumed to obey the Boussinesq form of the continuity equation, namely $\nabla\cdot\mathbf{U} = 0$.

A Reynolds decomposition in time is applied to each variable, breaking its instantaneous value into a low-passed part and a perturbation part that is defined to be the difference between the instantaneous value and the low-passed value (that is $\varphi \equiv \overline{\varphi} + \varphi'$). The standard conservation equation for the Eulerian-mean density and the conservation equation for density variance are found from (12) to be

$$\overline{\gamma}_t + \nabla\cdot(\overline{\mathbf{U}}\,\overline{\gamma}) = \overline{Q} - \nabla\cdot(\overline{\mathbf{U}'\gamma'}) \quad \text{and} \quad (13)$$

$$\overline{D}_t\overline{\phi} = \overline{Q'\gamma'} - \overline{\mathbf{U}'\gamma'}\cdot\nabla\overline{\gamma} + O(\alpha^3), \quad (14)$$

where $\overline{\phi} \equiv \frac{1}{2}\overline{(\gamma')^2}$ is again half the density variance measured at a fixed height. Here we have assumed the standard properties of Reynolds averaging, for example, that $\overline{\overline{\vartheta}} = \overline{\vartheta}$ and $\overline{\theta'\overline{\vartheta}} = 0$; however, this is only strictly true of ensemble averaging, not low-passed temporal averaging (Davis 1994). These properties are approximately true if a spectral gap exists between the slowly varying mean flow and the higher-frequency perturbation motions.

The mean density equation (13) can be rearranged as a conversation equation for the modified density $\tilde{\gamma}$ using (11) to find

$$\tilde{\gamma}_t + \nabla\cdot(\overline{\mathbf{U}}\tilde{\gamma}) = \overline{Q}^{\#} - \nabla\cdot\mathbf{F}^M + O(\alpha^3), \quad (15)$$

where the modified density flux, $\mathbf{F}^M$ is defined by

$$\mathbf{F}^M \equiv \overline{\mathbf{U}'\gamma'} + \overline{\mathbf{U}}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)_z + \mathbf{k}\left\{\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)_t - \frac{\overline{Q'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{Q}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right\} \quad (16)$$

and the modified density source term is

$$\overline{Q}^{\#} \equiv \overline{Q} + \left[-\frac{\overline{Q'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{Q}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right]_z. \quad (17)$$

The $Q$ terms in brackets in (17) cancel in (15) with the divergence of the last two terms of (16), but the terms are deliberately written in this fashion because we shall soon show that the total effect of the diabatic source terms on the $\tilde{\gamma}$ equation is contained in $\overline{Q}^{\#}$, while $-\nabla\cdot\mathbf{F}^M$ represents the forcing of the $\tilde{\gamma}$ equation by the adiabatic stirring of the eddy motions.

The density variance equation (14) is now used to eliminate $\overline{w'\gamma'}$ from the expression, (16) and, after considerable algebra and using the continuity equation, $\nabla_H\cdot\overline{\mathbf{V}} + \overline{w}_z = 0$, the modified density flux $\mathbf{F}^M$ can be expressed in the following two convenient forms:

$$\mathbf{F}^M = \tilde{\gamma}\mathbf{U}^+ + \mathbf{M} + O(\alpha^3) = -\mathbf{A}\nabla\tilde{\gamma} + \mathbf{N} + O(\alpha^3). \quad (18)$$

The quasi-Stokes velocity, $\mathbf{U}^+$, is defined in terms of the quasi-Stokes streamfunction of (4a) by

$$\mathbf{U}^+ \equiv \boldsymbol{\Psi}_z - \mathbf{k}(\nabla_H\cdot\boldsymbol{\Psi}) = \nabla\times(\boldsymbol{\Psi}\times\mathbf{k}) \quad (19)$$

and the density flux, $\mathbf{M}$, is nondivergent and is defined by

$$\mathbf{M} \equiv \left(-\boldsymbol{\Psi}\tilde{\gamma} + \frac{\overline{\mathbf{V}}\,\overline{\phi}}{\overline{\gamma}_z}\right)_z - \mathbf{k}\nabla_H\cdot\left(-\boldsymbol{\Psi}\tilde{\gamma} + \frac{\overline{\mathbf{V}}\,\overline{\phi}}{\overline{\gamma}_z}\right) = \nabla\times\left[\left(-\boldsymbol{\Psi}\tilde{\gamma} + \frac{\overline{\mathbf{V}}\,\overline{\phi}}{\overline{\gamma}_z}\right)\times\mathbf{k}\right]. \quad (20)$$

The antisymmetric matrix $\mathbf{A}$ in (18) is defined in terms of the two components of the quasi-Stokes streamfunction, $\boldsymbol{\Psi} = (\Psi^x, \Psi^y)$, as

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 0 & \Psi^x \\ 0 & 0 & \Psi^y \\ -\Psi^x & -\Psi^y & 0 \end{bmatrix} \quad (21)$$

and the nondivergent density flux $\mathbf{N}$ is given by

$$\mathbf{N} \equiv \left(\frac{\overline{\mathbf{V}}\,\overline{\phi}}{\overline{\gamma}_z}\right)_z - \mathbf{k}\nabla_H\cdot\left(\frac{\overline{\mathbf{V}}\,\overline{\phi}}{\overline{\gamma}_z}\right) = \nabla\times\left[\left(\frac{\overline{\mathbf{V}}\,\overline{\phi}}{\overline{\gamma}_z}\right)\times\mathbf{k}\right]. \quad (22)$$

The rearrangement of the Reynolds-averaged density equation (13) into the form (15)–(16) and the subsequent two simplified forms of the modified density flux, (18), lie at the heart of this paper and are the essential mathematical manipulations of the TRM theory. The same techniques will later be applied to the tracer conservation equation, and we will then allow the equation of state to be fully nonlinear. Substituting the two alternative expressions, (18), into the mean density conservation equation (15) gives the following two versions of the mean density equation,

$$\begin{aligned}\tilde{\gamma}_t + \nabla\cdot(\overline{\mathbf{U}}\tilde{\gamma}) &= \overline{Q}^{\#} - \nabla\cdot(\mathbf{U}^+\tilde{\gamma}) - \nabla\cdot\mathbf{M} + O(\alpha^3) \\ &= \overline{Q}^{\#} + \nabla\cdot(\mathbf{A}\nabla\tilde{\gamma}) - \nabla\cdot\mathbf{N} + O(\alpha^3). \quad (23)\end{aligned}$$

The cubic-order error terms here are not needed if the quasi-Stokes streamfunction is evaluated using the exact expression, (4a), but are needed if the approximate expression (4b) is used. By the word ‘‘exact’’ we mean that, if we had access to the exact Eulerian-mean velocity and if the quasi-Stokes streamfunction was equal to the exact expression (4a), then (23) is exactly the same conservation statement for density as is found by averaging in density coordinates (except for any cubic-order terms in the difference between $\overline{Q}^{\#}$ and the exact thickness-weighted form of $Q$, $\hat{Q}$). In the context of a forward numerical model, if we can regard the velocity that is carried by the model's momentum equations as the Eulerian-mean velocity, the use of (4a) for the quasi-Stokes streamfunction in the density equation means that the density that is carried by the model can be interpreted exactly as $\tilde{\gamma}$ so that the relationship between the averaging in density coordinates and in height coordinates is exact.

In proceeding from the original Reynolds-averaged density equation (13) through (15) to (23) we have so far not dropped any nondivergent fluxes so that, for example, the horizontal fluxes of density per unit height are identical in (13), in (15), and in both lines of (23); that is [noting that $-\mathbf{A}\nabla\tilde{\gamma} = -\tilde{\gamma}_z\boldsymbol{\Psi} + \mathbf{k}(\boldsymbol{\Psi}\cdot\nabla_H\tilde{\gamma})$],

$$\overline{\mathbf{V}}\,\overline{\gamma} + \overline{\mathbf{V}'\gamma'} = \overline{\mathbf{V}}\tilde{\gamma} + \mathbf{V}^+\tilde{\gamma} + \left(-\boldsymbol{\Psi}\tilde{\gamma} + \frac{\overline{\mathbf{V}}\,\overline{\phi}}{\overline{\gamma}_z}\right)_z = \overline{\mathbf{V}}\tilde{\gamma} - \boldsymbol{\Psi}\tilde{\gamma}_z + \left(\frac{\overline{\mathbf{V}}\,\overline{\phi}}{\overline{\gamma}_z}\right)_z. \quad (24)$$

By discarding the nondivergent flux, $\mathbf{M}$, from (23) one finds that the horizontal flux of density in this equation is $\tilde{\gamma}(\overline{\mathbf{V}} + \mathbf{V}^+)$, which is the same horizontal flux of density that is obtained by thickness-weighted averaging the instantaneous density flux, $\mathbf{V}\gamma$, in density coordinates [this can be shown by substituting $\mathbf{V}\gamma$ in place of $\mathbf{V}$ in the Taylor series analysis of section 2 or by substituting $\mathbf{V}\gamma$ in place of $A$ in Eq. (A5) of the appendix]. We will prove in section 5 that the three-dimensional velocity, $\overline{\mathbf{U}} + \mathbf{U}^+$, of (23) is the same as is found by averaging the conservation equations in density coordinates. Here we wish to emphasize that the key to making the density equation (23) behave as though it has been averaged in density coordinates is to simply recognize that these different averaged density equations differ by the nondivergent density flux **M**, which has no effect on the evolution of density.

It is clear from (24) that there are at least three ways in which one can evaluate the horizontal flux of density across an ocean section up to a fixed height in height coordinates. One could vertically and horizontally integrate $\overline{\mathbf{V}}\,\overline{\gamma} + \overline{\mathbf{V}'\gamma'}$ to find the straightforward height-coordinate estimate. (One complication with this approach is that, as we argue below, coarse-resolution ocean models actually carry $\tilde{\gamma}$ as their density variable rather than the Eulerian-averaged density $\overline{\gamma}$, so one would have to estimate $\overline{\gamma}$ in order to do the calculation with a coarse-resolution model.) Alternatively, by ignoring the nondivergent flux **M** in (23) one can integrate $\overline{\mathbf{V}}\tilde{\gamma} + \mathbf{V}^{+}\tilde{\gamma}$ across the section up to a fixed height. In so doing one has effectively done a temporal average of a vertical integration in density coordinates up to the undulating $\tilde{\gamma}$ density surface at each cast. The third alternative is to ignore the nondivergent flux **N** in (23) and to integrate $\overline{\mathbf{V}}\tilde{\gamma} - \boldsymbol{\Psi}\tilde{\gamma}_z$ across the ocean section. Here $-\boldsymbol{\Psi}\tilde{\gamma}_z$ is the horizontal component of the skew-diffusion of density. The point here is to note that, quite apart from the usual issues of having to arbitrarily choose a reference potential temperature in the calculation of heat flux, now the presence of the nondivergent fluxes, **M** and **N**, further complicate the issue and demonstrate that the concept of a heat flux is to some extent arbitrary until one integrates over a complete ocean section.

The density variance equation (14) can be written in terms of the modified density flux $\mathbf{F}^M$ as (subscripts denote differentiation)

$$\overline{D}_t\overline{\phi} = -\mathbf{F}^M\cdot\nabla\tilde{\gamma} + \left\{\overline{\gamma}_z\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)_t - \overline{\gamma}_t\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)_z + \left(\frac{\overline{Q}\,\overline{\phi}}{\overline{\gamma}_z}\right)_z\right\} + O(\alpha^3). \qquad (25)$$

Apart from the three terms within braces (the two unsteady terms and the diabatic term), it is apparent that the advection of density variance by the mean flow is only nonzero when the modified density flux has a component through the density surface, as illustrated in Figs. 2a and 2b. Furthermore, the vector decomposition, (18), shows that the diapycnal component of $\mathbf{F}^M$ is due only to the nondivergent flux **N** since the skew flux is, by definition, directed in the density surface so that $(\mathbf{A}\nabla\tilde{\gamma})\cdot\nabla\tilde{\gamma} = 0$ (see Fig. 2c). That is, apart from the diabatic and unsteady terms, we have $\overline{D}_t\overline{\phi} = -\mathbf{N}\cdot\nabla\tilde{\gamma} + O(\alpha^3)$. In the same three-dimensional turbulent situation as we are considering, Marshall and Shutts (1981) have found that the horizontal advection of variance is balanced by a nondivergent two-dimensional flux. This finding was subject to the assumption that the mean flow in the horizontal plane follows the mean density contours. Without having to invoke such

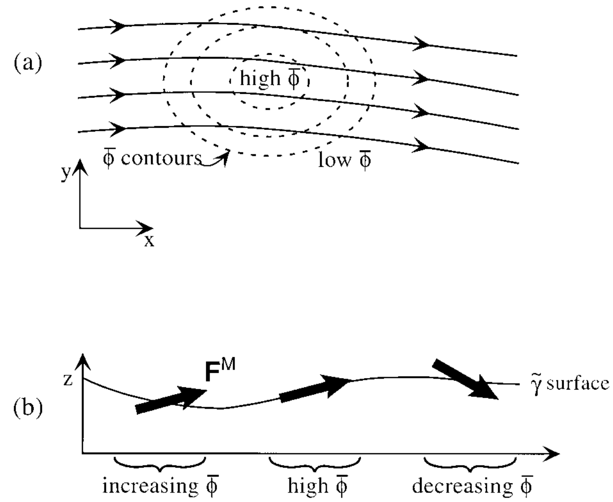


FIG. 2. As the mean flow moves through a region of increasing density variance (a) the modified density flux $\mathbf{F}^M$ must have an upward-directed diapycnal component (b) [where it is assumed that the unsteady and diabatic terms in (25) do not dominate]. This diapycnal component of $\mathbf{F}^M$ is supplied by only the nondivergent flux **N** since the skew-flux is directed in the density surface (c). Panel c also indicates the construction of the modified density flux $\mathbf{F}^M$ according to its definition (16). The TRM decomposition of the modified density flux $\mathbf{F}^M$ into the two parts, $-\mathbf{A}\nabla\tilde{\gamma}$ and **N**, is fascinating as it is only the skew-flux that affects the mean density equation while it is only the nondivergent flux that affects the density variance equation. (d) Just the horizontal components of the full three-dimensional vectors that are sketched in (c). (e) the decomposition of the modified density flux $\mathbf{F}^M$ into the advective part, $\tilde{\gamma}\mathbf{U}^{+}$, and the nondivergent part, **M**, while (f) shows the horizontal components of this decomposition.

an assumption, the present result generalizes Marshall and Shutt's finding to three dimensions, involving the advection of variance by the three-dimensional mean flow, $\overline{D}_t\overline{\phi}$, and the nondivergent part, **N**, of the three-dimensional modified density flux. However, this result appears to be of no real use but rather is in the nature of a truism because neither side of $\overline{D}_t\overline{\phi} = -\mathbf{N}\cdot\nabla\tilde{\gamma} + O(\alpha^3)$ involves the eddy density flux.

