Mass Transport in Deep-Water Waves

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ABSTRACT

The problem of mass transport induced by monochromatic waves in a viscous fluid of infinite depth and infinite lateral extent is examined. The fluid viscosity is assumed constant and the effects of Coriolis force and a nonzero surface shear stress are incorporated in the analysis. The solution shows the wave-induced surface drift to be finite, thus eliminating the apparent paradox of an infinite wave-induced surface drift predicted by Longuet-Higgins' classical solution. The nature of the present solution depends on the ratio of the Ekman depth δ to the wavelength L. The combined wind- and wave-induced drift velocity is found to be composed of a classical Ekman current and a wave-associated mass transport current. For large values of δ/L the wave-associated mass transport current is a superposition of Stokes' mass transport and the shear current arising from the unbalanced surface velocity gradient predicted by Longuet-Higgins' mass transport theory. For small values of δ/L the wave-associated mass transport velocity exhibits the features of a pure shear current corresponding to the surface velocity gradient induced by the wave motion, i.e., the mass transport becomes proportional to δ and approaches zero for an inviscid fluid in agreement with Ursell's finding. For all values of δ/L the wave-induced surface drift is found to be at an angle of approximately $\pi/4$ to the direction of wave propagation. The results show that a simple superposition of the Ekman current and Stokes' mass transport to find the combined surface drift of winds and waves is invalid. The extension of the present analysis to a fully developed sea, described by its spectrum, is discussed.

1. Introduction

Huang (1970) was one of the first to point out explicitly the seemingly paradoxical result that the deep-water limit of Longuet-Higgins' (1953) classical solution for the wave-induced mass transport in a viscous fluid showed the surface drift $u_{w,s}$ to be proportional to the water depth h, i.e., $u_{w,s} \to \infty$ as $h \to \infty$. Huang's (1970) analysis which removed this apparent paradox, however, was challenged by Ünlüata and Mei (1970) who showed that Longuet-Higgins' solution, given its underlying assumptions, indeed was correct. These assumptions were those of a steady state having been reached and the neglect of viscous attenuation of the waves.

As pointed out by Ünlüata and Mei (1970), the time scale for the establishment of a steady mass transport velocity, influenced by viscosity throughout the depth of fluid, is h^2/ν where ν is the fluid viscosity. Thus, as $h \to \infty$ steady state will never be attained. This conclusion is indirectly supported by the experimental results of Russell and Osorio (1957) who found good agreement with Longuet-Higgins' theory for intermediate depths, whereas Stokes' inviscid solution was superior to the viscous solution of Longuet-Higgins for deep-water waves. Milgram (1977)[1] has offered an explanation to this problem by considering the temporal development of the mass transport velocity as it becomes influenced by viscosity. Although explaining the discrepancy between Longuet-Higgins' theory and the experimental results by Russell and Osorio, Milgram's solution does not remove the paradox since his solution for high values of the fluid viscosity or for an infinitely long wave flume will approach that of Longuet-Higgins. Recently, Liu and Davis (1977) incorporated the effect of viscous damping of the waves in the analysis of wave-induced mass transport. By their own admission their analysis is incomplete, although it too offers a possible explanation of the discrepancy between Longuet-Higgins' theory and the experimental results of Russell and Osorio.

There is little doubt in this author's mind that a complete solution to the problem of wave-induced mass transport in a viscous fluid should consider the temporal development of the mass transport velocity as well as the viscous attenuation of the wave motion. However, given the fact that we con-

[1] Milgram, J. H., 1977: Mass transport of water and floating oil by gravity waves in deep water (personal communication).

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sider an infinite ocean we must, to be realistic, include Coriolis force in the analysis. Also, waves are generated by winds and the assumption of zero surface shear stress is therefore appropriate only for swell.

