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# A consistent derivation of the wave-energy equation from basic hydrodynamic principles

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**Abstract** Based on a decomposition of the velocity into mean flow, turbulent and wave components, momentum and hereafter a wave-energy equation is derived. It contains a turbulent energy dissipation term which is closed by applying a wave-related mixing length model and linear wave theory solutions. This closure produces a non-linear turbulent wave-energy dissipation including the wave energy in a 5/2 power law. The theory is able to predict correctly the shape of deep-water wave spectra according to Phillips' similarity law.

**Keywords** Wave energy spectrum · Wave energy dissipation · Reynolds decomposition

## 1 Introduction

Wave energy defined as the square of the wave height $H$ or amplitude $A$ is advected by the mean flow and the group velocity. It radiates energy to shear flows and is dissipated by bottom shear stresses, viscous and turbulent damping. Wave energy is lost also through breaking in shallow and white capping in deep water. When the wave-energy spectrum is regarded, it is believed that quadruplet non-linear interactions transfer energy within the spectrum.

For the description of these processes several wave energy equations have been derived. Initially, it was assumed that the observed changes in amplitude are mainly related to refraction and shoaling and the wave energy is obeying the law

$$\mathrm{div}\left[(\bar{\mathbf{u}} + \mathbf{c_g})E\right] = 0 \ ,$$

where $\bar{\mathbf{u}}$ is the mean flow and $\mathbf{c_g}$ the group velocity. $E$ is the wave energy defined as

$$E = \frac{1}{2}\varrho g A^2 = \frac{1}{T}\int_0^T \int_{z_B}^{z_S} \left(\frac{1}{2}\varrho \mathbf{u}^{\mathrm{w}}\mathbf{u}^{\mathrm{w}} + \varrho g z\right) \mathrm{d}z \, \mathrm{d}t \ ,$$

representing the kinetic and potential energy of Airy waves integrated over the wave period $T$ and water depth. Here $z_B$ and $z_S$ are the vertical coordinates of the bottom and the free surface, respectively, and $\mathbf{u}^{\mathrm{w}}$ is the orbital wave velocity.

Longuet–Higgins and Steward (1961a,b, 1962) proved for several flow situations that there is an exchange of energy between the mean flow and the waves, and that the resulting wave amplitude behaviour can be reproduced by:

$$\frac{\partial E}{\partial t} + \mathrm{div}\left[(\bar{\mathbf{u}} + \mathbf{c_g})E\right] + \frac{1}{2} S_{ij}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0 \ ,$$

where $S_{ij}$ is the so-called radiation stress. Because their approach was based on second-order Stokes theory, i.e. non-viscous flows, no mechanism for the dissipation of wave energy was found.

This changed in the derivation presented by Phillips in his text book on the dynamics of the upper ocean (Phillips 1966). He derived the total energy equation for the combined wave and mean flow by multiplying the Navier–Stokes equations with the velocity and adding the potential energy. The integration over the water depth was done without neglecting dispersion effects. After subtracting the depth-integrated energy equation for the mean flow, the wave energy no longer contains a potential energy component, i.e.

$$E = \frac{1}{4}\varrho g A^2 = \frac{1}{T}\int_0^T \int_{z_B}^{z_S} \frac{1}{2}\varrho \mathbf{u}^{\mathrm{w}}\mathbf{u}^{\mathrm{w}} \, \mathrm{d}z \, \mathrm{d}t$$

for related Airy waves. This change in the wave-energy definition is not dramatic unless the wave-energy equation is linear in the wave energy itself. Again, Phillips'

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derivation did not include viscosity, but he included viscous dissipation as a supplement.

Further improvements to this derivation refined the integration over the water depth introducing the boundary conditions at the bottom and the free surface whereby wind and bottom shear stresses as source and sink terms can be introduced straightforward (Mei 1989; Johnson 1997). Milbradt (1995) also added a sink term for wave breaking and showed that the resulting model is able to cope with large-scale coastal wave modelling.