The vector decomposition of the modified density flux, $\mathbf{F}^M$, into the advective component, $\tilde{\gamma}\mathbf{U}^{+}$, and the nondivergent flux, **M**, is sketched in Figs. 2e and 2f. Note that the quasi-Stokes velocity, $\mathbf{U}^{+}$, is in general not directed along the isopycnal surface but rather has a substantial diapycnal component.

The top line of (23) can be written more compactly as

$$\tilde{\gamma}_t + \nabla\cdot(\overline{\mathbf{U}}^{\#}\tilde{\gamma}) = \overline{Q}^{\#} + O(\alpha^3), \qquad (26)$$

where the nondivergent flux, **M**, has been ignored and the TRM velocity has been defined as the sum of the Eulerian-mean velocity and the quasi-Stokes velocity, $\overline{\mathbf{U}}^{\#} \equiv \overline{\mathbf{U}} + \mathbf{U}^{+}$. Equation (26) shows that, when the flow is adiabatic ($\overline{Q}^{\#} = 0$), the TRM velocity has no component through $\tilde{\gamma}$ surfaces to third order in perturbation quantities since $\overline{D}^{\#}_t\tilde{\gamma} \equiv \tilde{\gamma}_t + \overline{\mathbf{U}}^{\#}\cdot\nabla\tilde{\gamma} = O(\alpha^3)$. This contrasts sharply with the Eulerian-averaged equation, (13), where the advection of $\overline{\gamma}$ by the Eulerian-mean flow,

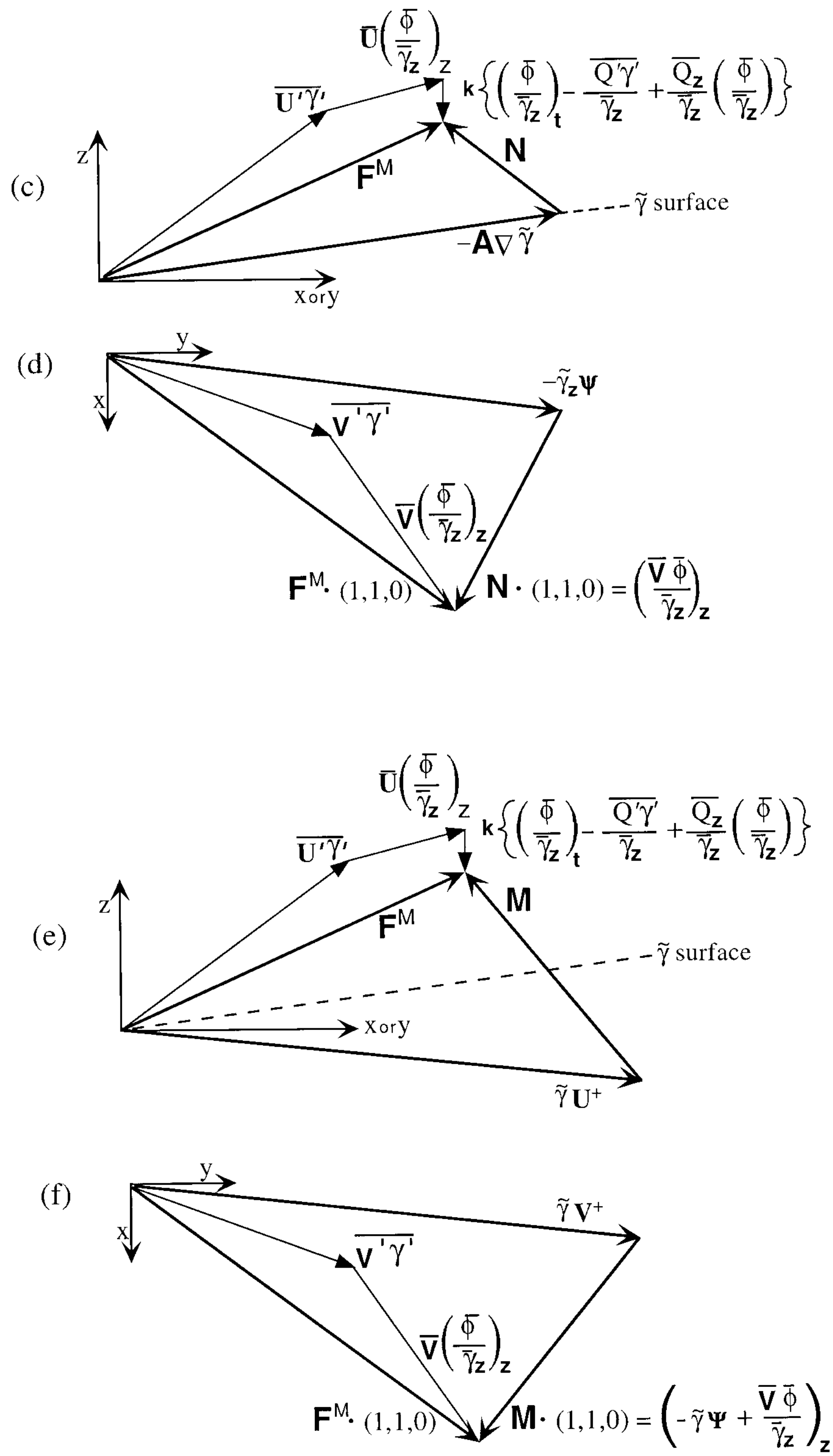


FIG. 2. (*Continued*)

$$\overline{D}_t\overline{\gamma} \equiv \overline{\gamma}_t + \overline{\mathbf{U}} \cdot \nabla\overline{\gamma},$$

is balanced by the divergence of the three-dimensional flux vector, $\nabla \cdot (\overline{\mathbf{U}'\gamma'})$, which is quadratic in perturbation quantities. By comparing (15) and (26) it is apparent that (up to a nondivergent density flux) the modified density flux $\mathbf{F}^M$ represents the forcing of the $\tilde{\gamma}$ equation (15) by the adiabatic stirring of the eddy motions. The TRM theory has revealed the following attributes of the modified density flux, $\mathbf{F}^M$ (see Fig. 2):

- $\mathbf{F}^M$ can be regarded as the sum of an advective flux, $\tilde{\gamma}\mathbf{U}^+$, plus some nongradient-terms, $\mathbf{M}$, which are nondivergent and so do not affect the mean density equation
- $\mathbf{F}^M$ can be also regarded as the sum of a skew flux, $-\mathsf{A}\nabla\tilde{\gamma}$, plus some nongradient-terms, $\mathbf{N}$, which are nondivergent and so do not affect the mean density equation
- with this decomposition of $\mathbf{F}^M$, its diapycnal component is due only to the nondivergent flux $\mathbf{N}$
- the skew flux, $-\mathsf{A}\nabla\tilde{\gamma}$, does not contribute to the density variance equation.

The above derivation of the TRM density equation (23) or (26) has involved various manipulations of (i) the mean density equation, (ii) the density variance equation, and (iii), the continuity equation, followed by a judicious neglect of one of two different nondivergent fluxes. Apart from simply writing these equations in different forms, the important things that have been achieved are (i) the realization that the relevant density variable is not the Eulerian-mean density but the modified density, (ii) the establishment of the physical meaning of the quasi-Stokes streamfunction as the contribution of eddies to the flux of water that is more dense than the modified density, and (iii) the reduction of the task of parameterization from the three-dimensional task in the Reynolds-averaged mean density equation (13) to the two-dimensional task of parameterizing the quasi-Stokes streamfunction, (4a). Moreover, if the Eulerian-mean velocity is regarded as known (from the momentum equations), an accurate parameterization of the quasi-Stokes streamfunction means that (23) and (26) are exact and are not restricted to being only accurate to cubic order in perturbation amplitude.

Gille and Davis (1999) have analyzed an eddy-resolving primitive equation model of a zonal channel with respect to the Eulerian-mean density $\overline{\gamma}$ rather than the modified density $\tilde{\gamma}$ and they used only the first term in the expression, (4b), for the quasi-Stokes streamfunction. Analyzing model data in this way leaves the unwanted term

$$(-\overline{\mathbf{U}'\gamma'} \cdot \nabla\overline{\gamma}/\overline{\gamma}_z)_z \qquad (27)$$

as an additional forcing in the mean density equation, and Gille and Davis (1999) conclude that this term is too large to ignore and needs to be parameterized along with the quasi-Stokes streamfunction. The present paper shows that this extra forcing term does not need to be parameterized because it does not arise when the full residual-mean transformation is performed so that the density of the eddyless forward model is interpreted as the modified density, $\tilde{\gamma}$. Rather, all that needs to be parameterized is the quasi-Stokes streamfunction.

## 5. Averaging the density equation in density coordinates

We have shown in (7) above that the horizontal components of the TRM velocity are the same as the thickness-weighted velocity of density coordinates; that is, $\hat{\mathbf{V}} = \overline{\mathbf{V}}^\# + O(\alpha^3)$. In this section we will show that the diapycnal component of the TRM velocity, $\overline{e}^\#$, is the same as the temporal average of the diapycnal velocity when averaged on a given density surface, $\tilde{e}$. This ensures that all three components of the TRM velocity correspond to the appropriately averaged velocity components in density coordinates and that the TRM density conservation equation (26) is the way of expressing in height coordinates the average density balance that applies when averaging in density coordinates.

First, we need a general expression for the thickness-weighted value of any quantity. This is obtained using a Taylor series approach as summarized in appendix A, where it is found that the thickness-weighted value $\hat{A}$ of any quantity $A$ obtained by averaging $A$ between a pair of closely spaced $\tilde{\gamma}$ surfaces is

$$\hat{A} \equiv \tilde{\gamma}_z\overline{\left(\frac{A}{\gamma_z}\right)\Big|_{\tilde{\gamma}}} \equiv \tilde{A} + \overline{A'|_{\tilde{\gamma}}\, z'_z}$$

$$= \overline{A} + \left(-\frac{\overline{A'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{A}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right)_z + O(\alpha^3), \qquad (28)$$

where $\tilde{A}$ is the value averaged on the density surface (not thickness-weighted) and $\overline{A'|_{\tilde{\gamma}} z'_z}$ is due to the correlations between perturbations evaluated on a density surface and the thickness between two closely spaced surfaces. As an example, with $A$ equal to the horizontal velocity $\mathbf{V}$, (28) shows that the thickness-weighted average velocity in isopycnal coordinates is simply $\overline{\mathbf{V}}^\#$, to third order in perturbation amplitude, as found earlier [Eq. (7)] by tracking the flux of water of certain density classes. Another interesting example is found with $A$ equal to $\gamma$ in (28), showing that the thickness-weighted mean density $\hat{\gamma}$ is the same as $\tilde{\gamma}$ (since $\gamma'|_{\tilde{\gamma}}$ is zero) and the right-hand side of (28) reduces to (11).

The instantaneous conservation statement for density is written as $D_t\gamma = Q$, where $Q$ represents source/sink terms and unresolvable mixing processes. If the material derivative is expressed with respect to density coordinates as $\gamma_t|_\gamma + \mathbf{V} \cdot \nabla_\gamma\gamma + e\gamma_z$, the temporal and isopycnal density gradient terms vanish, leaving a simple equation for the instantaneous value of the diapycnal velocity: $e = Q/\gamma_z$. The time mean of this equation following the density surface is $\tilde{e} \equiv \overline{(Q/\gamma_z)|_{\tilde{\gamma}}}$ and our thickness-weighting result, (28), is used to show that

$$\tilde{e} \equiv \overline{\left(\frac{Q}{\gamma_z}\right)\bigg|_{\tilde{\gamma}}} \equiv \frac{\hat{Q}}{\tilde{\gamma}_z} = \frac{\overline{Q}^{\#}}{\tilde{\gamma}_z} + O(\alpha^3), \qquad (29)$$

where the appropriate time-mean source/sink term, $\overline{Q}^{\#}$, was defined in (17) above. It is now apparent that this TRM source/mixing term, $\overline{Q}^{\#}$, is simply the thickness-weighted source/mixing term averaged in density coordinates, to $O(\alpha^3)$.

Writing the TRM density conservation equation (26) with respect to $\tilde{\gamma}$ coordinates as

$$\overline{D}_t^{\#}\tilde{\gamma} = \tilde{\gamma}_t|_z + \overline{\mathbf{V}}^{\#}\cdot\nabla_H\tilde{\gamma} + \overline{w}^{\#}\tilde{\gamma}_z$$
$$= \tilde{\gamma}_t|_{\tilde{\gamma}} + \overline{\mathbf{V}}^{\#}\cdot\nabla_{\tilde{\gamma}}\tilde{\gamma} + \overline{e}^{\#}\tilde{\gamma}_z = \overline{Q}^{\#} + O(\alpha^3), \qquad (30)$$

it is clear that this simplifies to $\overline{e}^{\#}\tilde{\gamma}_z = \overline{Q}^{\#} + O(\alpha^3)$. Comparison with (29) proves the equivalence [to $O(\alpha^3)$] of the TRM diapycnal velocity $\overline{e}^{\#}$ and the average diapycnal velocity $\tilde{e}$ through the $\tilde{\gamma}$ density surface; that is,

$$\tilde{e} = \overline{e}^{\#} + O(\alpha^3). \qquad (31)$$

We conclude from (7) and (31) that both the horizontal and the diapycnal components of the TRM velocity $\overline{\mathbf{U}}^{\#}$ are the same as those obtained by averaging the instantaneous flow with respect to $\tilde{\gamma}$ surfaces [within cubic errors in perturbation amplitude if the approximate expression (4b) is used for the quasi-Stokes streamfunction]. In this way, it is clear that the TRM theory is the way of representing in height coordinates the relevant terms that arise from averaging in density coordinates and it is the identification and then the discarding of the nondivergent flux **M** that makes this interpretation possible. The reason the TRM approach is needed in height coordinates is that, unless significant changes are made to the horizontal momentum equations, the resolved-scale velocity in height-coordinate models is best thought of as the Eulerian-mean velocity.

Observations of microscale mixing activity in mesoscale eddies have generally not shown a large amplification of microscale mixing activity in these mesoscale features, so ocean modelers parameterize the diapycnal source term in the TRM density equation $\overline{Q}^{\#}$ using the standard (small) values of the diapycnal diffusivity that are observed in the thermocline. As Tandon and Garrett (1996) have pointed out, this implies that the eddy kinetic energy of mesoscale eddies cannot be dissipated in the ocean interior but rather must be dissipated near the upper and/or lower boundaries.