The present paper considers the idealized case of wind and wave-induced drift currents. In the initial development, the waves are assumed monochromatic and propagating in the direction of the wind. The effect of the wind is modeled by assuming a constant shear stress to act on the free surface and by adopting an appropriate value for the constant turbulent eddy viscosity. Wave attenuation is neglected based on the assumption that the wind transfers sufficient energy to the wave motion to offset viscous dissipation. The present analysis may therefore be viewed as an idealized treatment of the wind- and wave-induced drift current corresponding to a fully developed sea. The inclusion of Coriolis effect in the analysis results in a finite wave-induced surface drift, thus removing the paradoxical result of Longuet-Higgins' solution. The combined wind- and wave-induced surface drift is found to be at an angle of approximately $\pi/4$ to the wind and wave direction, thus demonstrating that a simple superposition of the classical Ekman current and Stokes mass transport is invalid.

An extension of the monochromatic wave solution to the case of a fully developed sea, described by its spectrum, is discussed. It is shown that adopting the Pierson-Moskowitz (1964) spectrum to describe the fully developed sea state leads to an infinite surface drift. Thus, one paradoxical result is replaced by another. This latter paradox, however, appears to be somewhat artificial because of our lack of accurate knowledge of the high-frequency portion of the wave spectrum. Various tempting ways of removing this prediction of an infinite surface drift are discussed and dismissed as valid candidates on the grounds that they are inconsistent. An extremely simple analysis, however, does show that the wave-induced surface drift in a fully developed sea may be of the same order of magnitude as the 3% of the wind speed generally attributed to the effect of the wind shear stress.

2. Analysis for monochromatic waves

Ünlüata and Mei (1970) demonstrated clearly that an analysis of wave-induced mass transport was facilitated greatly when the perturbation equations in their Lagrangian form, as derived by Pierson (1962), were employed. Since the effect of earth's rotation is to be included in the present analysis, in addition to viscous effects, the expression for fluid accelerations in the x^*, y^* and z^* directions become

$$\left.\begin{aligned} a_{x^*} &= \frac{\partial^2 x}{\partial t^2} - f\frac{\partial y}{\partial t} + \hat{f}\frac{\partial z}{\partial t} \\ a_{y^*} &= \frac{\partial^2 y}{\partial t^2} + f\frac{\partial x}{\partial t} \\ a_{z^*} &= \frac{\partial^2 z}{\partial t^2} - \hat{f}\frac{\partial x}{\partial t} \end{aligned}\right\}, \tag{1}$$

in which x, y and z denote the location of a fluid particle which is tagged by its position (ξ,ϵ,ζ), i.e., $x = x(\xi,\epsilon,\zeta,t)$ and analogous for y and z; and f and $\hat{f}$ are the Coriolis parameters defined by $(f,\hat{f}) = 2\omega_e(\sin\phi_e, \cos\phi_e)$, where ω_e is the radian frequency of earth's rotation and ϕ_e the latitude.

Introducing the fluid accelerations expressed by (1) in the dynamic equations the derivation of the first- and second-order perturbation equations follows readily from Pierson's (1962) development since the Coriolis terms are linear. Thus, the first-order equations become

$$\left.\begin{aligned} &\frac{\partial^2 x_1}{\partial t^2} - f\frac{\partial y_1}{\partial t} + \hat{f}\frac{\partial z_1}{\partial t} + g\frac{\partial z_1}{\partial \xi} + \frac{1}{\rho}\frac{\partial p_1}{\partial \xi} - \nu_T\nabla^2\left(\frac{\partial x_1}{\partial t}\right) = 0 \\ &\frac{\partial^2 y_1}{\partial t^2} + f\frac{\partial x_1}{\partial t} + g\frac{\partial z_1}{\partial \epsilon} + \frac{1}{\rho}\frac{\partial p_1}{\partial \epsilon} - \nu_T\nabla^2\left(\frac{\partial y_1}{\partial t}\right) = 0 \\ &\frac{\partial^2 z_1}{\partial t^2} - \hat{f}\frac{\partial x_1}{\partial t} + g\frac{\partial z_1}{\partial \zeta} + \frac{1}{\rho}\frac{\partial p_1}{\partial \zeta} - \nu_T\nabla^2\left(\frac{\partial z_1}{\partial t}\right) = 0 \\ &\frac{\partial x_1}{\partial \xi} + \frac{\partial y_1}{\partial \epsilon} + \frac{\partial z_1}{\partial \zeta} = 0 \end{aligned}\right\}, \tag{2}$$

in which ρ is the fluid density, g gravity, ν_T the constant turbulent eddy viscosity and $\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\epsilon^2 + \partial^2/\partial\zeta^2$.