Spectral wave-energy models start from the conservation of wave-action density $N = 2|A|^2/\omega$ given as:

$$\frac{\partial N}{\partial t} + \mathrm{div}\left[(\bar{\mathbf{u}} + \mathbf{c_g})N\right] = 0 \ ,$$

where $\bar{\mathbf{u}}$ is the mean flow and $\mathbf{c_g}$ the group velocity. This equation is derived from the classical kinematic wave theory, which analyzes the behaviour of a harmonic function dependent on an arbitrary phase function (Whitham 1965; Bretherton and Garrett 1969; Hayes 1970). The same derivation process leads to the spectral wave-energy equation when the wave-action density is assumed to be also a function of the wave number (Willebrand 1975). The main characteristic of this approach is the fact that it is originally a kinematic and not a hydrodynamic approach. Therefore sources and sinks for the energy of surface waves do not appear in the derivation of the wave-action density equation and have to be taken into account empirically. These were obtained from the JONSWAP dataset, whereby Hasselmann et al. (1973) concluded that turbulent energy dissipation is a minor important process while surface effects like white capping are main dissipation source of waves in deep water. This kind of model is one of the most successful approaches for ocean wave modelling. The WAM model as it is presented by Komen et al. (1994) did not contain a mechanism for wave-energy dissipation due to turbulence. Further developments of the model included a quadratic dissipation mechanism (Schneggenburger 1999) which was attributed to the dissipation related with the eddy viscosity (Rosenthal 1989).

Independently of wave-energy modelling, several attempts were made to study wave dynamics using a decomposition of the flow field into a mean flow, the periodic wave motion and turbulent fluctuations. For the separation of the three kinds of motions two different averaging processes have to be performed. Svendsen and Lorenz (1989) used this decomposition to improve the theory of undertow and long-shore currents neglecting viscous effects. They started their decomposition with an ensemble averaging for the turbulence and continued with an averaging over a wave period. You et al. (1991) studied in this way the vertical velocity distribution in a combined flow of waves and mean currents. They separated mean and wave motions by time and phase averaging; but none of them proceeded to a wave-energy equation. Also very often a decomposition into the mean and the turbulent components is found whereby the mean flow is identified with the wave motion. In this way, Teixeira and Belcher (2002) for example, showed how kinetic wave energy is transformed into turbulent kinetic energy as the wave propagates.

In this paper a new approach to derive the wave-energy equation is presented. First, the flow field is decomposed into three components i.e. mean, turbulent and wave motions (Malcherek 2001). Second, the Navier–Stokes equations are split into three sets of momentum equations for the three kinds of movements using a long- and a short-term time-averaging process. It will be shown that the decomposition is consistent with the original equations, i.e. when adding the resulting momentum equations for the three components the original Navier–Stokes equations are reobtained. From the momentum equations of the wave field three-dimensional kinetic wave-energy equations can be derived straightforwardly. Double averaging over the wave period and water depth leads to an equation for the wave energy with some unknown correlation terms. These terms are closed using linear wave theory and a new mixing length model for the wave-induced turbulence. Finally, a wave-energy equation is obtained where the energy is propagated in a velocity field, being the sum of mean and group velocity. Energy dissipation processes due to viscosity, turbulence and bottom shear stress as well as wind input are clearly identified in this derivation.

The presented theory states that wave-energy dissipation related to wave-induced turbulence is the most important process which is able to explain correctly the shape of the spectrum according to Phillips' similarity law (Phillips 1978).

## 2 Consistent Reynolds decomposition of the velocity field in mean flow, waves and turbulence

For the following derivation the Navier–Stokes equations are written in their conservative form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathrm{div}(\mathbf{u} \otimes \mathbf{u}) = -\frac{1}{\varrho}\mathrm{grad}\, p + \mathrm{div}\, P + \mathbf{f} \ ,$$

where $\mathbf{u}$ is the combined flow velocity, $\varrho$ is the fluid density, $P$ is the viscous tensor

$$P = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and $\nu$ the molecular viscosity. The tensor product $\otimes$ is given as:

$$\mathbf{u} \otimes \mathbf{v} = \left(u_i v_j\right)_{i,j=1,3}$$

and the divergence of a tensor produces the vector:

$$\mathrm{div}\, P = \left(\sum_{i=1}^{3} P_{ij}\right)_{j=1,3} \ .$$

The mean flow velocity is defined as a time average of the actual velocity of a time scale $\Delta t_l$, which is longer than the wave periods. For wind-generated waves this time scale lies in the range of 1 min:

$$\bar{\mathbf{u}}(x,y,z,t) = \frac{1}{\Delta t_l} \int_t^{t+\Delta t_l} \mathbf{u}(x,y,z,t)\mathrm{d}t \ .$$