## 6. The continuity equation

The height-coordinate form of the continuity equation for the TRM velocity is $\nabla_H\cdot\overline{\mathbf{V}}^{\#} + \overline{w}_z^{\#} = 0$, which can be written with respect to the $\tilde{\gamma}$ density surfaces as (using the standard coordinate transformations)

$$\left(\frac{1}{\tilde{\gamma}_z}\bigg|_{\tilde{\gamma}}\right)_t + \nabla_{\tilde{\gamma}}\cdot\left(\frac{\overline{\mathbf{V}}^{\#}}{\tilde{\gamma}_z}\right) + \frac{\partial\overline{e}^{\#}}{\partial\tilde{\gamma}} = 0. \qquad (32)$$

In density coordinates, the instantaneous continuity equation is

$$\left(\frac{1}{\gamma_z}\bigg|_{\gamma}\right)_t + \nabla_{\gamma}\cdot\left(\frac{\mathbf{V}}{\gamma_z}\right) + \frac{\partial e}{\partial\gamma} = 0, \qquad (33)$$

and the time-mean of this equation at constant density $\gamma = \tilde{\gamma}$ is

$$\left(\frac{1}{\tilde{\gamma}_z}\bigg|_{\tilde{\gamma}}\right)_t + \nabla_{\tilde{\gamma}}\cdot\left(\frac{\hat{\mathbf{V}}}{\tilde{\gamma}_z}\right) + \frac{\partial\tilde{e}}{\partial\tilde{\gamma}} = 0, \qquad (34)$$

where the diapycnal velocity, $\tilde{e} \equiv \overline{e|_{\tilde{\gamma}}}$, is the result of averaging $e$ on a $\tilde{\gamma}$ surface, and so is not thickness weighted. Because $\hat{\mathbf{V}} = \overline{\mathbf{V}}^{\#} + O(\alpha^3)$ and $\overline{e}^{\#} = \tilde{e} + O(\alpha^3)$, each term in the TRM continuity equation (32) corresponds to its opposite number in the continuity equation that has been averaged in density coordinates, (34).

## 7. The TRM scalar conservation equation

In order to motivate the derivation of the TRM tracer equation, it is convenient to begin by averaging the tracer conservation equation in density coordinates. The instantaneous tracer conservation equation in density coordinates is

$$\left(\frac{\tau}{\gamma_z}\bigg|_{\gamma}\right)_t + \nabla_{\gamma}\cdot\left(\frac{\mathbf{V}\tau}{\gamma_z}\right) + (e\tau)_{\gamma} = \frac{X}{\gamma_z}, \qquad (35)$$

where tracer $\tau$ has a source term, $X$, which includes local production or consumption, as well as the flux divergence of unresolved mixing processes including molecular diffusion (see, e.g., deSzoeke and Bennett 1993). Temporally averaging (35) between $\tilde{\gamma}$ density surfaces gives

$$\left(\frac{\hat{\tau}}{\tilde{\gamma}_z}\bigg|_{\tilde{\gamma}}\right)_t + \nabla_{\tilde{\gamma}}\cdot\left(\frac{\hat{\mathbf{V}}\hat{\tau}}{\tilde{\gamma}_z}\right) + (\tilde{e}\hat{\tau})_{\tilde{\gamma}}$$
$$= \frac{\hat{X}}{\tilde{\gamma}_z} - \overline{\left(\frac{Q''\tau''}{\gamma_z}\right)}_{\tilde{\gamma}} - \nabla_{\tilde{\gamma}}\cdot\overline{\left(\frac{\mathbf{V}''\tau''}{\gamma_z}\right)}, \qquad (36)$$

where both overbars on the right represent quantities averaged at constant density and subscripts denote differentiation. In deriving (36) from (35) we have written $e$ as $Q/\gamma_z$ and used (29) to express $\hat{Q}/\tilde{\gamma}_z$ as $\tilde{e}$. The mean value of the tracer here is the thickness-weighted tracer value, and the double primed variables are the deviation of the instantaneous variables from their thickness-weighted values. These variables obey

$$\hat{\tau} \equiv \tilde{\gamma}_z\overline{\left(\frac{\tau}{\gamma_z}\bigg|_{\tilde{\gamma}}\right)} \quad \text{and} \quad \overline{\frac{\tau''}{\gamma_z}\bigg|_{\tilde{\gamma}}} \equiv 0. \qquad (37)$$

In a forward ocean model the first two terms on the right-hand side of (36) will be parameterized with a diapycnal diffusivity so that these terms amount to $(1/\tilde{\gamma}_z)$ times the divergence of a diapycnal diffusivity, $D$, operating on the diapycnal gradient of $\hat{\tau}$. The epipycnal tracer flux (i.e., the flux directed along a density surface) in (36) is also normally assumed to take a Fickian form so that

$$\overline{\left(\frac{\mathbf{V}''\tau''}{\gamma_z}\right)} = -\frac{K}{\tilde{\gamma}_z}\nabla_{\tilde{\gamma}}\hat{\tau} \quad \text{and}$$

$$\frac{\hat{X}}{\tilde{\gamma}_z} - \overline{\left(\frac{Q''\tau''}{\gamma_z}\right)}_{\tilde{\gamma}} = \frac{(D\hat{\tau}_z)_z}{\tilde{\gamma}_z}, \qquad (38)$$

where $K$ is the isopycnal diffusivity and $D$ the diapycnal diffusivity. These assumptions mean that the right-hand side of (36) can be written in height coordinates as $(1/\tilde{\gamma}_z)\nabla \cdot (\mathbf{S}\nabla\hat{\tau})$, where $\mathbf{S}$ is the symmetric diffusion tensor of Redi (1982) and the diffusion tensor is rotated into the local neutral tangent plane at each of the six faces of the model's tracer box.

In order to make progress with the $z$-coordinate TRM tracer conservation equation, the turbulent fluxes on the right-hand side of (36) need to be expressed in terms of quantities that are evaluated in height coordinates. The instantaneous value of any quantity (e.g., $\tau$) on a density surface can be expressed in both the thickness-weighted form and the nonthickness-weighted form so that $\hat{\tau} + \tau'' = \tilde{\tau} + \tau'|_{\tilde{\gamma}}$, and, since the two mean quantities differ by an amount that is quadratic in perturbation quantities [i.e., $\hat{\tau} - \tilde{\tau} = O(\alpha^2)$ see, e.g., (60) below], it follows that the perturbation quantities also differ by a quadratic term so that $\tau'' - \tau'|_{\tilde{\gamma}} = O(\alpha^2)$. This means that at leading order we can replace the double primed quantities in (36) with single primed quantities evaluated on the density surface, and a simple Taylor series expression shows that this in turn is related to perturbations at fixed height by

$$\tau'' = \tau'|_{\tilde{\gamma}} + O(\alpha^2) = \tau' - \overline{\tau}_z\frac{\gamma'}{\overline{\gamma}_z} + O(\alpha^2). \qquad (39)$$

Hence we have

$$\left.\overline{\left(\frac{Q''\tau''}{\gamma_z}\right)}\right|_{\tilde{\gamma}} = \frac{\overline{Q'\tau'}}{\overline{\gamma}_z} - \overline{\tau}_z\frac{\overline{Q'\gamma'}}{\overline{\gamma}_z^2} - \overline{Q}_z\frac{\overline{\gamma'\tau'}}{\overline{\gamma}_z^2} + 2\overline{Q}_z\overline{\tau}_z\frac{\overline{\phi}}{\overline{\gamma}_z^3} + O(\alpha^3) \quad \text{and} \qquad (40)$$

$$\left.\overline{\left(\frac{\mathbf{V}''\tau''}{\gamma_z}\right)}\right|_{\tilde{\gamma}} = \frac{\overline{\mathbf{V}'\tau'}}{\overline{\gamma}_z} - \overline{\tau}_z\frac{\overline{\mathbf{V}'\gamma'}}{\overline{\gamma}_z^2} - \overline{\mathbf{V}}_z\frac{\overline{\gamma'\tau'}}{\overline{\gamma}_z^2} + 2\overline{\mathbf{V}}_z\overline{\tau}_z\frac{\overline{\phi}}{\overline{\gamma}_z^3} + O(\alpha^3), \qquad (41)$$

where the overbars on the left are taken at constant density while those on the right are taken at constant height. The epipycnal flux divergence term in (36) is expressed in height coordinates by noting that for any two-dimensional vector, $\mathbf{C}$, the thickness-weighted epipycnal flux divergence is equal to the three-dimensional divergence of a three-dimensional flux that is directed in the density surface, according to

$$\tilde{\gamma}_z\nabla_{\tilde{\gamma}} \cdot \left(\frac{\mathbf{C}}{\tilde{\gamma}_z}\right) = \nabla \cdot \left(\mathbf{C} - \mathbf{k}\frac{\mathbf{C}\cdot\nabla_H\tilde{\gamma}}{\tilde{\gamma}_z}\right). \qquad (42)$$

We can now construct the TRM tracer conservation equation by rearranging the Eulerian-mean tracer equation,

$$\overline{\tau}_t + \nabla \cdot (\overline{\mathbf{U}}\,\overline{\tau}) = \overline{X} - \nabla \cdot (\overline{\mathbf{U}'\tau'}), \qquad (43)$$

into the form

$$\hat{\tau}_t + \nabla\cdot(\overline{\mathbf{U}}\hat{\tau}) = \overline{X}^{\#} - \tilde{\gamma}_z\overline{\left(\frac{Q''\tau''}{\gamma_z}\right)}_{\tilde{\gamma}} - \tilde{\gamma}_z\nabla_{\tilde{\gamma}}\cdot\overline{\left(\frac{\mathbf{V}''\tau''}{\gamma_z}\right)} - \nabla\cdot\mathbf{F}^{M\tau} + O(\alpha^3), \qquad (44)$$

where the modified tracer flux, $\mathbf{F}^{M\tau}$, is defined by

$$\begin{aligned}\mathbf{F}^{M\tau} \equiv{}& \overline{\mathbf{U}'\tau'} - \overline{\mathbf{U}}\left(-\frac{\overline{\tau'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{\tau}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right)_z \\ &- \left(\overline{\mathbf{V}'\tau'} - \overline{\tau}_z\frac{\overline{\mathbf{V}'\gamma'}}{\overline{\gamma}_z} - \overline{\mathbf{V}}_z\frac{\overline{\gamma'\tau'}}{\overline{\gamma}_z} + 2\overline{\mathbf{V}}_z\overline{\tau}_z\frac{\overline{\phi}}{\overline{\gamma}_z^2}\right) \\ &+ \mathbf{k}\left\{-\left(-\frac{\overline{\tau'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{\tau}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right)_t - \frac{\overline{X'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{X}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right\} \\ &- \mathbf{k}\left\{\frac{\overline{Q'\tau'}}{\overline{\gamma}_z} - \overline{\tau}_z\frac{\overline{Q'\gamma'}}{\overline{\gamma}_z^2} - \overline{Q}_z\frac{\overline{\gamma'\tau'}}{\overline{\gamma}_z^2} + 2\overline{Q}_z\overline{\tau}_z\frac{\overline{\phi}}{\overline{\gamma}_z^3}\right\} \\ &+ \mathbf{k}\left\{\frac{\nabla_H\tilde{\gamma}}{\tilde{\gamma}_z}\cdot\left(\overline{\mathbf{V}'\tau'} - \overline{\tau}_z\frac{\overline{\mathbf{V}'\gamma'}}{\overline{\gamma}_z} - \overline{\mathbf{V}}_z\frac{\overline{\gamma'\tau'}}{\overline{\gamma}_z} + 2\overline{\mathbf{V}}_z\overline{\tau}_z\frac{\overline{\phi}}{\overline{\gamma}_z^2}\right)\right\},\end{aligned} \qquad (45)$$

and $\overline{X}^{\#}$ is the height-coordinate version of the thickness-weighted source term; namely,

$$\overline{X}^{\#} \equiv \overline{X} + \left[-\frac{\overline{X'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{X}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right]_z. \qquad (46)$$

While the modified tracer flux, (45), looks quite daunting, progress is made by forming the conservation equation for the temporal correlation, $\overline{\tau'\gamma'}$. This is found by first multiplying the perturbation density equation by the tracer perturbation, then multiplying the perturbation tracer equation by the density perturbation, and finally adding these two parts and averaging, yielding

$$(\overline{\tau'\gamma'})_t + \overline{\mathbf{U}} \cdot \nabla(\overline{\tau'\gamma'}) = \overline{Q'\tau'} + \overline{X'\gamma'} - \overline{\mathbf{U}'\gamma'} \cdot \nabla\overline{\tau} - \overline{\mathbf{U}'\tau'} \cdot \nabla\overline{\gamma} + O(\alpha^3). \quad (47)$$

This equation is now used to eliminate $\overline{w'\tau'}$ from the expression (45), and after considerable algebra and using the density variance equation and the continuity equation, the modified tracer flux $\mathbf{F}^{M\tau}$ can be expressed in the two convenient forms

$$\mathbf{F}^{M\tau} = \hat{\tau}\mathbf{U}^+ + \mathbf{M}^\tau + O(\alpha^3) = -\mathsf{A}\nabla\hat{\tau} + \mathbf{N}^\tau + O(\alpha^3). \quad (48)$$

The quasi-Stokes velocity $\mathbf{U}^+$ and the antisymmetric diffusion tensor $\mathsf{A}$ are the same as in section 4, while the tracer fluxes, $\mathbf{M}^\tau$ and $\mathbf{N}^\tau$ are both nondivergent and are defined by

$$\mathbf{M}^\tau \equiv \nabla \times \left\{\left[-\boldsymbol{\Psi}\hat{\tau} - \overline{\mathbf{V}}\left(-\frac{\overline{\tau'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{\tau}_z\overline{\phi}}{\overline{\gamma}_z^2}\right)\right] \times \mathbf{k}\right\} \quad (49)$$

and

$$\mathbf{N}^\tau \equiv \nabla \times \left\{\left[-\overline{\mathbf{V}}\left(-\frac{\overline{\tau'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{\tau}_z\overline{\phi}}{\overline{\gamma}_z^2}\right)\right] \times \mathbf{k}\right\}. \quad (50)$$

Substituting the two alternative expressions, (48), into the mean tracer conservation equation (44) gives the following two versions of the mean tracer equation,

$$\begin{aligned}\hat{\tau}_t + \nabla \cdot (\overline{\mathbf{U}}\hat{\tau}) &= \overline{X}^\# - \tilde{\gamma}_z\overline{\left(\frac{Q''\tau''}{\gamma_z}\right)}_{\tilde{\gamma}} - \tilde{\gamma}_z\nabla_{\tilde{\gamma}} \cdot \overline{\left(\frac{\mathbf{V}''\tau''}{\gamma_z}\right)} - \nabla \cdot (\mathbf{U}^+\hat{\tau}) - \nabla \cdot \mathbf{M}^\tau + O(\alpha^3)\\ &= \overline{X}^\# - \tilde{\gamma}_z\overline{\left(\frac{Q''\tau''}{\gamma_z}\right)}_{\tilde{\gamma}} - \tilde{\gamma}_z\nabla_{\tilde{\gamma}} \cdot \overline{\left(\frac{\mathbf{V}''\tau''}{\gamma_z}\right)} + \nabla \cdot (\mathsf{A}\nabla\hat{\tau}) - \nabla \cdot \mathbf{N}^\tau + O(\alpha^3). \quad (51)\end{aligned}$$

The modified tracer flux $\mathbf{F}^{M\tau}$ represents the total effects of mesoscale eddies in the thickness-weighted tracer equation apart from the passive mixing of tracer along isopycnals. This is in contrast to the usual tracer eddy flux $\overline{\mathbf{U}'\tau'}$ which also includes contributions from diapycnal mixing. It is apparent from the vector decomposition (48), or equivalently by comparing (44) and (51), that (up to a nondivergent tracer flux) the modified tracer flux $\mathbf{F}^{M\tau}$ can be regarded as representing either an extra advective flux of tracer, $\hat{\tau}\mathbf{U}^+$, or an extra skew-diffusive flux of tracer, $-\mathsf{A}\nabla\hat{\tau}$. Such a simple interpretation is not available for the original tracer flux $\overline{\mathbf{U}'\tau'}$ in the Reynolds-averaged tracer equation (43).