To find a periodic solution of radian frequency ω corresponding to wind waves, i.e., $\omega = O(1\ \mathrm{s}^{-1})$, it would appear that the Coriolis terms in (2) may safely be neglected since these terms are of the order $f/\omega \approx 10^{-4}$ compared to the leading terms. The validity of this argument may be challenged based on the finding of Pollard (1970) regarding the necessity of including Coriolis effects in the first-order solution in order to obtain the correct second-order solution. It should, however, be recalled that the finding of Pollard was based on a perturbation analysis of the governing equations in their Eulerian form and that the problem arose in the second-order perturbation equations as a result of convective acceleration terms not present in an analysis employing the Lagrangian form of the governing equations. As a consequence of employing Lagrangian coordinates it is therefore sufficiently accurate to take the first-order solution for the periodic motion as the solution to (2) neglecting the Coriolis terms, i.e., the solution obtained by Ünlüata and Mei (1970). This will ensure that the second-order, steady, wave-induced current is obtained to the accuracy of the square of the wave steepness.

Therefore we seek a solution to the second-order, time-averaged equations of motion corresponding to a first-order wave motion whose surface profile is given by

$$\eta_1 = ae^{i(k\xi-\omega t)}, \tag{3}$$

in which a is the wave amplitude, k the wavenumber, and only the real part of (3) constitutes the solution.

Since the fluid is assumed viscous, a decay of the wave motion in time or space may be introduced. In the present paper, however, we consider the combined effect of wind and waves and we may justify a neglect of wave decay by assuming an energy transfer from the wind to the wave motion equal to the rate of energy dissipation, i.e., we are simulating a condition corresponding to a fully developed sea. With these assumptions we have for infinitely deep water

$$k = \frac{\omega^2}{g}. \tag{4}$$

This form of the dispersion relationship neglects the effects of earth's rotation, which has already been justified, and the presence of the steady wave- and wind-induced current, the solution for which is to be obtained from the second-order equations. The absence of the steady current in the dispersion relationship is clearly a convenient assumption which, however, may be partially justified by recognizing that observations show a surface drift, whether wave- or wind-induced or both, of the order 3% of the wind speed. For a fully developed sea this is considerably below the first-order wave orbital velocities and suggests that the surface drift may be considered of second-order in wave steepness, thus justifying the assumption.

For the general case of any ratio of water depth to wavelength, the Lagrangian equations governing the second-order steady streaming were given by Ünlüata and Mei (1970) for a pure wave motion without considering the Coriolis effect. From the preceding discussion it follows that earth's rotation will manifest itself in the second-order equations in the same manner as it appears in (2), the first-order equations. Therefore, the time-averaged, second-order Langrangian equations [Eq. (45) in Ünlüata and Mei (1970)] are readily generalized for a fluid of infinite depth and infinite lateral extent to account for Coriolis effect. For infinite depth we obtain in our notation

$$\nu_T \frac{d^2u}{d\zeta^2} = -fv + 4\nu_T\omega k^3a^2e^{2k\zeta}, \tag{5}$$

$$\nu_T \frac{d^2v}{d\zeta^2} = +fu, \tag{6}$$

in which $u = \partial\bar{x}_2/\partial t$ and $v = \partial\bar{y}_2/\partial t$ are the second-order steady Lagrangian velocity components in the horizontal plane.

Up to this point the influence of the wind blowing over the water has been introduced as a justification for the neglect of wave decay. In terms of the second-order streaming associated with the combined effect of wind and waves, the wind enters the problem through the surface boundary condition to be satisfied by (5) and (6). Whereas the pure wave solution by Ünlüata and Mei (1970) assumed a zero shear on the free surface, we account for the effect of wind by assuming a constant surface shear stress to act in the direction of wave propagation, i.e.,

$$\tau_s = \tau_{s,\xi} = \rho u_*^2 \quad \text{at} \quad \zeta = 0, \tag{7}$$

where u_* is the shear velocity.