Surface waves and turbulent fluctuations are superimposed on the mean flow whereby the turbulent fluctuations are assumed to occur on a shorter time scale, $\Delta t_s$, than the wave motions. In this case

$$\mathbf{u}^w(x,y,z,t) = \frac{1}{\Delta t_s} \int_t^{t+\Delta t_s} \mathbf{u}(x,y,z,t)\mathrm{d}t - \bar{\mathbf{u}}(x,y,z,t)$$

is a good definition for the wave-velocity components. It should be pointed out that $\mathbf{u}^w$ up to now contains the full spectrum of wave frequencies.

Finally, the short-scale turbulent fluctuations are given as:

$$\mathbf{u}' = \mathbf{u} - \bar{\mathbf{u}} - \mathbf{u}^w \ .$$

It should be mentioned that this approach does not exclude the existence of an overlap between the higher wave and lower turbulence frequencies, but it treats modes with periods larger than $t_s$ and shorter than $t_l$ as gravity wave-like motions. In the same way, the pressure $p$ is decomposed into mean, wave and turbulent fluctuations. After inserting the decompositions

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' + \mathbf{u}^w$$

$$p = \bar{p} + p' + p^w$$

into the Navier–Stokes equations and time averaging over the long-term period, the following equations for the dynamics of the mean currents are obtained:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \mathrm{div}\,(\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) = -\frac{1}{\varrho}\,\mathrm{grad}\bar{p} + \mathrm{div}\,(\bar{P} + T_l + R + S) + \mathbf{f}. \tag{1}$$

They contain the molecular viscosity tensor of the average flow field

$$\bar{P} = \nu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \ ,$$

the long-term Reynolds stress tensor

$$T_l = -\overline{\mathbf{u}' \otimes \mathbf{u}'} \ ,$$

the radiation stress tensor

$$S = -\overline{\mathbf{u}^\mathbf{w} \otimes \mathbf{u}^\mathbf{w}}$$

and another nameless tensor

$$R = -\overline{\mathbf{u}' \otimes \mathbf{u}^\mathbf{w}} - \overline{\mathbf{u}^\mathbf{w} \otimes \mathbf{u}'} \ ,$$

which describes the reaction on the interaction between turbulence and waves. Therefore the average flow field is affected by what turbulence and waves do with each other and the tensor $R$ can be characterized as a jealous one.

Subtracting the average flow field from the Navier–Stokes equations leads to dynamic equations for the turbulent and wave-induced fluctuations:

$$\begin{aligned}\frac{\partial \mathbf{u}' + \mathbf{u}^\mathbf{w}}{\partial t} &+ \mathrm{div}\,(\mathbf{u} \otimes \mathbf{u} - \bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) \\ &= -\frac{1}{\varrho}\mathrm{grad}\,(p' + p^w) \\ &\quad + \mathrm{div}\,(\nu\,\mathrm{grad}\,\mathbf{u} - \bar{P} - T_l - R - S) \ .\end{aligned}$$