In proceeding from the original Reynolds-averaged tracer equation (43) through (44) to (51) we have so far not dropped any nondivergent fluxes so that, for example, the horizontal fluxes of tracer per unit height are identical in (43), in (44), and in both lines of (51); that is [noting that $-\mathsf{A}\nabla\hat{\tau} = -\hat{\tau}_z\boldsymbol{\Psi} + \mathbf{k}(\boldsymbol{\Psi} \cdot \nabla_H\hat{\tau})$],

$$\begin{aligned}\overline{\mathbf{V}}\,\overline{\tau} + \overline{\mathbf{V}'\tau'} &= \overline{\mathbf{V}}\hat{\tau} + \mathbf{V}^+\hat{\tau} + \tilde{\gamma}_z\overline{\left(\frac{\mathbf{V}''\tau''}{\gamma_z}\right)}\Bigg|_{\tilde{\gamma}} + \left(-\boldsymbol{\Psi}\hat{\tau} - \overline{\mathbf{V}}\left(-\frac{\overline{\tau'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{\tau}_z\overline{\phi}}{\overline{\gamma}_z^2}\right)\right)_z + O(\alpha^3)\\ &= \overline{\mathbf{V}}\hat{\tau} - \boldsymbol{\Psi}\hat{\tau}_z + \tilde{\gamma}_z\overline{\left(\frac{\mathbf{V}''\tau''}{\gamma_z}\right)}\Bigg|_{\tilde{\gamma}} + \left(-\overline{\mathbf{V}}\left(-\frac{\overline{\tau'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{\tau}_z\overline{\phi}}{\overline{\gamma}_z^2}\right)\right)_z + O(\alpha^3). \quad (52)\end{aligned}$$

By discarding the nondivergent flux $\mathbf{M}^\tau$ from (51) one finds that the horizontal flux of tracer is $\overline{\mathbf{V}}\hat{\tau} + \mathbf{V}^+\hat{\tau} + \tilde{\gamma}_z\overline{(\mathbf{V}''\tau''/\gamma_z)|}_{\tilde{\gamma}}$, which is the same horizontal tracer flux that is obtained by thickness-weighted averaging the instantaneous tracer flux $\mathbf{V}\tau$ in density coordinates [this can be shown by substituting $\mathbf{V}\tau$ in place of $\mathbf{V}$ in the Taylor series analysis of section 2 or by substituting $\mathbf{V}\tau$ in place of $A$ in Eq. (A5) of the appendix and using (41)]. Just as in the case of the mean density equation, the key to making the horizontal and vertical fluxes in the tracer equation (51) the same as those that appear when the averaging is performed in density coordinates is to simply ignore the nondivergent density flux $\mathbf{M}^\tau$. Furthermore, just as in the case of the density equation, the reasoning above shows that the concept of a tracer flux at a given height is to some extent arbitrary. Just as we found that $\boldsymbol{\Psi}$ represents the amount of extra volume transport that occurs on integrating instantaneously up to the temporally undulating $\tilde{\gamma}$ density surface (as opposed to vertically integrating to the fixed mean height), so it can be shown that $\boldsymbol{\Psi}\tilde{\gamma} - \overline{\mathbf{V}}\overline{\phi}/\overline{\gamma}_z$ and $\boldsymbol{\Psi}\hat{\tau} + \overline{\mathbf{V}}(-\overline{\tau'\gamma'}/\overline{\gamma}_z + \overline{\tau}_z\overline{\phi}/\overline{\gamma}_z^2)$ in (20) and (49) are the extra horizontal fluxes of density and of tracer that are found on vertically integrating up to the undulating $\tilde{\gamma}$ density surface.

When the Fickian rotated diffusion tensor is adopted for the first three terms on the right-hand side of (51) [as discussed below (38)] and on dropping the nondivergent fluxes, the mean tracer equation becomes

$$\begin{aligned}\hat{\tau}_t + \nabla \cdot (\overline{\mathbf{U}}\hat{\tau}) &= \nabla \cdot (\mathsf{S}\nabla\hat{\tau}) - \nabla \cdot (\mathbf{U}^+\hat{\tau}) + O(\alpha^3)\\ &= \nabla \cdot ([\mathsf{S} + \mathsf{A}]\nabla\hat{\tau}) + O(\alpha^3). \quad (53)\end{aligned}$$

These two forms of the tracer conservation equation are exactly the same as are solved in modern oceanic GCMs. Griffies (1998) pointed out that adopting the skew-diffusion approach in the second line of (53) is preferable numerically because the quasi-Stokes streamfunction is spatially differentiated one time less than in the advective approach. It is worth noting that while the skew flux,

$$-\mathsf{A}\nabla\hat{\tau} = -\hat{\tau}_z\boldsymbol{\Psi} + \mathbf{k}(\boldsymbol{\Psi}\cdot\nabla_H\hat{\tau}),$$

has no component in the direction of $\nabla\hat{\tau}$, the horizontal and vertical components of the skew flux are often downgradient and upgradient respectively (see Plumb 1979).

In section 4 when motivating the TRM theory using the density equation we restricted attention to a linear equation of state. This restriction can now be relaxed because the only place in the TRM tracer conservation (53) where the nonlinear nature of the equation of state enters is in the calculation of the slope of the local neutral tangent plane in order to correctly rotate the symmetric diffusion tensor. Griffies et al. (1998) have shown how this can be done in height-coordinate models. Since ocean models do not carry a conservation equation for density but rather for thickness-weighted potential temperature $\hat{\theta}$ and salinity $\hat{S}$, it is clear that the TRM development that has led to (53) is quite consistent with a nonlinear equation of state. In particular, cabbeling and thermobaricity will still arise in the normal way because, while the epineutral (i.e., along neutral tangent plane) gradients of potential temperature $\hat{\theta}$ and salinity $\hat{S}$ are balanced in density terms, the epineutral divergence of the epineutral fluxes of potential temperature $\hat{\theta}$ and salinity $\hat{S}$ are not balanced in density terms and this imbalance causes the dianeutral motion of cabbeling and thermobaricity (McDougall 1987). Since the quasi-Stokes streamfunction,

$$\boldsymbol{\Psi} = -\overline{\mathbf{V}'\gamma'}/\overline{\gamma}_z + \overline{\mathbf{V}}_z\overline{\phi}/\overline{\gamma}_z^2 + O(\alpha^3),$$

is evaluated at fixed height, perturbations of in situ density are sufficient to evaluate $\overline{\mathbf{V}'\gamma'}$ and $\overline{\phi}$ while $\overline{\gamma}_z$ should be evaluated as the vertical gradient of locally referenced potential density so that the nonlinear nature of the equation of state does not pose a problem for the evaluation of $\boldsymbol{\Psi}$.

It is important to realize that the mean tracer that appears in the TRM conservation equation (51) or (53) is not the Eulerian-averaged tracer value. If one wanted to insist that the model's tracer value was the Eulerian-mean tracer value $\overline{\tau}$, then one would need to impose an additional flux divergence in the tracer equation as follows

$$\overline{\tau}_t|_z + \nabla_H\cdot(\overline{\mathbf{V}}\,\overline{\tau}) + (\overline{w}\,\overline{\tau})_z = \nabla\cdot([\mathsf{S} + \mathsf{A}]\nabla\overline{\tau}) - \nabla\cdot\mathbf{E} + O(\alpha^3), \quad (54)$$

where the additional flux, $\mathbf{E}$ is [using (28)],

$$\mathbf{E} = \mathbf{k}\left[-\frac{\overline{\tau'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{\tau}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right]_t + \overline{\mathbf{U}}\left[-\frac{\overline{\tau'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{\tau}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right]_z. \quad (55)$$

Such nongradient terms would be very difficult to parameterize. It is perhaps surprising, but also fortunate, that the tracer that is naturally carried by an eddyless height-coordinate model should be interpreted in exactly the same way as the tracer carried by an eddyless density-coordinate model, namely, as the thickness-weighted tracer value $\hat{\tau}$. Lozier et al. (1994) have drawn attention to the damage that can be done to water masses by averaging salinity and potential temperature data at fixed height. The theoretical result presented in this section dictates that tracer data should be averaged between density surfaces not only for the purpose of forming atlases of hydrographic data, but also for comparing with the output of ocean models, for the assimilation of observed data into ocean models, and before using data in inverse models. In particular, Fig. 4 of McDougall and McIntosh (1996) demonstrates how a coarse-resolution ocean model is able to model a thin injection of tracer in the ocean even though the Eulerian-mean tracer values would be vertically smoothed by the vertical excursions of isopycnals.

It is concluded that, up to error terms that are cubic in perturbation quantities, the conservation statement for tracers in an eddyless height-coordinate model is the same as the isopycnally averaged conservation statement, so long as the advection is by the TRM velocity, $\overline{\mathbf{U}}^\# = \overline{\mathbf{U}} + \mathbf{U}^+$, and a nondivergent tracer flux is ignored. In addition, we have found that the tracers (such as potential temperature, salinity, or tritium) must be interpreted in a specific fashion. These tracers are not the Eulerian-averaged values, but are the thickness-weighted values averaged between $\tilde{\gamma}$ surfaces. If the quasi-Stokes streamfunction is evaluated using the exact expression (4a) rather than the approximate expression (4b), the TRM tracer conservation statement is identical to the thickness-weighted density coordinate version. That is, if we assume that the momentum equations of a coarse-resolution model deliver accurate estimates of the Eulerian-mean velocity components, then the use of an exact quasi-Stokes streamfunction leads to exactly the same tracer conservation equation as appears under thickness-weighted averaging in density coordinates.

## 8. Boundary conditions on the quasi-Stokes streamfunction

The physical interpretation of the quasi-Stokes streamfunction, as described in section 2 above, provides guidance on the boundary conditions that should be imposed at the top and bottom of the ocean. Figures 3a–c displays the temporal variations in the heights of three different $\tilde{\gamma}$ surfaces when the ocean's density field displays harmonic temporal variations. When a density surface outcrops, it is assumed to still exist right at the ocean surface. The modified density, $\tilde{\gamma}$, appropriate to each height has the property (by definition) that the height perturbation of this $\tilde{\gamma}$ surface averages to zero, as is indicated by the shading in Fig. 3 (the shaded fluid appearing below the mean height is equal to the shaded fluid above the mean height). As the sea surface (or the

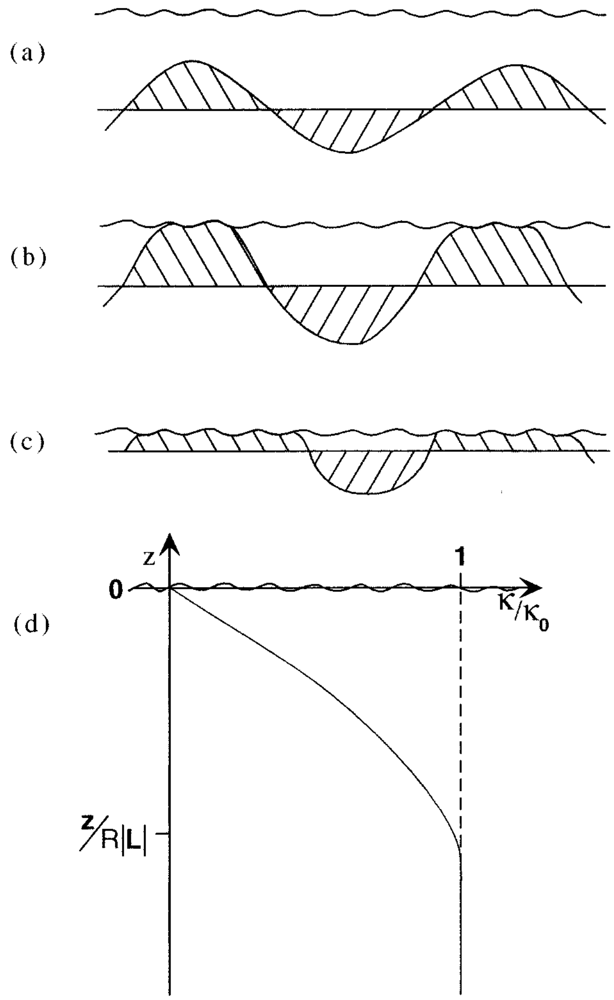


FIG. 3. Sketch of the temporal variation of the heights of three different surfaces as the sea surface is approached. (d) the implication for the vertical tapering of the diffusivity.

ocean floor) is approached, the shaded area reduces to zero, so the correlation of velocity and thickness in this shaded region also must tend to zero. To be more specific, the vertical integral on the right-hand side of (4a), that is, the contribution of eddies to the transport of water that is more dense than $\tilde{\gamma}$, reduces to zero as the sea surface (or ocean floor) is approached. This transport is the quasi-Stokes streamfunction: the streamfunction that together with the Eulerian-mean velocity provides the exact connection between averaging in density and height coordinates. In order to preserve this very important connection between averaging in density and in height coordinates, we must taper the quasi-Stokes streamfunction to zero at the top and bottom of the ocean even though the expression on the right of (4b) does not go to zero at these locations. The Taylor series approach that led to (4b) is inaccurate at the boundaries because of the nonlinear clipping of the isopycnal heights at the boundary. For the case of zonal averaging, McIntosh and McDougall (1996) also found that the close correspondence between the zonal residual mean and the zonal thickness-weighted mean velocities broke down near the sea surface. As explained by Killworth (2001), the relationship (11) between the Eulerian-mean and modified densities becomes inaccurate at the top and bottom boundaries. The difference between these densities there is first order in perturbation density rather than quadratic.