In addition to this explicit account for the action of wind, the presence of the wind will also manifest itself implicitly through its effect on the value of the turbulent eddy viscosity. Thus, with increasing wind speed it would be reasonable to assume an increasing value of ν_T.

The surface boundary condition to be satisfied by the second-order Lagrangian velocity was derived by Ünlüata and Mei (1970). The effect of a non-zero surface shear is readily incorporated and the free surface boundary condition to be satisfied by (u,v) for infinitely deep water is

$$\frac{\partial u}{\partial \zeta} = 4\omega k^2a^2 + \frac{u_*^2}{\nu_T} \quad \text{at} \quad \zeta = 0. \tag{8}$$

The first term on the right-hand side is the familiar velocity gradient obtained for a pure wave motion and, as pointed out numerous times, it is twice the value of the surface gradient predicted by Stokes' inviscid solution.

Combining (5) and (6) using the complex velocity variable

$$w = u + iv, \tag{9}$$

we obtain the governing equation

$$\frac{d^2w}{d\zeta^2} - i\frac{f}{\nu_T}w = 4\omega k^3a^2e^{2k\zeta} \tag{10}$$

which, in addition to (8), must satisfy the boundary condition of remaining bounded as $\zeta \rightarrow -\infty$, i.e.,

$$\frac{dw}{d\zeta} = 4\omega k^2 a^2 + \frac{u_*^2}{\nu_T}. \quad (11)$$

The solution to the homogeneous form of (10), which satisfies $w \rightarrow 0$ as $\zeta \rightarrow -\infty$, is

$$w_0 = Ae^{(1+i)\zeta/\delta} \quad (12)$$

in which A is an arbitrary constant and

$$\delta = (2\nu_T/f)^{1/2} \quad (13)$$

is the scale of the Ekman layer.

A particular solution of (10) is

$$w_p = \frac{\omega k a^2}{1 - \frac{1}{2}i(k\delta)^{-2}} e^{2k\zeta}. \quad (14)$$

Requiring the complete solution, $w = w_0 + w_p$, to satisfy the boundary condition given by (11) leads to a determination of the constant

$$A = \frac{\delta}{1+i}\left[\frac{u_*^2}{\nu_T} + 2\omega k^2 a^2 \frac{1 - i(k\delta)^{-2}}{1 - \frac{1}{2}i(k\delta)^{-2}}\right]. \quad (15)$$

The complete solution, therefore, for the wind- and wave-induced current is given by

$$w = u + iv = \frac{\delta}{1+i}\frac{u_*^2}{\nu_T} e^{(1+i)\zeta/\delta} + \frac{\omega k a^2}{1 - \frac{1}{2}i(k\delta)^{-2}} \times \left[\frac{2k\delta}{1+i}(1 - i(k\delta)^{-2})e^{(1+i)\zeta/\delta} + e^{2k\zeta}\right]. \quad (16)$$

Written in this form the solution clearly identifies the current associated with the surface wind shear to be the classic Ekman current. The wave-induced current is seen to consist of two components, the first of which has a variation with depth indicative of a shear current, whereas the second term exhibits a depth variation typical of the Stokes drift. It is quite revealing to examine the nature of the wave-induced current as determined here with the effect of Coriolis force included in the analysis. The solution is clearly governed by the ratio of the Ekman depth δ to the wavelength L, i.e., $k\delta$. The Ekman depth determines the length scale of the spiraling current direction. Thus, if δ is much greater than L, the wave motion is virtually unaffected by the tendency of the current to turn with depth. For $k\delta$ large the wave-associated portion of (16) gives

$$w_w = \omega k a^2\{[2k\delta/(1+i)]e^{(1+i)\zeta/\delta} + e^{2k\zeta}\}, \quad (17)$$

which is merely a combination of the Stokes inviscid mass transport solution and a shear current corresponding to the portion of the surface velocity gradient which is unbalanced by the gradient of Stokes' solution $dw/d\zeta = 2\omega k^2 a^2$.