It is assumed that the turbulent time scales are much smaller than the wave periods. Therefore an equation for the wave momentum can be obtained by an averaging over the turbulent time scales:

$$\begin{aligned}\frac{\partial \mathbf{u}^\mathbf{w}}{\partial t} &+ \mathrm{div}[\mathbf{u}^\mathbf{w} \otimes (\bar{\mathbf{u}} + \mathbf{u}^\mathbf{w})] = -\frac{1}{\varrho}\mathrm{grad}\,p^w \\ &+ \mathrm{div}\,(P^w - R - S - T_l + T_s - \bar{\mathbf{u}} \otimes \mathbf{u}^\mathbf{w}]\end{aligned} \tag{2}$$

with

$$P^w = \nu\left(\frac{\partial u_i^w}{\partial x_j} + \frac{\partial u_j^w}{\partial x_i}\right)$$

and the Reynolds stress tensor $T_s$ related to the short-term averaging of the non-linear advection terms:

$$T_s = -\frac{1}{\Delta t_s} \int_t^{t+\Delta t_s} \mathbf{u}'(x,y,z,t) \otimes \mathbf{u}'(x,y,z,t)\,\mathrm{d}t \ .$$

When this distinction between a long- and a short-term averaging is not made, i.e. $T_l = T_s$, the wave-energy equation will not contain a mechanism for the dissipation of wave energy by turbulence.

The momentum due to waves is advected with the average and the wave velocity field itself. Otherwise, there is no advection due to turbulence as expected because turbulence is a smaller-scale motion.

When the wave-momentum equation is subtracted from the equation for wave and turbulent fluctuations, an equation for the turbulent fluctuations is left:

$$\begin{aligned}\frac{\partial \mathbf{u}'}{\partial t} &+ \mathrm{div}\,(\mathbf{u}' \otimes \mathbf{u}) \\ &= -\frac{1}{\varrho}\mathrm{grad}\,p' + \mathrm{div}\,(P' - T_s - \bar{\mathbf{u}} \otimes \mathbf{u}' - \mathbf{u}^\mathbf{w} \otimes \mathbf{u}') \ .\end{aligned} \tag{3}$$

Turbulent fluctuations are advected with the total velocity consisting of average, wave and turbulent components. The last two terms on the right-hand side describe turbulence production by currents and waves, respectively.

When the three dynamic Eqs. (1), (2) and (3) are added together, the initial Navier–Stokes equations are obtained. Therefore, these equations form a consistent decomposition of the flow field in average, periodic and turbulent components.

## 3 The kinetic wave energy equation

Because the kinetic energy of the wave field is defined as

$$k^w = \frac{1}{2}\overline{\mathbf{u}^\mathbf{w}\mathbf{u}^\mathbf{w}} \ ,$$

Eq. (2) will be multiplied with the wave velocity and averaged over the wave period. The terms with the tensors $R$, $S$ and $T_l$ are cancelled because they are constant on the wave time scale:

$$\frac{\partial k^w}{\partial t} + \mathrm{div}\left(\bar{\mathbf{u}}k^w + \overline{\mathbf{u}^\mathbf{w} k^w} + \frac{\overline{\mathbf{u}^\mathbf{w}}}{\varrho} p^w\right) = \overline{\mathbf{u}^\mathbf{w}\,\mathrm{div}\,(P^w + T_s - \bar{\mathbf{u}} \otimes \mathbf{u}^\mathbf{w})} \ .$$

For ideal Airy as well as Stokes waves the viscous term $\overline{\mathbf{u}^\mathbf{w}\,\mathrm{div}\,P^w}$ vanishes. Therefore, it will be neglected. We will see later that even if it is not zero it will be overwhelmed by turbulent viscosity. The last term can be written as:

$$\overline{\mathbf{u}^\mathbf{w}\,\mathrm{div}\,(\bar{\mathbf{u}} \otimes \mathbf{u}^\mathbf{w})} = \overline{\mathbf{u}^\mathbf{w} \otimes \mathbf{u}^\mathbf{w}} : \mathrm{grad}\,\bar{\mathbf{u}} = -\mathrm{grad}\,\bar{\mathbf{u}} : S$$

Whereby : denotes the scalar product of two tensors A and B;

$$\mathrm{A{:}B} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\ b_{ji}$$

The short-term turbulence term is split into flux and source/sink contingents creating the wave-energy equation:

$$\frac{\partial k^w}{\partial t} + \mathrm{div}\ \Phi_{k^w} = -\overline{\mathrm{grad}\,\mathbf{u}^\mathbf{w} : T_s} + \mathrm{grad}\,\bar{\mathbf{u}} : S \tag{4}$$

with the kinetic wave energy flux:

$$\Phi_{k^w} = \bar{\mathbf{u}}k^w + \overline{\mathbf{u}^\mathbf{w} k^w} + \frac{1}{\varrho}\overline{\mathbf{u}^\mathbf{w} p^w} - \overline{\mathbf{u}^\mathbf{w} T_s} \ .$$