Our TRM approach involves the insistence that the scalar variables in an ocean model are exactly the thickness-weighted scalars averaged in density coordinates and that the density surfaces in which these variables are averaged have exactly zero mean perturbation height. This insistence persists right up to the ocean surface and the ocean floor. With these interpretations of the tracer variables and the density surfaces, it becomes clear that the real benefit that the TRM approach delivers is the ability to represent in height coordinates exactly the same quantities and property transports that would be obtained if the averaging had occurred with respect to instantaneous density surfaces. This aim dictates that the quasi-Stokes streamfunction must be tapered to zero at the top and bottom of the ocean as occurs when using the exact definition of the quasi-Stokes streamfunction, (4a), even though the definition of the quasi-Stokes streamfunction expression, (4b), would not go to zero in this fashion (see Killworth 2001). In other words, we value more highly the physical interpretation, (4a), of the quasi-Stokes streamfunction as the eddy contribution to the transport of water of a certain density class than the Taylor series expression, (4b), for this streamfunction. We note that this requirement to taper the quasi-Stokes streamfunction as the seafloor is approached also applies when the ocean floor is sloped.

The method of Killworth (1997) has delta functions of horizontal velocity at the top and bottom of the ocean and the reason these delta functions appear is that the density variable that is used to transform between height coordinates and density coordinates is $\overline{\gamma}(z)$ rather than $\tilde{\gamma}(z)$ that we advocate in this paper. While Killworth's (1997) method involves a parameterization of thickness-weighted transport in density coordinates, it is implemented in height coordinates and so this method effectively finds a streamfunction for the eddy-induced flow in height coordinates. The aim of these parameterization schemes is to represent in an eddyless height-coordinate model processes that occur in an eddy-resolving model (or in reality), and it is clear that, when such an eddy-resolved model is analyzed in density coordinates, one does not obtain a delta function of velocity in $\tilde{\gamma}(z)$ coordinates at the sea surface and at the ocean bottom. By the same token, it is clearly incorrect to imagine the eddy-induced velocity to actually flow vertically through the sea surface or through the ocean floor when the model's density is interpreted as the modified density.

Further evidence against the existence of delta functions at the top and bottom of the ocean can be gleaned from the work of Edmon et al. (1980), who contoured the divergence of the Eliassen–Palm flux of the zonally averaged flow and showed that for a finite amplitude situation (their Fig. 3d), the contours are not all bunched into surface boundary layers. These several lines of evidence lead us to conclude that, in addition to numerical convenience, there are good physical justifications for the tapering of the quasi-Stokes streamfunction over several hundred meters when approaching the sea surface or the ocean floor. Held and Schneider (1999) have shown that much of the zonal-averaged equatorward flow in the troposphere occurs in the potential temperature layers that clip the ground at some longitude. This demonstrates the importance of the volume transport that occurs in the near-surface region where the quasi-Stokes streamfunction is tapered toward zero.

In the absence of a horizontal boundary one would estimate that a fluid parcel in a mesoscale eddy would undergo vertical excursions (of height) that would scale with the Rossby radius, $R$, multiplied by the magnitude of the slope of the density surface, $|\mathbf{L}| = |\nabla_H \tilde{\gamma}/\tilde{\gamma}_z|$. Both $R$ and $|\mathbf{L}|$ can be estimated locally at every point in a GCM. One would estimate that, if $R|\mathbf{L}|$ were greater than the depth of a certain gridpoint, then the density surface, $\tilde{\gamma}$, appropriate to that point would spend part of its time clipping the sea surface as indicated in Figs. 3b and 3c. This suggests that the quasi-Stokes streamfunction (or a diffusivity that is used to parameterize the quasi-Stokes streamfunction) should be tapered to zero according to the scaled height, $z/(R|\mathbf{L}|)$ (as sketched in Fig. 3d) and such a procedure is already in use (see appendix B of Large et al. 1997). The physical interpretation of the quasi-Stokes streamfunction makes it possible to justify this procedure. Indeed, it would be unphysical to not taper the streamfunction in this fashion: we regard delta functions of eddy-induced velocity at the top and bottom of the ocean (in the last vertical grid point) as erroneous as they are not seen when an eddy-resolved model is averaged in $\tilde{\gamma}$ coordinates. The observed vertical heaving of density surfaces in the Southern Ocean in the mesoscale frequency range is about 200 m at mean depths of 420 and 1150 m (H. Phillips 2000, personal communication). The magnitude of this vertical movement of isopycnals demonstrates that the quasi-Stokes streamfunction should be tapered to zero over a vertical distance that is considerably larger than the typical vertical grid spacing in a numerical model and also over a greater distance than the mixed layer depth.

## 9. The diapycnal nature of the Eulerian-mean velocity

The advection of $\tilde{\gamma}$ by the Eulerian-mean flow is given by [from (23) after ignoring various nondivergent fluxes, and for simplicity, reverting to consider a linear equation of state]

$$\begin{aligned}\overline{D}_t\tilde{\gamma} &\equiv \tilde{\gamma}_t + \overline{\mathbf{U}}\cdot\nabla\tilde{\gamma} = \overline{Q}^{\#} + \nabla\cdot(\mathbf{A}\nabla\tilde{\gamma}) + O(\alpha^3)\\ &= \overline{Q}^{\#} - \mathbf{U}^{+}\cdot\nabla\tilde{\gamma} + O(\alpha^3)\\ &= \overline{Q}^{\#} + \tilde{\gamma}_z\nabla_{\tilde{\gamma}}\cdot\mathbf{\Psi} + O(\alpha^3),\end{aligned} \tag{56}$$

where $\nabla_{\tilde{\gamma}}$ is the spatial gradient operator in a $\tilde{\gamma}$ surface. Expressing the left-hand side of (56) with respect to $\tilde{\gamma}$ surfaces, the diapycnal component of the Eulerian-mean velocity is clearly

$$\overline{e} = \overline{Q}^{\#}/\tilde{\gamma}_z + \nabla_{\tilde{\gamma}}\cdot\mathbf{\Psi} + O(\alpha^3). \tag{57}$$

This is a compact way of expressing how the eddy forcing affects the Eulerian-mean velocity: namely by the epineutral divergence of the quasi-Stokes streamfunction, $\nabla_{\tilde{\gamma}}\cdot\mathbf{\Psi}$. Since $\mathbf{\Psi}$ is the contribution of eddy motions to the horizontal transport of water that is denser than $\tilde{\gamma}$, it is perhaps not surprising that its epineutral divergence should be a diapycnal advection. (Note that we use the word ‘‘diapycnal’’ to mean the part of the vertical velocity that flows through the $\tilde{\gamma}$ surface, not through the $\overline{\gamma}$ surface.) In contrast to the Eulerian-mean flow, the diapycnal component of the TRM velocity, $\overline{e}^{\#}$ is [from simply rewriting (30)]

$$\overline{e}^{\#} = \overline{Q}^{\#}/\tilde{\gamma}_z + O(\alpha^3). \tag{58}$$

The vector relationship between the three-dimensional velocity vectors $\overline{\mathbf{U}}$, $\mathbf{U}^{+}$, and $\overline{\mathbf{U}}^{\#}$ is illustrated in Fig. 4a. The positions of a given $\tilde{\gamma}$ density surface are shown at an initial time and at a later time, and the labeled velocity vectors are actually the displacements achieved by those velocities in this time interval. The vertical component of the Eulerian-mean velocity, $\overline{w}$, is due to the sum of (i) the sliding along the density surface due to its slope, $-\overline{\mathbf{V}}\cdot\nabla_H\tilde{\gamma}/\tilde{\gamma}_z$; (ii) the vertical motion of the density surface, $-\tilde{\gamma}_t/\tilde{\gamma}_z$; (iii) diabatic mixing processes, $\overline{Q}^{\#}/\tilde{\gamma}_z$; and (iv) the diapycnal eddy forcing of the Eulerian-mean flow, $\nabla_{\tilde{\gamma}}\cdot\mathbf{\Psi}$. This last component arises from the instantaneously epipycnal motions of mesoscale eddies and is not inherently diapycnal: it only appears to be diapycnal when the flow is examined with respect to the Eulerian-mean velocity. The vertical component of the TRM velocity, $\overline{w}^{\#}$, has no component caused by mesoscale eddy forcing but rather is the sum of just three terms: (i) the sliding along the density surface due to its slope, $-\overline{\mathbf{V}}^{\#}\cdot\nabla_H\tilde{\gamma}/\tilde{\gamma}_z$; (ii) the vertical migration of the density surface, $-\tilde{\gamma}_t/\tilde{\gamma}_z$; and (iii) the diabatic mixing processes, $\overline{Q}^{\#}/\tilde{\gamma}_z$. When the density surfaces are not migrating vertically, Fig. 4a simplifies to Fig. 4c. The cubic terms in perturbation quantities are not included in Fig. 4 as they are of higher order and are believed to be unimportant.

The eddy forcing of the mean density equation can be expressed as $-\mathbf{U}^{+}\cdot\nabla\tilde{\gamma}$ [see the middle line of (56)]. The relative contribution of the vertical quasi-Stokes velocity to this is

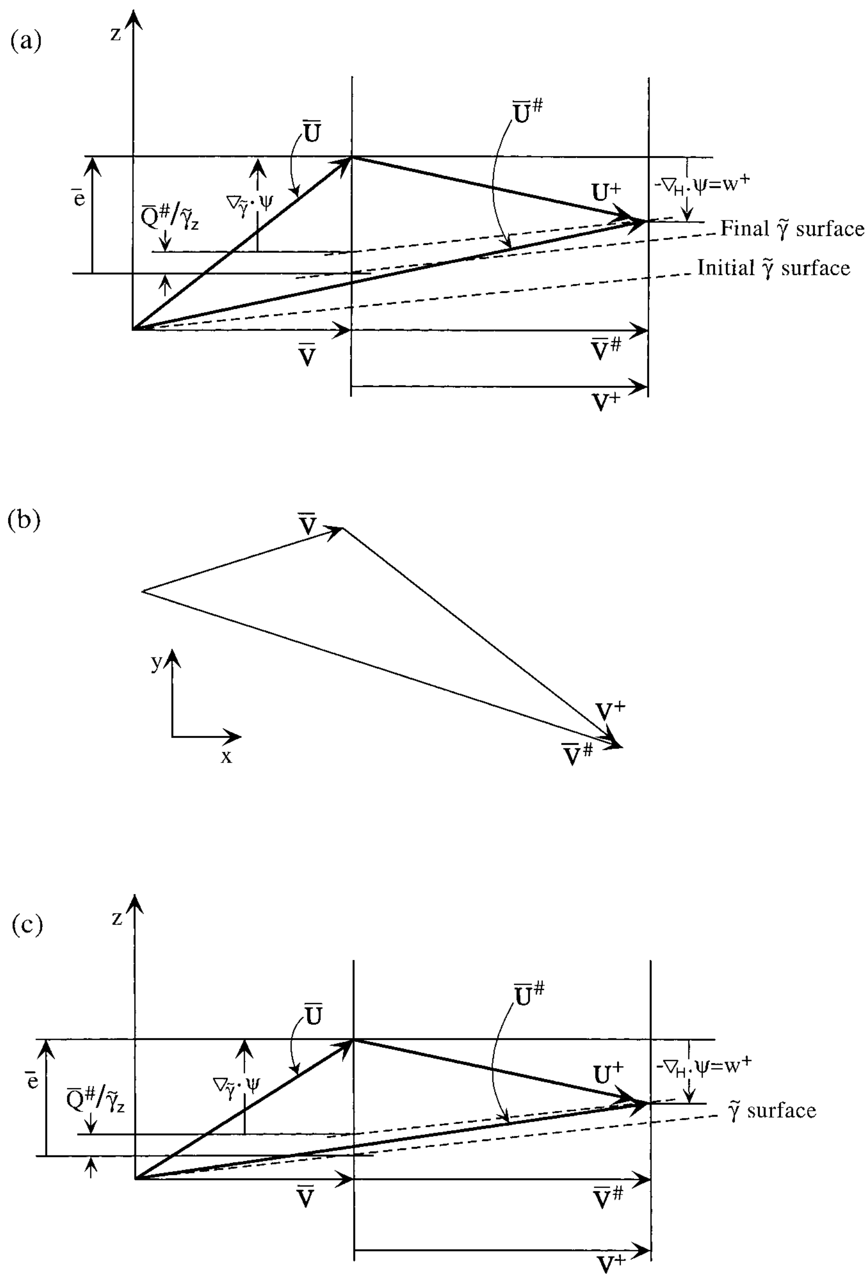


FIG. 4. (a) Sketch of the Eulerian-mean velocity $\overline{\mathbf{U}}$, the quasi-Stokes velocity $\mathbf{U}^+$, and the TRM velocity $\overline{\mathbf{U}}^{\#}$. (b) The horizontal components of the same three velocities. (c) A simpler version of (a) when the density surfaces are assumed to not be moving vertically.

$$\frac{-w^+\tilde{\gamma}_z}{-\mathbf{U}^+\cdot\boldsymbol{\nabla}\tilde{\gamma}} = \frac{\boldsymbol{\nabla}_H\cdot\boldsymbol{\Psi}}{\boldsymbol{\nabla}_{\tilde{\gamma}}\cdot\boldsymbol{\Psi}}. \qquad (59)$$

In the quasigeostrophic (QG) limit this ratio is assumed to be unity because only the vertical component of the extra velocity appears as eddy forcing in the density equation (see Treguier et al. 1997). Equation (59) shows that the QG limit is equivalent to assuming that the quasi-Stokes streamfunction varies much more strongly with horizontal position than with height so that $\boldsymbol{\nabla}_H\cdot\boldsymbol{\Psi} \approx \boldsymbol{\nabla}_{\tilde{\gamma}}\cdot\boldsymbol{\Psi}$. To the extent that QG theory is applicable to the real ocean, the implication would be that the horizontal quasi-Stokes velocity is unimportant. While this will often be the case as a point-by-point balance in the ocean, the following argument suggests that this implication of QG theory is misleading. McDougall (1995) pointed out that while the eddy-induced horizontal velocity $\mathbf{V}^+$ might be small (rarely larger than 1 mm s$^{-1}$), the fact that it tends to point in a direction across the mean epipycnal property gradients means that it can be, and often is, the dominant term in a conservation equation. Also, GCM experiments with the Gent and McWilliams (1990) scheme show that the quasi-Stokes circulation accounts for up to 0.5 PW of meridional heat flux in the Southern Ocean (see Danabasoglu and McWilliams 1995). This contribution to the meridional heat flux is achieved by the northward quasi-Stokes velocity $v^+$. Hence we must conclude that the horizontal components of the quasi-Stokes velocity cannot be ignored in favor of just the vertical component.