For the scale of the turning δ small relative to the wavelength, i.e., $k\delta \ll 1$, the normal Stokes drift cannot develop since its scale is larger than the scale of turning due to earth's rotation. This is shown by the fact that the wave-associated portion of (16) simplifies to

$$w_w = 4\omega k^2 a^2[\delta/(1+i)]e^{(1+i)\zeta/\delta} \quad (18)$$

for $k\delta \ll 1$. This solution corresponds to the shear current solution satisfying the surface velocity gradient condition $dw/d\zeta = 4\omega k^2 a^2$. It is further noticed that the resulting magnitude of the wave-associated current is of the order $k\delta$ times the Stokes drift, i.e., as $k\delta \rightarrow 0$ the Lagrangian drift approaches zero as predicted based on inviscid theory by Ursell (1950) and Pollard (1970).

Asymptotic expressions for the magnitude $|w_{w,s}|$ of the wave-associated surface drift may be obtained from (17) and (18) by setting $\zeta = 0$. In this manner we obtain

$$|w_{w,s}| = \begin{cases} 2\sqrt{2}\omega k^2 a^2 \delta, & k\delta \ll 1 \\ \sqrt{2}\omega k^2 a^2 \delta + \frac{1}{2}\sqrt{2}\omega k a^2, & k\delta \gg 1. \end{cases} \quad (19)$$

The asymptotic expression for the angle between the wave-associated surface drift and the direction $\theta_{w,s}$ of wave propagation may similarly be obtained for large values of $k\delta$ from (17), i.e.,

$$\theta_{w,s} = \frac{\pi}{4} - \frac{1}{2\sqrt{2}k\delta}, \quad k\delta \gg 1. \quad (20)$$

For all values of $k\delta$ the magnitude and direction of the wave-associated surface drift may be obtained from (16). As shown in Fig. 1 the magnitude of the surface drift may vary from much less than the value of Stokes drift, $\omega k a^2$, to several times larger. The direction of the surface drift is shown in Fig. 2 to deviate only slightly from $\pi/4$ (to the right on the Northern Hemisphere). The asymptotic expressions given by (19) and (20) are seen to represent the complete solution to (16) quite accurately. In particular, (19) represents the complete solution over the entire range if $k\delta = \frac{1}{2}$ is chosen as the transition value between the two asymptotic expressions.

The nature of the solution obtained here clearly demonstrates that a simple superposition of the Ekman shear current and the Stokes surface drift, as has been suggested by some investigators in the context of oil slick trajectory prediction, is invalid.

The present solution may be viewed as a highly idealized representation of the wave-induced mass transport in a fully developed sea. If viewed as such, the question to ask is: what is the appropriate value of $k\delta$? As seen from Fig. 1 the surface drift may still be substantially greater than the Stokes drift if $k\delta \gg 1$. To answer this we assume that the value of the turbulent eddy viscosity may be determined by requiring that a purely wind-induced surface drift takes on the value of 3% of the wind speed. From this criterion and assuming a value of 3.2×10^{-3} for the friction factor relating wind shear stress and wind velocity, we obtain (Stolzenbach *et al.*, 1977) a representative value for the turbulent eddy viscosity

$$\nu_T = 2.3\ 10^{-5} \frac{W^2}{\sin\phi_e}, \tag{21}$$

where ν_T is in square meters per second if W, the wind speed (at, say, 19.5 m above the sea surface) is in meters per second. Combining this with a value of Coriolis parameter $f = 1.46\ 10^{-4} \sin\phi_e$ [s^{-1}], Eq. (13) yields the following estimate of the Ekman depth (m):

$$\delta = (2\nu_T/f)^{1/2} = 0.56 \frac{W}{\sin\phi_e}. \tag{22}$$

To obtain a representative value of the wavenumber k, for a fully developed sea, we choose that corresponding to a wave whose phase speed is equal to the wind velocity W. This choice is consistent with the Pierson-Moskowitz (1964) spectrum for a fully developed sea as well as the SMB wave forecasting technique (U. S. Army, 1973). From this, since $\omega/k = g/\omega = W$, we obtain

$$k = g/W^2. \tag{23}$$

Therefore, a representative value of the important dimensionless parameter $k\delta$ is

$$k\delta = 0.56 \frac{g}{W \sin\phi_e} \tag{24}$$

where g and W are in SI units. Expression (24) shows that a representative value of $k\delta$, corresponding to a fully developed sea, is of the order unity or less. Thus, the wave-associated surface drift is seen from Fig. 2 to remain bounded and to be of the same