As distinct from the kinetic wave-energy equation as it is derived, for example, by Phillips, two additional terms appear in Eq. (4). First of all, the wave energy is also transported through short-term turbulent fluctuations ($-\overline{\mathbf{u}^\mathbf{w} T_s}$), second waves exchange (i.e. they produce) turbulent kinetic energy through the term $\overline{\mathrm{grad}\,\mathbf{u}^\mathbf{w}: T_s}$.

## 4 Derivation of the wave energy equation

The integral of the wave energy—however it is defined—over the water depth is for Airy waves proportional to the square of wave height or amplitude and therefore a very easily measurable quantity. In this paper the term wave energy is defined as the average kinetic energy integrated over the water depth:

$$e^w = \int_{z_B}^{z_S} k^w\,\mathrm{d}z = \frac{1}{4} g A^2 \ .$$

The dynamics of this quantity is obtained by the integration of Eq. (4) over the water depth. Here, the following integration formula for the wave-energy fluxes (Malcherek 2001) is very helpful

$$\int_{z_B}^{z_S} \mathrm{div}\,\Phi\,\mathrm{d}z = \mathrm{div}\int_{z_B}^{z_S} \Phi\,\mathrm{d}z + \Phi_S - \Phi_B \ ,$$

where $\Phi_S$ and $\Phi_B$ are the respective fluxes through the free surface and the bottom and the divergence on the right-hand side is meant to be a two-dimensional divergence. The advective flux of the kinetic wave energy with the mean flow can be integrated as

$$\int_{z_B}^{z_S} \mathrm{div}\,\bar{\mathbf{u}}k^w\,\mathrm{d}z = \mathrm{div}\int_{z_B}^{z_S} \bar{\mathbf{u}}k^w\,\mathrm{d}z \simeq \mathrm{div}\,\bar{\mathbf{u}}\int_{z_B}^{z_S} k^w\,\mathrm{d}z = \mathrm{div}\,\bar{\mathbf{u}}\,e^w.$$

The second equality stems from the fact that there exists no advective flux of kinetic wave energy through the bottom and the free surface. The third equality assumes a constant flow velocity over the depth. This approximation has to be improved when the influence of mean currents on waves is investigated.

Additionally, the relationship

$$\int_{z_B}^{z_S}\left(\overline{\mathbf{u}^\mathbf{w} k^w} + \frac{1}{\varrho}\overline{\mathbf{u}^\mathbf{w} p^w}\right)\mathrm{d}z = \mathbf{c_g} e^w \ ,$$

which holds true for linear waves, can be applied. Using all these relations, we end up with the wave-energy equation:

$$\frac{\partial e^w}{\partial t} + \mathrm{div}\left((\bar{\mathbf{u}} + \mathbf{c_g})e^w - \int_{z_B}^{z_S}\overline{\mathbf{u}^\mathbf{w} T_s}\,\mathrm{d}z\right) = -\Phi_S + \Phi_B - \int_{z_B}^{z_S}\overline{\mathrm{grad}\,\mathbf{u}^\mathbf{w} : T_s}\,\mathrm{d}z + \int_{z_B}^{z_S}\mathrm{grad}\,\mathbf{u} : S\,\mathrm{d}z \ .$$

This describes on the left-hand side the transport of wave energy by mean and group velocity. It further contains a wave-energy turbulent diffusion term. The first two terms on the right-hand side are the energy fluxes through the bottom and the free surface. They are used to model energy input by wind and its dissipation by the bottom shear stress. The last two terms are turbulent dissipation and energy radiation to the mean flow.