Less than a decade ago it was commonplace in ocean modeling to assume that in the Eulerian-mean density conservation equation (13), the horizontal density flux $\overline{\mathbf{V}'\gamma'}$ could be parameterized as a downgradient Fickian flux while the vertical turbulent flux $\overline{w'\tilde{\gamma}'}$ was ignored except for the part that was taken to be small-scale diapycnal mixing. This parameterization caused serious problems in ocean simulations due to the fictitious diapycnal flux of density that results from the exactly horizontal mixing (Veronis 1975). The temporal-residual-mean theory has demonstrated the inadequacy of this approach and it provides a route forward that avoids the fictitious diapycnal density fluxes of the Veronis effect because the skew-diffusive density flux,

$$-\mathsf{A}\nabla\tilde{\gamma} = -\tilde{\gamma}_z\boldsymbol{\Psi} + \mathbf{k}(\boldsymbol{\Psi}\cdot\nabla_H\tilde{\gamma}),$$

lies in the density surface. The prior approach is akin to parameterizing the horizontal part of this skew flux with a downgradient diffusion of density and ignoring the vertical component.

## 10. The interpretation of the Gent–McWilliams scheme

### *a. The nature of the bolus velocity*

Gent et al. (1995) and many subsequent authors have described the Gent and McWilliams (1990) scheme as a parameterization of the bolus velocity for eddyless models. Also, in Gent et al. (1995) we took the model's resolved-scale velocity to be ‘‘an observed velocity that has been filtered by a low-pass projection operator in time and space at constant density.’’ We will show that these interpretations of these velocities are incorrect at leading order and that, while this difference applies to both the horizontal velocities and the three-dimensional velocities, the different interpretation is most important for the three-dimensional velocities. Having said that, it is not our intention to take issue with the Gent et al. (1995) scheme itself. On the contrary, it is a valuable parameterization scheme for including the effects of mesoscale eddies in eddyless models. We do, however, disagree with our prior interpretation of the velocity components in that paper and we propose an alternative explanation of these velocities. We suggest that the Gent and McWilliams (1990) eddy parameterization scheme is the community's first attempt at parameterizing the quasi-Stokes streamfunction of the TRM theory and that the resolved-scale velocity be interpreted as the Eulerian-mean velocity.

In what follows we argue that the bolus velocity does not have a diapycnal component and that it is divergent. Then we discuss the implications of these two properties for the interpretation of the Gent et al. (1995) method.

The bolus velocity, $\mathbf{V}^B = \overline{\mathbf{V}'|_{\tilde{\gamma}}(z')_z}$, [see (7)] is defined to be that part of the flow between two neighboring density surfaces that is caused by the correlation between velocity and thickness perturbations. Here the velocity perturbation is interpreted as describing motion at constant density even though it is an exactly horizontal velocity in density coordinates [Bleck (1978) and appendix A of McDougall (1995)]. Hence the bolus transport is confined between density surfaces and so is fundamentally isopycnal. This is exactly analogous to forming any weighted average of the horizontal velocity in $z$ coordinates; the result always lies in the horizontal plane. Just as this weighted horizontal velocity in $z$ coordinates has no vertical component, so the bolus velocity has no diapycnal component.

The continuity equation (34) demonstrates that, in the simplest case of a steady state and in the absence of diapycnal diffusion, the thickness-weighted velocity in density coordinates is nondivergent; that is, $\nabla_{\tilde{\gamma}}\cdot(\hat{\mathbf{V}}/\tilde{\gamma}_z) = 0$. There is no requirement that the two parts of $\hat{\mathbf{V}}$, namely $\tilde{\mathbf{V}}$ and $\mathbf{V}^B$, should individually be nondivergent. We conclude that, in general, the (three-dimensional) bolus velocity is divergent. It is this divergent nature of the two velocities that resolves the ‘‘curious point’’ raised by Treguier et al. (1997, p. 571). Note that, while the bolus velocity has a zero vertical component in density coordinates, when transformed to Cartesian coordinates it has a vertical component.

In Gent et al. (1995) we stated that the extra eddy-induced advection [$\mathbf{U}^+$ in the present terminology and $\mathbf{u}^* + w^*\mathbf{k}$ in the terminology of Gent et al. (1995)] was the bolus velocity. The vertical component of this velocity, $w^*$, was calculated by assuming that $\mathbf{u}^* + w^*\mathbf{k}$ was nondivergent. Since the bolus velocity is divergent in general, this argues that the eddy-induced velocity of Gent et al. (1995) is not the bolus velocity. In addition, section 9 and Fig. 4 demonstrate that $\mathbf{U}^+$ has a large diapycnal component. In contrast, the bolus velocity has no diapycnal component, so this adds weight to the argument that the eddy-induced velocity of Gent et al. is not the bolus velocity.

In Gent et al. (1995) we also assumed that the resolved-scale horizontal velocity of an eddyless $z$ coordinate ocean model was the velocity averaged on density surfaces, $\tilde{\mathbf{V}}$. Since their eddy-induced velocity is nondivergent, then their resolved-scale velocity must also be nondivergent. Since $\tilde{\mathbf{V}}$ can in general be divergent, it cannot be the resolved-scale velocity of Gent et al. Also, under adiabatic conditions, $\tilde{\mathbf{V}}$ has no diapycnal component, however Gent and McWilliams (1990) realized that their suggested parameterization of eddies caused the resolved-scale velocity to have a diapycnal component, and for several years this feature was considered a weakness of the scheme. This was considered a sufficiently negative feature that it was emphasized that the areal average of this diapycnal velocity component was zero (see, e.g., McWilliams and Gent 1994). We conclude that the resolved-scale horizontal velocity cannot be $\tilde{\mathbf{V}}$.

This conclusion is confirmed by numerical experiments. Ocean GCM simulations using the Gent et al. (1995) scheme have found large diapycnal transports in the Southern Ocean for both the resolved-scale flow and the eddy-induced circulation. For example, Hirst and McDougall (1998) plotted the zonally averaged streamfunctions of both the resolved-scale velocity and the eddy-induced velocity in density coordinates specifically to illustrate the diapycnal nature of both circulations (their Fig. 6). They found about 14 Sv (Sv $\equiv 10^6$ $m^3\ s^{-1}$) of zonally averaged diapycnal transport in the Southern Ocean in both the resolved-scale velocity field and the so-called eddy-induced circulation. The sum of these circulations had a much smaller diapycnal transport.

We now interpret the resolved-scale velocity of the Gent et al. (1995) scheme to be the Eulerian-mean velocity, and the TRM theory demonstrates [see (57)] that the Eulerian-mean velocity is expected to have the diapycnal component $\nabla_{\tilde{\gamma}} \cdot \mathbf{\Psi}$ induced by the mesoscale motions that are themselves instantaneously adiabatic. Furthermore we interpret the eddy-induced advection of Gent et al. as including the bolus velocity, but also accounting for the difference between horizontal velocities averaged on density and $z$ coordinates.

### *b. Comparing the horizontal components, $\mathbf{\Psi}_z$ and $\mathbf{V}^B$*

Here we examine the similarities and differences between $\mathbf{\Psi}_z$ and the horizontal component of the bolus velocity $\mathbf{V}^B$ by taking the vertical derivative of $\mathbf{\Psi}$ [from (3)] obtaining

$$\begin{aligned}\mathbf{V}^+ &= \mathbf{\Psi}_z \\ &= \overline{(\mathbf{V}' + \overline{\mathbf{V}}_z z')z'_z} + \overline{\left(\mathbf{V}_z + \frac{1}{2}\overline{\mathbf{V}}_{zz} z'\right)z'} + O(\alpha^3) \\ &= \qquad \mathbf{V}^B \qquad + \qquad (\tilde{\mathbf{V}} - \overline{\mathbf{V}}) \qquad + O(\alpha^3).\end{aligned} \tag{60}$$

The expression $(\mathbf{V}' + \overline{\mathbf{V}}_z z')$ is the first-order Taylor series expansion for the velocity perturbation at constant density, $\mathbf{V}'|_{\tilde{\gamma}}$, so that the first term of (60) is simply the horizontal component of the bolus velocity $\mathbf{V}^B$ to within an error that is third order in perturbation quantities. At leading order, the expression $(\mathbf{V}_z + \frac{1}{2}\overline{\mathbf{V}}_{zz} z')$ is the instantaneous value of the shear half way between the fixed height and the instantaneous height of the $\tilde{\gamma}$ density surface so that the second term in (60) is the difference between the horizontal velocities averaged at constant density and at constant height, $(\tilde{\mathbf{V}} - \overline{\mathbf{V}})$. It is apparent from (60) that the task of the quasi-Stokes streamfunction is not only to parameterize the horizontal bolus velocity, but also to take account of $(\tilde{\mathbf{V}} - \overline{\mathbf{V}})$.

Using the above identification of $(\tilde{\mathbf{V}} - \overline{\mathbf{V}})$ with $\overline{(\mathbf{V}_z + \frac{1}{2}\overline{\mathbf{V}}_{zz} z')z'}$, the following approximate expression can be found for $(\tilde{\mathbf{V}} - \overline{\mathbf{V}})$ using the thermal wind relation,

$$\mathbf{V}_z = -(g/f\rho_0)\mathbf{k} \times \nabla_H \gamma,$$

(which is used for both the mean and the perturbation fields),

$$(\tilde{\mathbf{V}} - \overline{\mathbf{V}}) \approx \left(\frac{g}{f\rho_0}\right)\mathbf{k} \times \nabla_H\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right). \tag{61}$$

Taking a maximum expected value of half the density variance of $10^{-2}$ $kg^2\ m^{-6}$ (equivalent to a root-mean-square vertical heaving by mesoscale eddies of about 150 m) and assuming the horizontal gradient of $(\overline{\phi}/\overline{\gamma}_z)$ to vary by its own magnitude in a horizontal distance of $10^6$ m, we find from (61) that $(\tilde{\mathbf{V}} - \overline{\mathbf{V}})$ is about $10^{-3}$ $m\ s^{-1}$. While this estimate of $(\tilde{\mathbf{V}} - \overline{\mathbf{V}})$ is as large as the horizontal bolus velocity, it must be noticed that this velocity is directed along the (usually closed) contours of $(\overline{\phi}/\overline{\gamma}_z)$ in the horizontal plane. Because of this, when the normal component of $(\tilde{\mathbf{V}} - \overline{\mathbf{V}})$ is integrated along a closed path (e.g., a path that encircles the globe passing through Drake Passage) then the average of the normal component of $(\tilde{\mathbf{V}} - \overline{\mathbf{V}})$ is zero. This term can contribute to zonally averaged tracer budgets if there is a correlation between the tracer values and the locations where $(\tilde{\mathbf{V}} - \overline{\mathbf{V}})$ is northward or southward, but this is unlikely to be of leading order importance.

Another consequence of the form (61) is that the horizontal divergence of $(\tilde{\mathbf{V}} - \overline{\mathbf{V}})$ is zero except for a small term due to the beta effect [as pointed out by Treguier et al. (1997) for the quasigeostrophic case]. This implies that if the quasi-Stokes streamfunction was made to satisfy $\mathbf{\Psi}_z = \mathbf{V}^B$ rather than (60), this approximation would not affect the determination of the correct vertical quasi-Stokes velocity, $w^+$. Hence we conclude that while it may well be sufficiently accurate for many purposes to regard the horizontal quasi-Stokes velocity of the TRM circulation as the bolus velocity, the eddy forcing of the mean density equation (56) is caused by the part of the three-dimensional quasi-Stokes velocity that is directed normal to the density surfaces, $-\mathbf{U}^+ \cdot \nabla\tilde{\gamma}$, and this is just the part that would be zero (in a steady state) if the three-dimensional quasi-Stokes velocity were in fact the bolus velocity. It is the construction of the vertical component of the quasi-Stokes velocity that causes the full three-dimensional quasi-Stokes velocity to have a component normal to density surfaces. This construction uses the nondivergent nature of the three-dimensional quasi-Stokes velocity, and this is the key attribute that makes it different to the bolus velocity, which is three-dimensionally divergent.

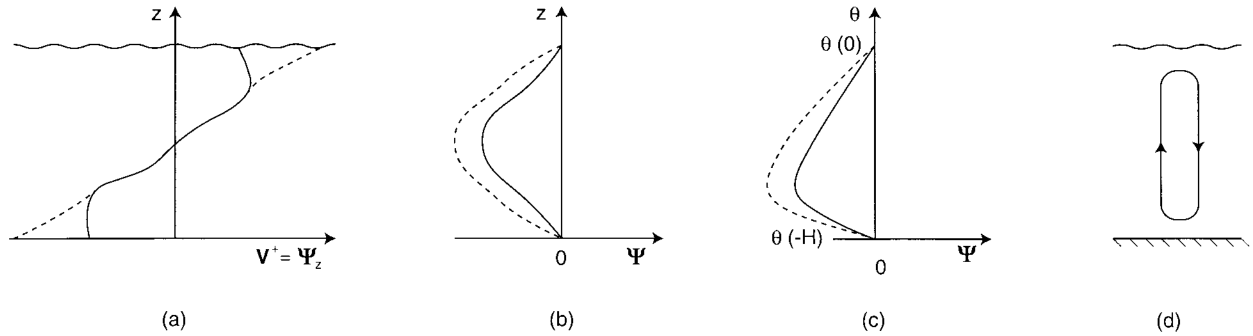


FIG. 5. (a) The horizontal quasi-Stokes velocity, $\mathbf{V}^+ = \mathbf{\Psi}_z$, is sketched at a cast as a function of height. The dashed curve represents a different choice for the quasi-Stokes velocity involving a different assumption about how the quasi-Stokes transport is distributed between the upper and lower boundaries. (b) The vertical distribution of the quasi-Stokes streamfunction for the two cases shown in (a). (c) The two versions of the quasi-Stokes streamfunction plotted as a function of potential temperature. The area under this curve is the horizontal flux of potential temperature on this cast caused by the horizontal quasi-Stokes transport. (d) Sketch of the vertical overturning circulation of the three-dimensional quasi-Stokes velocity caused by the uncertainty in near-boundary values of $\mathbf{\Psi}_z$ on just one cast.

### *c. On parameterizing mesoscale eddies in density-coordinate models*

The task of parameterizing the effects of mesoscale eddies is very different for isopycnal models than for height-coordinate models. In an isopycnal model the extra horizontal velocity that is needed is indeed the bolus velocity. The sum of the bolus velocity and the resolved-scale horizontal velocity of an isopycnal model, $\tilde{\mathbf{V}}$, gives the thickness-weighted velocity of isopycnal coordinates, $\hat{\mathbf{V}}$. In stark contrast to the large diapycnal component of the quasi-Stokes velocity of height-coordinate models, the bolus advection that is needed in isopycnal models has zero diapycnal component.