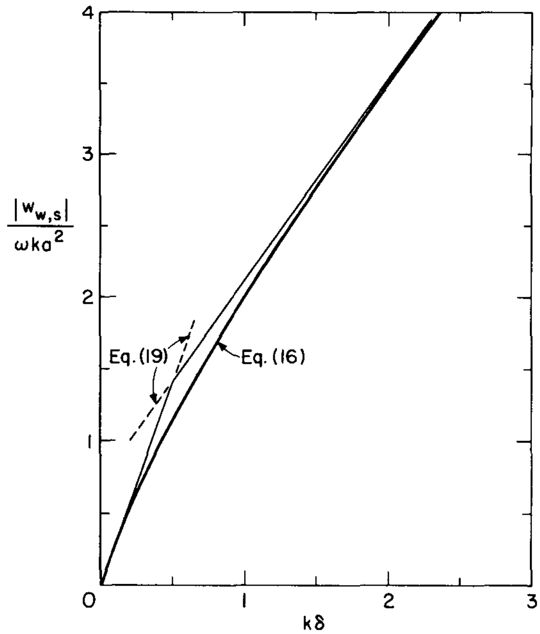

FIG. 1. Magnitude of the wave-induced surface drift as a function of $k\delta$.

order as the Stokes surface drift. The present simplified analysis therefore removes the paradoxical infinite surface drift predicted by Longuet-Higgins' solution for wave-induced mass transport in deep-water waves.

The preceding highly simplified argument applies only to the combined action of wind and waves. The analysis assumed monochromatic waves which may be approximately realized only in the case of swell. For swell the effect of wind must be negligible and the value of ν_T given by (21) must consequently be considered high. With a much smaller value of ν_T, corresponding to swell, the

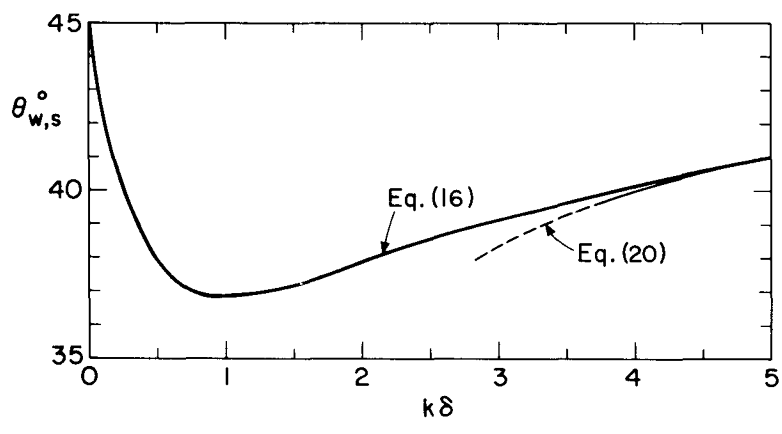

FIG. 2. Deflection angle of wave-induced surface drift relative to the wave and wind direction (deflection to the right on the Northern Hemisphere) as a function of $k\delta$.

value of $k\delta$ is again found to be less than unity and the wave-induced surface drift is found to remain bounded; possibly being considerably smaller than the Stokes drift predicted from the wave characteristics.

In both cases discussed above the direction of the wave-induced surface drift is approximately at an angle of $\pi/4$ to the direction of wave propagation. In the case of a fully developed sea it should be recalled that the Ekman surface drift (in the present analysis forced to be 3% of the wind speed) should be added to the wave-induced surface drift. To estimate the magnitude of the wave-induced surface drift we assume a value of $k\delta \approx 1/2$ and a representative wave amplitude of half the significant wave height predicted by the SMB method. For these assumptions Fig. 2 shows that $|w_{w,s}| \approx \omega k a^2$ or $|w_{w,s}| \approx 0.02W$, i.e., 2% of the wind speed, which is comparable to the assumed value of the wind-induced surface drift. Although highly simplified, the present arguments do point out the importance of considering waves as well as wind when calculating surface drift velocities.