## 5 Turbulent wave energy diffusion and dissipation

For the application of the wave-energy equation the closure of the short-term Reynolds stress tensor $T_s$ is crucial. It consists of the turbulent fluctuations on a time scale which is shorter than the wave period. Therefore, we assume that it can be modelled applying an eddy viscosity principle whereby the turbulent fluctuations are induced by the wave orbital motions:

$$T_s = v_t\left(\frac{\partial u_i^w}{\partial x_j} + \frac{\partial u_j^w}{\partial x_i}\right).$$

Now the closure of the last flux term can be carried out:

$$\int_{z_B}^{z_S} \overline{\mathbf{u^w} T_s}\, \mathrm{d}z = \int_{z_B}^{z_S} \nu_t \, \mathrm{grad}\, k^w \mathrm{d}z \simeq \overline{\nu_t}\, \mathrm{grad}\, e^w \ ,$$

which can therefore be identified as turbulent diffusion of kinetic wave energy by a depth-averaged diffusion coefficient $\overline{\nu_t}$.

A scalar multiplication of the short term turbulence tensor and the wave velocity gradient tensor gives a positive scalar. Therefore the term

$$\epsilon^w = \int_{z_B}^{z_S} \overline{\mathrm{grad}\, \mathbf{u^w} : T_s}\, \mathrm{d}z \quad = \int_{z_B}^{z_S} \overline{\nu_t \sum_{i,j} \frac{\partial u_i^w}{\partial x_j} \left( \frac{\partial u_i^w}{\partial x_j} + \frac{\partial u_j^w}{\partial x_i} \right)} \mathrm{d}z$$

can be interpreted as the turbulent wave-energy dissipation. For its explicit calculation the turbulent viscosity profile has to be known.

We assume the correctness of the mixing length model in its general form and set

$$\nu_t = l_m^2 \left[ 2\left(\frac{\partial u^w}{\partial x}\right)^2 + 2\left(\frac{\partial v^w}{\partial y}\right)^2 + 2\left(\frac{\partial w^w}{\partial z}\right)^2 \left(\frac{\partial v^w}{\partial x}\frac{\partial u^w}{\partial y}\right)^2 + \left(\frac{\partial w^w}{\partial x} + \frac{\partial u^w}{\partial z}\right)^2 + \left(\frac{\partial w^w}{\partial y} + \frac{\partial v^w}{\partial z}\right)^2 \right]^{1/2} \ .$$

For the analytical quantification of the wave-related turbulent viscosity the deep-water approximation ($kh \gg 1$) of the orbital velocities

$$u^w = A\omega e^{k(z-h)} \sin(kx - \omega t)$$
$$v^w = 0$$
$$w^w = -A\omega e^{k(z-h)} \cos(kx - \omega t)$$

are used. It is postulated that the mixing length is proportional to the wave orbital radius:

$$l_m = \kappa^w A e^{k(z-h)}$$

Here a "wave-related Karman constant", $\kappa^w$, is introduced which will be specified later. Now the eddy viscosity profile can be determined as:

$$\nu_t = 2(\kappa^w)^2 A^3 k\omega e^{3k(z-h)}. \tag{5}$$

The turbulent viscosity is proportional to the wave amplitude and increases with decreasing wave length and period. Coming to the free surface, the viscosity increases rapidly. The shape of this profile is qualitatively confirmed by velocity field measurements under breaking waves (Melville et al. 2002).

Using this result for the turbulent wave-energy dissipation the three-dimensional turbulent deep-water wave-energy dissipation can be calculated as:

$$\overline{\mathrm{grad}\, \mathbf{u^w} : T_s} = 8(\kappa^w)^2 A^5 k^3 \omega^3 e^{5k(z-h)} \ .$$

The integration over the depth leads to a new formulation for the wave-energy dissipation rate:

$$\begin{aligned} \epsilon^w &= \frac{8}{5}(\kappa^w)^2 A^5 \left(1 - e^{-5kh}\right) k^2 \omega^3 \\ &= \frac{256}{5}(\kappa^w)^2 \left(\frac{e^w}{g}\right)^{5/2} \left(1 - e^{-5kh}\right) k^2 \omega^3 \\ &\simeq \frac{256}{5}(\kappa^w)^2 \left(\frac{e^w}{g}\right)^{5/2} k^2 \omega^3 \\ &= \frac{256}{5}(\kappa^w)^2 \left(\frac{e^w}{g}\right)^{5/2} g^{-2} \omega^7 \ . \end{aligned} \tag{6}$$