## 11. Downgradient potential vorticity parameterization

Several authors have suggested that, rather than making a Fickian assumption on the horizontal flux of density [as is done by the Gent et al. (1995) scheme], a plausible alternative is to assume that potential vorticity is fluxed epineutrally down the epineutral gradient of potential vorticity. Such a parameterization naturally leads to an expression for the horizontal bolus velocity (Killworth 1997); then this is assumed equal to the horizontal quasi-Stokes velocity, $\mathbf{V}^+ = \mathbf{\Psi}_z$, of the Gent et al. (1995) method. The arguments following (61) above suggest that the error involved with equating the horizontal bolus velocity with $\mathbf{\Psi}_z$, while not being small locally, is small when considering a spatial average. Even if $\mathbf{\Psi}_z$ were estimated rather accurately in the ocean interior by this procedure, it will be poorly known near the ocean floor and near the sea surface (see Fig. 5a). This uncertainty in the near-boundary values of $\mathbf{\Psi}_z$ causes a depth-independent uncertainty of $\mathbf{\Psi}$ in the ocean interior (Fig. 5b).

By contrast, the TRM theory and also the Gent et al. (1995) scheme recognizes the need to specify $\mathbf{\Psi}$ at every point in the ocean and so avoids the depth-independent uncertainty that characterizes the downgradient potential vorticity approach. The key here is the difference between having knowledge in the ocean interior of $\mathbf{\Psi}$ versus only knowing its vertical derivative. The downgradient potential vorticity parameterization, which is suggested by the modeling study of Marshall et al. (1999), only specifies the vertical derivative of the quasi-Stokes streamfunction; extra information from near the boundaries is needed to close the problem and so provide the quasi-Stokes streamfunction everywhere. The main conclusion of the present paper is that what needs to be parameterized for a forward eddyless model is the quasi-Stokes streamfunction (4a). The task of finding this quasi-Stokes streamfunction is more demanding than simply parameterizing its vertical derivative.

The different vertical profiles of the quasi-Stokes streamfunction shown in Fig. 5b have rather direct implications for the horizontal flux of potential temperature on a cast, since (using integration by parts)

$$\int_{-H}^{0} \mathbf{V}^+ \theta \, dz = -\int_{\theta(-H)}^{\theta(0)} \mathbf{\Psi} \, d\theta \qquad (62)$$

so that our relative ignorance of how the bolus velocity should be distributed in the vertical near the upper and lower boundaries, and apportioned between the upper and lower boundaries, causes an offset to the quasi-Stokes streamfunction in the ocean interior, which in turn directly affects the horizontal flux of potential temperature. By contrast, the TRM scheme requires knowledge of the quasi-Stokes streamfunction at each height in the ocean interior, so if one is able to directly parameterize this streamfunction rather than just its vertical derivative, then the horizontal flux of potential temperature is not sensitive to the way the streamfunction approaches zero at the ocean boundaries.

It can be shown that in the downgradient potential vorticity mixing approach a change to the near-boundary structure of $\mathbf{\Psi}_z$ on just one cast causes an adiabatic vertical overturning cell of the quasi-Stokes velocity that extends all the way from the bottom boundary to the top boundary where it must close, as sketched in Fig. 5d. By contrast, in the TRM scheme, any uncertainty in $\boldsymbol{\Psi}$ near an ocean boundary only affects the flow locally near that boundary.

The present tension then is between (i) a downgradient potential vorticity mixing parameterization that seems well supported by modeling studies but is not yet able to provide enough information to close the eddy-parameterization problem (because ocean models need not only $\boldsymbol{\Psi}_z$ but also $\boldsymbol{\Psi}$) and (ii) the Gent et al. (1995) scheme that is equivalent to the assumption that horizontal eddy density flux is directed down the horizontal density gradient in some area-averaged sense. This assumption does not have as much theoretical or modeling support as the downgradient potential vorticity assumption, but it does constrain the full quasi-Stokes streamfunction rather than simply its vertical derivative. The present choice then seems to be between having a good vertical profile of the quasi-Stokes streamfunction but an uncertain offset versus having less confidence in the vertical structure of the quasi-Stokes streamfunction but having the depth-integrated value of the quasi-Stokes streamfunction in the ocean interior known with more confidence.

This discussion draws attention to the importance of the way eddies interact with bottom topography and, in particular, to what extent this interaction may cause a depth-independent offset to the quasi-Stokes streamfunction in the ocean interior. To date the only work that uses the downgradient potential vorticity assumption and that also specifies the quasi-Stokes streamfunction everywhere is Killworth (1997), where linear stability analysis is used to estimate the ratio of the bolus transport that is carried near the top and the bottom of the ocean. It should be noted that at the large horizontal scales appropriate to global circulation problems the quasi-Stokes streamfunction can be parameterized independently of the ''Neptune Effect'' as described by Holloway (1997). This is because at these scales the Neptune effect forces a barotropic Eulerian flow whereas the quasi-Stokes circulation of the TRM theory is baroclinic, having no depth-integrated transport at each location.

## 12. The TRM form of the horizontal momentum equations

Gent and McWilliams (1996) have recently discussed the horizontal momentum equations for use with the Gent et al. (1995) eddy parameterization scheme. In this regard we must caution against their use of the words ''Eliassen–Palm flux'' since this type of flux has a specific meaning in the zonal averaging literature that has not been demonstrated under other types of averaging. An Eliassen–Palm flux has the very important property that, though it is of second order in perturbation quantities, its divergence is of higher order (cubic order) under steady conditions and for small amplitude perturbations. Hence the divergence of the Eliassen–Palm flux can be ignored under nonacceleration conditions. (Our nondivergent fluxes, $\mathbf{M}$, $\mathbf{N}$, $\mathbf{M}^\tau$, and $\mathbf{N}^\tau$ have the same property that the individual components are second order in perturbations but the divergences of these fluxes can be ignored.) Andrews and McIntyre (1976) found an Eliassen–Palm flux for which this property held under zonal averaging in a fluid without meridional boundaries. Neither Lee and Leach (1996) nor Gent and McWilliams (1996) have proved this property under either temporal or ensemble averaging and neither has any other author, so the Eliassen–Palm flux for three-dimensional flows remains to be discovered—if indeed it exists.

The default approach when using the Gent et al. (1995) eddy parameterization scheme has been to use the Reynolds-averaged horizontal momentum equations with a Fickian assumption on the Reynolds fluxes. The momentum equations are then taken to be prognostic equations for the horizontal Eulerian-mean velocity while the skew diffusion (using the quasi-Stokes streamfunction) is added to the conservation equations for scalars but not to the momentum equations. Here we derive the TRM form of the horizontal momentum equations, building on the derivation of the tracer equation in section 7 of this paper.

The mean conservation equation for the eastward (or northward) velocity component can be written as (43) with $u$ in place of the tracer $\tau$ and replacing the source term $X$ with $X - (1/\rho_0)p_x + f\upsilon$. The same derivation process as in (44)–(51) can then be followed so that the top line of (51) becomes

$$\hat{u}_t + \boldsymbol{\nabla}\cdot(\overline{\mathbf{U}}^{\#}\hat{u}) - f\overline{\upsilon}^{\#} = -\frac{1}{\rho_0}\overline{(p_x)}^{\#} + \overline{X}^{\#} - \tilde{\gamma}_z\left(\overline{\frac{Q''u''}{\gamma_z}}\right)_{\tilde{\gamma}} - \tilde{\gamma}_z\boldsymbol{\nabla}_{\tilde{\gamma}}\cdot\left(\overline{\frac{\mathbf{V}''u''}{\gamma_z}}\right) - \boldsymbol{\nabla}\cdot\mathbf{M}^u + O(\alpha^3). \quad (63)$$

The nondivergent flux $\mathbf{M}^u$ is defined as in (49) with $u$ in place of $\tau$. The second and third terms on the right of (63) are traditionally parameterized with a vertical viscosity, while the fourth term is the transport of momentum along the isopycnals. It is not obvious that a downgradient Fickian assumption is appropriate for this epipycnal flux of momentum since momentum is not a passive tracer, and it is well known that eddies can act to accelerate the Eulerian-mean flow. The TRM form of the horizontal pressure gradient in (63) is defined [similarly to (46)] as

$$\overline{(p_x)}^{\#} \equiv \overline{p}_x + \left[-\frac{\overline{p_x'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{p}_{xz}}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right]_z, \quad (64)$$

and it is this form of the horizontal pressure gradient that embodies the eddy forcing of the mean flow. The new feature of (63) is that, unlike the work of Gent and

McWilliams (1996) and Greatbatch (1998), the left-hand side is written in terms of the one velocity vector, namely the thickness-weighted velocity (since we know that the thickness-weighted velocity is the same as the TRM velocity, to third order in perturbation amplitude).

From our result (28) above we know that the TRM form of the horizontal pressure gradient, $\overline{(p_x)}^{\#}$, can be interpreted as the thickness-weighted horizontal pressure gradient that arises when the momentum equation is averaged in density coordinates (e.g., Lee and Leach 1996; Smith 1999). To parameterize the quadratic perturbation terms on the right-hand side of (64) we assume that the perturbations are geostrophic so that $fv' = \rho_0^{-1}p'_x$ and (64) becomes (also using the hydrostatic equation and the leading order geostrophic balance for the mean flow, $f\overline{v} = \rho_0^{-1}\overline{p}_x$)

$$\frac{1}{\rho_0}\overline{(p_x)}^{\#} \approx \frac{1}{\rho_0}\overline{p}_x + f\left[-\frac{\overline{v'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{v}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right]_z$$

$$= \frac{1}{\rho_0}\overline{p}_x + f\Psi^y_z + O(\alpha^3). \quad (65)$$

Assuming that the Reynolds stresses can be parameterized in a Fickian form, that is, that $\overline{X}^{\#} - \tilde{\gamma}_z\overline{(Q''u''/\gamma_z)}_{\tilde{\gamma}} - \tilde{\gamma}_z\nabla_{\tilde{\gamma}}\cdot\overline{(\mathbf{V}''u''/\gamma_z)}$ can be written using a symmetric diffusion tensor as $\nabla\cdot(\mathbf{S}\nabla\hat{u})$, the eastward momentum equation can be written compactly as

$$\overline{u}^{\#}_t + \overline{\mathbf{U}}^{\#}\cdot\nabla\overline{u}^{\#} - f\overline{v}^{\#}$$

$$= -\frac{1}{\rho_0}\overline{p}_x + \nabla\cdot(\mathbf{S}\nabla\overline{u}^{\#} - \mathbf{k}f\Psi^y) + O(\alpha^3). \quad (66)$$

As noted above, only one type of velocity appears in this TRM momentum equation, namely the TRM velocity, and this form of the equation is the natural extension to three dimensions of the transformed Eulerian mean zonal momentum equation of the zonal averaging literature. By analogy then the flux, $\mathbf{S}\nabla\overline{u}^{\#} - \mathbf{k}f\Psi^y$, is the three-dimensional version of the Eliassen–Palm flux of the zonal-averaging theory. The physical interpretation of the quasi-Stokes streamfunction we have found in section 2 provides information on how the extra stress should behave as the ocean surface and floor are approached (see section 8).

One of the features of the Eliassen–Palm flux of the zonal-averaging problem is that its divergence can be approximated by the northward flux of potential vorticity. While this has never been proven theoretically for the three-dimensional case of temporal averaging, Lee and Leach (1996) provide strong circumstantial evidence that this result may carry over to three dimensions. This would imply that

$$-\tilde{\gamma}_z\nabla_{\tilde{\gamma}}\cdot\overline{(\mathbf{V}''u''/\gamma_z)} - f\Psi^y_z$$

could be approximated by the northward flux of isopycnal potential vorticity. Wardle (1999) shows that a downgradient assumption on the potential vorticity flux can be more general that the normal Fickian assumption on the Reynolds stresses because in many locations where the Reynolds stress is directed up the relevant velocity gradient (equivalent to a negative diffusion coefficient) the potential vorticity flux is still directed down its gradient.

Greatbatch (1998), building on the earlier paper of Greatbatch and Lamb (1990), has also recommended that the effects of unresolved eddy motions be parameterized in the horizontal momentum equations rather than in the tracer equations [as occurs in the Gent et al. (1995) method]. Our accurate derivation of the TRM momentum equations above, where we have retained all terms that are quadratic in perturbation amplitude, supports this suggestion as an alternative method of parameterizing eddy motions in eddyless models, although it should be noted that the geostrophic assumption had to be made to arrive at (66). Moreover, the TRM theory provides an expression in height coordinates, (4b), for the quasi-Stokes streamfunction (that appears multiplied by $f$ as the extra stress in the momentum equations) and this should act as a convenient target for the parameterization task. In this approach one would retain separate parameterizations for the Reynolds stress and for the thickness-weighted horizontal pressure gradient; this contrasts with the potential vorticity approach where one faces only one parameterization task (except that the epineutral diffusion of tracers would still need to be parameterized). Equation (23) of Gent et al. (1995) is similar to our TRM momentum equation. That paper went on to make a geostrophic approximation so that the extra stress was parameterized using a vertical viscosity and the difference between the Eulerian-mean and the TRM velocity was also ignored in this viscous term. The TRM momentum equation derived above shows that the stress is best left as the Coriolis parameter multiplied by the physically motivated quasi-Stokes streamfunction and that the whole momentum equation then contains only one type of velocity, namely the TRM velocity.

If a parameterization is implemented in the horizontal momentum equations, then the whole system of equations (momentum, continuity, and tracer equations) carries only the one type of velocity variable, namely the TRM velocity, so there is no need to include skew-diffusion or quasi-Stokes advection in the tracer equations. This procedure of parameterizing the unresolved eddy motions in the horizontal momentum equations needs to be thoroughly tested and compared with the Gent et al. (1995) scheme where the quasi-Stokes streamfunction is used to construct a skew flux of tracer in the tracer equation but the standard momentum equation is assumed to time step the Eulerian-mean horizontal velocity. In appendix B we discuss a rather small issue related to the evaluation of the Eulerian mean of the horizontal pressure gradient.

## 13. Discussion

In this paper we have derived a new temporal-residual-mean circulation that has the desirable property that for adiabatic flow, the TRM velocity has no component through the appropriately defined mean density surfaces. Unlike the results of McDougall and McIntosh (1996), this ''adiabatic'' result holds regardless of whether the low-passed time-averaged flow is evolving or not. The definition of the TRM velocity field has changed slightly and we have found that the mean density conservation equation must be written for a very specific density variable: it is the density variable whose surface is, on average, at the height of the averaging. It is intended that the definition of the density surfaces and the TRM velocity used in this paper replace the definitions in McDougall and McIntosh (1996).