3. Approximate analysis for a fully developed sea

To extend, in a more rigorous manner, the analysis of wave-induced mass transport to the case of a fully developed random sea we assume for the purpose of the following discussion the sea state to be described by the Pierson-Moskowitz (1964) spectrum

$$S_{\eta\eta}(\omega) = \alpha g^2 \omega^{-5} \exp[-\beta(\omega_0/\omega)^4], \tag{25}$$

in which $\alpha = 0.0081$, $\beta = 0.74$ and $\omega_0 = g/W$, with W being the wind speed at 19.5 m above the sea surface.

From Ünlüata and Mei (1970) we obtain the expected value of the wave-induced surface drift by replacing a^2 in the monochromatic solution by $2S_{\eta\eta}(\omega)$ and integrating over all frequencies. Realizing that the surface drift deviates only slightly from 45° to the wind direction for all values of $k\delta$, i.e., for all frequencies, we may simplify the analysis by assuming the wave-induced surface drift to be at an angle of $\pi/4$ to the wind, independent of frequency. Furthermore, we adopt the asymptotic expressions given by (19) for the magnitude of the surface drift. With these assumptions we may obtain the expected value of the surface drift from

$$\langle |w_{w,s}| \rangle = \int_0^\infty 2\sqrt{2}\omega k^2 \delta S_{\eta\eta}(\omega)d\omega$$

$$+ \int_0^{\omega_1} 2\sqrt{2}\omega k^2 \delta S_{\eta\eta}(\omega)d\omega + \int_{\omega_1}^\infty \sqrt{2}\omega k S_{\eta\eta}(\omega)d\omega, \tag{26}$$

in which ω_1 is given by (4) corresponding to the value of $k\delta = 0.5$, i.e.,

$$\omega_1 = \left(\frac{g}{2\delta}\right)^{1/2} \approx \left(\frac{g \sin\phi_e}{1.12W}\right)^{1/2}, \tag{27}$$

where (22) has been used.

When substituting ω^2/g for k and introducing $S_{\eta\eta}(\omega)$ from (25) the integration may be performed. It is, however, evident that the first integral in (26) is divergent. Retaining only the first integral in (26) we have

$$\langle |w_{w,s}| \rangle \approx 2\sqrt{2}\alpha\delta \int_0^\infty \exp[-\beta(\omega_0/\omega)^4]d\omega \to \infty. \tag{28}$$

The problem is associated with the upper limit. Several possible ways to remove the unboundedness of this integral may be explored. First, the dispersion relationship for the high-frequency wave components is not given by (4). Surface tension effects become important and we know that $k \propto \omega^{2/3}$ for capillary waves. Thus, introducing $k \propto \omega^{2/3}$ for the high frequencies would remove the unboundedness of the first integral in (26). However, to do this would not be consistent since our analysis of monochromatic waves is based on the assumption of gravity waves for which (4) is valid. Only if we adopted a solution for mass transport in capillary waves for the high-frequency end of the spectrum would this approach be consistent.

A second approach would be to consider the fact that the expected value of the surface mass transport velocity in a random sea is finite. For large frequencies (i.e., $k\delta \gg 1$) the troublesome term has a depth variation similar to the Ekman current. The high-frequency wave components would therefore behave as if they were superimposed on a steady current. The ξ component of this current, $u_{w,s}$, would not be of second order compared to the phase velocity of the high-frequency waves and would therefore alter the dispersion relationship for the high-frequency wave components. Thus, for the highest frequencies the appropriate form of the dispersion relationship would be $k = \omega/u_{w,s}$. This again would remove the unboundedness of the first integral in (26), but the procedure would again be inconsistent since our analysis of the monochromatic wave did not include the effect of a superimposed steady current.