This is proportional to the power of 5/2 of the wave energy or to the fifth power of the wave amplitude. Viscous wave energy dissipation is therefore a highly non-linear process. With this expression, the final wave-energy equation gets the form:

$$\frac{\partial e^w}{\partial t} + \mathrm{div}\left[ (\mathbf{u} + \mathbf{c_g}) e^w - \nu_t \, \mathrm{grad}\, e^w \right] = -\Phi_S + \Phi_B - \epsilon^w + \int_{z_B}^{z_S} \mathrm{grad}\, \mathbf{u} : S \, \mathrm{d}z \ .$$

This equation states that there exists a diffusion process for wave energy and that turbulent dissipation can damp wave energy significantly. The latter process should play a dominant role, especially in deep waters, where the bottom friction does not anticipate the dissipation process.

Actually, the expression for the viscous wave-energy dissipation was obtained using a single monochromatic wave. Therefore, it is straightforward to test it in a spectral wave-energy model. This will be done in the next section for the deep-water equilibrium wave-energy spectrum.

## 6 The deep-water equilibrium wave-energy spectrum

The new theory on the interactions between mean flow, turbulence and waves leads to the hypothesis that the turbulent energy dissipation should not be neglected in the wave-energy equation. This hypothesis will be verified in this section, presenting a new formulation for the deep-water wave-energy spectrum under equilibrium conditions.

Assuming that there is a homogeneous and stationary wave-energy distribution over deep waters and that there are no mean currents, the wave-energy equation simplifies to:

$$-\Phi_S - \epsilon^w = 0 \ .$$

The energy input at the free surface by wind is usually modelled as (Komen et al. 1994):

$$-\Phi_S := F_W \omega e^w \ ,$$

where $F_W$ is a dimensionless wind input function depending on the wind shear stress and the wave's phase speed. Introducing both relations and using the deep-water dispersion relation, the wave energy turns out to be:

$$e^w_{eq} = \left(\frac{5}{256}\right)^{2/3} (\kappa^w)^{-4/3} F_W^{2/3} \left(1 - e^{-5kh}\right)^{-2/3} g^3 \omega^{-4}$$
$$\simeq \left(\frac{5}{256}\right)^{2/3} (\kappa^w)^{-4/3} F_W^{2/3} g^3 \omega^{-4} \ .$$

The wave energy as it is defined here contains the gravitation acceleration. The wave action density can be obtained as:

$$N_{eq} = \frac{8e^w}{g\omega} = 8\left(\frac{5}{256}\right)^{2/3} (\kappa^w)^{-4/3} F_W^{2/3} g^2 \omega^{-5} \tag{7}$$

and Phillips' similarity law (Phillips 1978) is confirmed stating that the wave action density is proportional to $g^2$ and inversely proportional to the fifth power of the frequency.

For a quantitative evaluation of the theory the Snyder–Cox wind input function $F_W$ is chosen (Komen et al. 1994):

$$F_W = 0.0003 \max\left(28\sqrt{C_D}\frac{u_{10}}{c} - 1, 0\right) \ , \tag{8}$$

where $u_{10}$ is the wind speed 10 m above the water surface, $c = g\omega$ is the phase velocity of the deep-water waves and the wind and wave directions are identical. For the following the drag coefficient $C_D$ is calculated according to Smith and Banke (1975).

The last coefficient which has to be calibrated is the wave related Karman constant, $\kappa^w$. This can be done by comparing the resulting peak energy densities with the corresponding values from the Pierson–Moskowitz spectrum (Pierson and Moskowitz 1964). It turns out that $\kappa^w \simeq 0.108$ is a good choice. Figure 1 presents the resulting equilibrium deep-water wave-energy spectra for some wind speeds. The dependency of the peak frequency $\nu_p = \omega_p/2\pi$ of the wind shear stress velocity $u_* = \sqrt{C_D}u_{10}$ can be calculated analytically as:

$$\nu_p = \frac{1}{152}\frac{g}{u_*} \ . \tag{9}$$

The proportionality constant $1/152$ in not unrealistic; it lies in between the values $1/239.8$ for the fully developed spectrum proposed by the Coastal Engineering Manual (2001) and the value $1/127$ derived by Günther and Rosenthal (1995).