Since much of our intuition about ocean mixing and modeling comes from thinking in density coordinates and because the rate of diapycnal mixing is so much less than the mixing that occurs along density surfaces, we have pursued the relationship between averaging in height coordinates and in density coordinates. Using a Taylor series approach we have found the following quasi-Lagrangian interpretation for the quasi-Stokes streamfunction $\boldsymbol{\Psi}$ of the TRM circulation: it is the contribution of perturbations to the average horizontal volume flux of water that is denser than the model's density, $\tilde{\gamma}$, at the depth of averaging. Because of this layered interpretation of this $z$-coordinate streamfunction, it follows that the TRM velocity gives the same horizontal transport of fluid between a pair of resolved-scale density surfaces as occurs when the averaging is done in density coordinates between the same pair of density surfaces. It has also been proved that the TRM velocity has the same diapycnal component as is found by averaging the instantaneous diapycnal velocity in density coordinates, so the volume conservation statement written in height coordinates using the TRM velocity is the same as is obtained by averaging in density coordinates. The TRM theory of this paper applies to models whose vertical coordinate does not vary in height as a function of time. In this way the TRM theory applies to both eddyless height-coordinate models and to eddyless sigma-coordinate models such as SPEM. The Taylor series approach of our section 2 has been used recently by Kushner and Held (1999) to quantify the total flow between potential vorticity surfaces (rather than potential temperature surfaces).

The rather surprising result of McDougall and McIntosh (1996) that the TRM velocity advects the isopycnally averaged tracer value has been confirmed. This result has been extended to include a general tracer that is now allowed to vary arbitrarily along density surfaces. We conclude that the tracer in eddyless height-coordinate models should be interpreted as the thickness-weighted tracer that would result from averaging in density coordinates.

It has been shown that the eddy-induced velocities that are needed in eddyless models are quite different in isopycnal models compared with height-coordinate models. In isopycnal models the extra velocity that is needed is the bolus velocity, which has no diapycnal component and is three-dimensionally divergent. In stark contrast, in height-coordinate models the eddy-induced velocity, $\mathbf{U}^{+}$, is nondivergent and has a substantial diapycnal component. While the horizontal components of the bolus velocity and the quasi-Stokes velocity are not materially different, their vertical components are very different.

When the approximate expression, (4b), is used for the quasi-Stokes streamfunction, the above results have errors that are cubic in the amplitude of perturbation quantities, so the question arises whether these errors may be significant. On the basis of the corresponding zonal-residual-mean theory, we have reason to believe that the cubic terms will be quite small. McIntosh and McDougall (1996) extended the zonal-residual-mean theory of Andrews and McIntyre (1976) to the oceanic situation where the continents interrupt the zonal integrals; they showed that the zonally thickness-weighted circulation (evaluated by zonally averaging in density coordinates) was well approximated by the zonal-residual-mean circulation (which was evaluated in height coordinates). This close correspondence occurred even though at a constant latitude in the Southern Ocean, density surfaces vary in height by about 1500 m. In the TRM case of temporal averaging at a given latitude and longitude, density surfaces undulate much less than 1500 m, with root-mean-square vertical excursions of 150 m being more typical. Hence we expect that ignoring the cubic terms in perturbation amplitude will be quite adequate for our purposes in the TRM theory. When the accurate expression, (4a), is used for the quasi-Stokes streamfunction, then all the terms in the tracer conservation equations have a one-to-one correspondence to the corresponding terms when thickness-weighted averaging in density coordinates (this assumes that the momentum equations are prognostic equations for the Eulerian-mean velocity). Hence the use of (4a) circumvents the issue of the cubic-order approximation.

We have pointed out a very practical difficulty in using the down-epineutral-gradient of potential vorticity approach, namely that the uncertainty in the near-boundary regions affects the solution throughout the whole water column. In this way the section-integrated contribution of eddy-induced advection to the horizontal transport of heat is sensitive to the uncertain distribution of eddy potential vorticity fluxes at the top and bottom of the ocean. By having parameterized knowledge of the quasi-Stokes streamfunction itself (rather than just of its vertical derivative), the TRM theory avoids this uncertainty. The eddy-resolving numerical calculations of Marshall et al. (1999) show that the bolus velocity is more closely directed up the isopycnal gradient of potential vorticity than down the gradient of thickness, although the difference (being $\beta\kappa^q/f$) is quite small compared to the dominant bolus velocity in the southern ocean. This type of study is very useful in showing what is required of the bolus velocity, and to sufficient accuracy, probably what is also required of $\boldsymbol{\Psi}_z$, (although this type of study has only been performed to date for zonal averaging) but only the work of Killworth (1997) has to date addressed the practical issue of finding the three-dimensional eddy-induced velocity. Killworth has used linear theory to say something about the question raised by our Fig. 5, namely, the relative fluxes of bolus transport near the sea surface versus near the ocean bottom.

When it comes to parameterizing the Reynolds stress, $\overline{\mathbf{U}'u'}$, in the $\overline{u}$ momentum equation, it has been traditional in the use of the Gent and McWilliams (1990) method to simply assume Fickian diffusion with different horizontal and vertical viscosities. Using this procedure, the velocity that appears in the horizontal momentum equation is taken to be the Eulerian-mean velocity and consequently the tracer equations need the skew-diffusion (or quasi-Stokes advection) derived from the parameterized quasi-Stokes streamfunction. However, just as we now realize that mesoscale eddy activity drives a skew-diffusive flux of tracer in the tracer equations, so it is reasonable to expect that this will be a feature of the momentum equations. This leads to a procedure whereby an ocean model can include an extra horizontal stress in the momentum equation and that this need be the only place in all the model equations that the dynamical effects of mesoscale eddies are included. There is however one important caveat to this picture, namely, that the eddies had to be assumed to be in geostrophic balance in order to arrive at the TRM form of the momentum equation, (66). Operationally this procedure is very different to the Gent et al. (1995) implementation of the temporal residual mean but importantly, the TRM theory tells us what the quasi-Stokes streamfunction and the vertical form drag must be in terms of quantities that are evaluated at fixed height. In both cases the problem reduces to parameterizing the quasi-Stokes streamfunction of (4). In both cases the physical interpretation of this streamfunction demands that it must be smoothly tapered to zero on all boundaries, and this tapering avoids unphysical delta functions of horizontal velocity.

There is always the issue of whether the present implementation of the TRM circulation in ocean models is displaying improvements for the correct reasons. Several authors have described substantial improvements including greatly improved deep-water masses, less unwanted deep convection, and less drift in coupled atmosphere–ocean models (see Hirst et al. 1996). An important common element of these improvements is that the bottom water of the world's oceans has been able to sink from the surface to the ocean bottom with very little dilution. In fact, in the work of Hirst and McDougall (1996) it was found that there was insufficient diapycnal mixing occurring in the overflow regions. Previously such a result had only been possible with a density-coordinate model. In this way, a height-coordinate model has been shown to be sufficiently ''adiabatic'' for the purposes of climate modeling.

In practice this vertical motion of the deep and bottom water occurs in canyons and across sills that are not part of the coarse resolution models. The TRM advection scheme achieves this ''adiabatic'' sinking motion because of two almost equal effects, as demonstrated by Hirst and McDougall (1996). First, the unwanted horizontal diffusion is eliminated and, second, an extra advection (or skew diffusion) is added that assists in the transport of water from the surface to the deep. The elimination of horizontal diffusion is thought to be physically required, but the extra advection at the bottom of the ocean seems to be more an artefact of the bottom boundary condition on the quasi-Stokes streamfunction than a representation of the actual boundary current mechanisms that achieve the transport of bottom water. In this way it may be that half of the benefits that we are seeing to date have been obtained for the wrong reasons. If so, what fraction of this half will respond incorrectly to changing boundary conditions associated with, for example, climate change?

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## APPENDIX A

### Thickness-Weighted Averaging

The thickness between two closely spaced density surfaces differing in density by $\Delta\tilde{\gamma}$ is given by $h = \Delta\tilde{\gamma}/\gamma_z$ in the limit as $\Delta\tilde{\gamma} \to 0$. The thickness-weighted average of any quantity, $A$, following the $\tilde{\gamma}$ density surface is $\hat{A} = \overline{(Ah)|_{\tilde{\gamma}}}/\overline{h|_{\tilde{\gamma}}}$, which in terms of $\gamma$ becomes $\hat{A} = \overline{(A/\gamma_z)|_{\tilde{\gamma}}}/\overline{(1/\gamma_z)|_{\tilde{\gamma}}}$. The average thickness between two closely spaced $\tilde{\gamma}$ surfaces is clearly the same as the difference between the average heights of the same two surfaces, and hence it is apparent that, without approximation, the nature of our $\tilde{\gamma}$ surfaces ensures that $\overline{(1/\gamma_z)|_{\tilde{\gamma}}} = 1/\tilde{\gamma}_z$. This follows because the average perturbation height of any $\tilde{\gamma}(z)$ surface about the height $z$ is, by definition, zero. In order to evaluate $(A/\gamma_z)|_{\tilde{\gamma}}$ we begin by writing a vertical Taylor series about the fixed height, $z_a$,

$$\left.\left(\frac{A}{\gamma_z}\right)\right|_{\tilde{\gamma}} = \left.\left(\frac{A}{\gamma_z}\right)\right|_{z_a} + z_a'\left(\frac{A}{\gamma_z}\right)_z + \frac{1}{2}(z_a')^2\left(\frac{A}{\gamma_z}\right)_{zz} + O(\alpha^3), \qquad \text{(A1)}$$

where it is understood that the first- and second-order partial derivatives of $(A/\gamma_z)$ are evaluated at the constant height, $z = z_a$, and that these partial derivatives have both mean and perturbation parts. The next step is to find the mean and perturbation parts of $A/\gamma_z$ by first performing the Reynolds decomposition

$$\left.\left(\frac{A}{\gamma_z}\right)\right|_{z_a} = \frac{(\overline{A} + A')}{\overline{\gamma}_z}\left(1 - \frac{\gamma_z'}{\overline{\gamma}_z} + \frac{\gamma_z'^2}{\overline{\gamma}_z^2}\right) + O(\alpha^3) \quad \text{(A2)}$$

so that the mean and perturbation parts are

$$\overline{\left.\left(\frac{A}{\gamma_z}\right)\right|_{z_a}} = \frac{\overline{A}}{\overline{\gamma}_z}\left(1 + \frac{\overline{[\gamma_z']^2}}{\overline{\gamma}_z^2}\right) - \frac{\overline{A'\gamma_z'}}{\overline{\gamma}_z^2} + O(\alpha^3) \quad \text{(A3)}$$

and

$$\left.\left(\frac{A}{\gamma_z}\right)'\right|_{z_a} = \frac{A'}{\overline{\gamma}_z} - \frac{\overline{A}\gamma_z'}{\overline{\gamma}_z^2} + O(\alpha^2). \quad \text{(A4)}$$

Taking the time average of (A1), multiplying by $\tilde{\gamma}_z$, and using (10), (11), (A3), and (A4) it can be shown after considerable algebra that

$$\begin{aligned}\hat{A} &\equiv \tilde{\gamma}_z\overline{\left.\left(\frac{A}{\gamma_z}\right)\right|_{\tilde{\gamma}}} \equiv \tilde{A} + \overline{A'|_{\tilde{\gamma}}z_z'} \\ &= \overline{A} + \left[-\frac{\overline{A'\gamma'}}{\overline{\gamma}_z} + \frac{\overline{A}_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right]_z + O(\alpha^3). \quad \text{(A5)}\end{aligned}$$

## APPENDIX B

### The Hydrostatic Equation Using Modified Density

Here we discuss issues related to the evaluation of the Eulerian mean of the horizontal pressure gradient. In appendix B of McDougall and McIntosh (1996) we pointed out that there are two reasons why $\rho_0^{-1}\nabla_H\overline{p}$ is unavailable to an eddyless model. The most important reason was that one does not know the horizontal variation of temperature variance in an eddyless model. The model's estimate of the horizontal gradient of the Eulerian-mean density, $\nabla_H\overline{\rho}$, is therefore incomplete [see Eq. (B3) of that paper] and, because these models vertically integrate the hydrostatic equation, this implies an incomplete evaluation of $\rho_0^{-1}\nabla_H\overline{p}$. This problem is not associated with the use of an eddy parameterization scheme as such, but rather is a problem inherent to an eddyless model. While this can result in a misestimation of the geostrophic velocity by up to 1 mm s$^{-1}$ locally, this has not been regarded as a problem because the error occurs in the form of a horizontal divergence. Therefore any local overestimation in a region of high eddy activity is matched nearby with a region of underestimation so that no persistent errors can accumulate.

Now we address a third (but equally unimportant) reason why $\rho_0^{-1}\nabla_H\overline{p}$ is unavailable to an eddyless model. In this paper we point out that the tracers of an eddyless model should be interpreted as the thickness-weighted tracers evaluated on a specific density surface, namely the surface whose height is on average at the height concerned. In the ocean interior, this density is different to the Eulerian-mean density by the amount $-(\overline{\phi}/\overline{\gamma}_z)_z$ (plus higher order terms). The use of the hydrostatic equation to evaluate the model's pressure field,

$$p_z^m = -g\rho(\hat{S}, \hat{\theta}, p) = -g\rho^m,$$

means that the horizontal pressure gradient that is available to the model, $-\rho_0^{-1}p_x^m$, is different from $-\rho_0^{-1}\overline{p}_x$. Since $\rho^m - \overline{\rho} = -(\overline{\phi}/\tilde{\gamma}_z)_z$, the true Eulerian-mean hydrostatic equation, $\overline{p}_z = -g\overline{\rho}$, is used together with the model's hydrostatic equation to show that $-\rho_0^{-1}(\overline{p} - p^m)_x = (g/\rho_0)(\overline{\phi}/\tilde{\gamma}_z)_x$ so that (66) becomes

$$\begin{aligned}&\overline{u}_t^\# + \overline{\mathbf{U}}^\# \cdot \nabla\overline{u}^\# - f\overline{v}^\# \\ &\quad = -\frac{1}{\rho_0}p_x^m + \frac{g}{\rho_0}\left(\frac{\overline{\phi}}{\tilde{\gamma}_z}\right)_x - f\Psi_z^y + \nabla\cdot(\mathbf{S}\nabla\overline{u}^\#) \\ &\qquad + O(\alpha^3). \quad \text{(B1)}\end{aligned}$$

This new term is equivalent to a velocity difference of no more than 1 mm s$^{-1}$ in the term involving the Coriolis parameter in (B1). Hence we can conclude that there is no more loss of accuracy in estimating the Eulerian-mean pressure field by vertically integrating the model's density field, $\rho^m = \rho(\hat{S}, \hat{\theta}, p)$, than has traditionally been incurred in eddyless models due to the lack of the larger cabbeling term [as described in the appendix of McDougall and McIntosh (1996)]. Furthermore, as these error terms occur as horizontal divergences, they do not cause persistent errors. From (61) we note that if the perturbations are assumed to be geostrophic, the extra term in (B1),

$$\frac{g}{\rho_0}(\overline{\phi}/\tilde{\gamma}_z)_x,$$

is equal to $f(\tilde{v} - \overline{v})$ and from (60) this is equal to $f(\Psi_z^y - v^B)$ so that the sum of the second and third terms on the right of (B1) amount to $-fv^B$ if the perturbations are assumed to be in geostrophic balance. Since there is a larger term missing from (B1) due to cabbeling, we do not attach much store to the difference between $-f\Psi_z^y$ and $-fv^B$ on the right-hand side.

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