Figure 1 also shows a comparison of the resulting equilibrium spectra with the Pierson–Moskowitz spectra using the peak frequencies according to Eq. (9). It can be seen that the agreement is excellent in the higher frequency tail and in the absolute values of the peak energies. The latter result indicates that the wave-related Karman constant does not depend on the peak frequency, i.e. it seems to be in fact a constant.

Otherwise, the comparison is poor in the low-frequency range. As Günther and Rosenthal (1995) point out, this is due to the sharp cut in the wind input function which does not allow the growth of waves with phase velocities higher than the surface wind speed. They showed that the application of a Gaussian distribution for the fluctuations of the wind speed significantly increases the width of the peak to the low-frequency end.

## 7 Conclusions

A consistent Reynolds decomposition of the Navier–Stokes equations to obtain dynamic equations for the mean, wave and turbulent components of the velocity field is presented. These equations are used to derive the kinetic energy equations of the mean and turbulent and wave components. The derivation emphasizes the role of

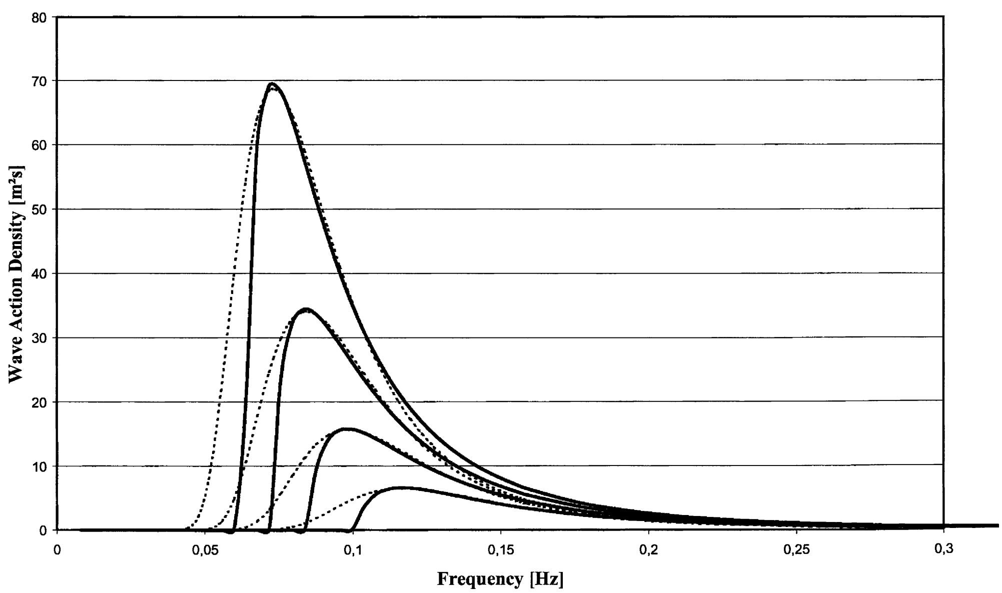


**Fig. 1** Equilibrium frequency spectra for the wind speeds 14, 16, 18 and 20 m s$^{-1}$. Solid lines: Calculations according to (7) and (8), dashed lines: Pierson-Moskowitz Spectra

turbulent diffusion and dissipation of wave energy. Using a wave related mixing length model, the turbulent wave-energy dissipation turns out to be a non-linear mechanism which is able to generate a wave-energy spectrum as the most important counterpart to wind-energy input when breaking is excluded. The resulting wave-energy equation is able to predict the shape of the equilibrium deep water wave-energy spectrum according to Phillips' similarity law as well as the decrease of the peak frequency with increasing wind shear.

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