Without a significant theoretical development of a mass transport theory for a viscous fluid with proper attention being paid to surface tension and a superimposed current, there does not appear to be a rigorous way of removing the singularity of the wave-induced surface drift in a fully developed random sea. As an admittedly more convenient than rigorous elimination of this problem we may take recourse to our lack of knowledge about the high-frequency end of the wave spectrum. Most wave measurements are analyzed based on data digitized at 0.5 s intervals. This precludes us from obtaining any knowledge about the wave spectrum for radian frequencies greater than $2\pi\ s^{-1}$. This practical limitation eliminates considerations of surface tension effects. At the same time the phase velocity of waves of greater than 1 s period (1.6 m s^{-1}) is such that a significant effect of a superimposed current of the order 5% of the wind speed would affect the dispersion relationship significantly only for very high wind speeds. Therefore, it would appear reasonable to limit the validity of (25) to values of $\omega < \omega_m$, where ω_m is of the order $2\pi\ s^{-1}$, and to take $S_{\eta\eta}(\omega) = 0$ for $\omega > \omega_m$. With this assumption we obtain from (28)

$$\langle |w_{w,s}| \rangle \approx 2\sqrt{2}\delta\alpha \int_0^{\omega_m} \exp[-\beta(\omega_0/\omega)^4]d\omega$$

$$= 2\sqrt{2}\delta\alpha\beta^{1/4}\omega_0\left(y_m{}^{-1/4}e^{-y_m} - \int_{y_m}^{\infty} y^{-1/4}e^{-y}dy\right), \quad (29)$$

in which

$$y = \beta(\omega_0/\omega)^4 \quad (30)$$

and y_m is the value of y for $\omega = \omega_m$.

For $\omega_0 = g/W$ and $\omega_m = O(2\pi\ s^{-1})$ it is evident that $y_m \ll 1$ for all cases of practical interest. The integral in (29) is therefore essentially the Gamma-function and the solution may be written

$$\langle |w_{w,s}| \rangle \approx 2\sqrt{2}\delta\alpha\beta^{1/4}\omega_0\left[\frac{\omega_m}{\beta^{1/4}\omega_0} - \Gamma(3/4)\right]. \quad (31)$$

For wind speeds >5–10 m s^{-1} the first term in the square brackets dominates and we obtain the simple solution

$$\langle |w_{w,s}| \rangle \approx 2\sqrt{2}\delta\alpha\omega_m = 0.0128 \frac{W}{\sin\phi_e}\omega_m, \quad (32)$$

where (22) was introduced.

For a value of $\omega_m = 2\pi\ s^{-1}$ the wave-induced surface drift predicted by this model approaches 8% of the wind speed. The present model is admittedly simplified and approximate in nature. However, even if the validity of the Pierson-Moskowitz spectrum is limited to waves of periods greater than 2 s, in which case we may be relatively confident in the validity of the analysis, we still end up with a predicted surface drift due solely to the low frequency portion ($\omega < \pi\ s^{-1}$) of the same order as the surface drift generally attributed to the effect of the wind shear stress.

4. Concluding remarks

An analysis of mass transport in deep-water, monochromatic waves has revealed that the inclusion of Coriolis force in the analysis effectively removes the unrealistic prediction of an infinite surface drift obtained from Longuet-Higgins' solution. The combined effect of wind and waves is found to produce a drift current which is a combination of Ekman's classical wind-induced current and a wave-associated drift current. The wave-associated surface drift is found to remain finite and to be in a direction of approximately $\pi/4$ to the direction of wave propagation. The analysis shows that a simple superposition of Ekman's surface drift and Stokes mass transport to model the combined effect of wind and waves is invalid.

An approximate analysis of the wave-induced mass transport in a fully developed sea shows that the wave-induced surface drift is likely to be of the same order as the 3% of the wind speed generally attributed to the effect of wind shear on the surface. Since wind blowing over water necessarily is associated with the generation of waves this finding raises the age-old question of whether the observed surface drift currents were wave- or wind-induced or both. The present analysis does not pretend to answer this question, although it indicates the wind- and wave-induced surface drift currents are likely to be equally important.

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REFERENCES

Huang, N. E., 1970: Mass transport induced by wave motion. *J. Mar. Res.*, **29**, 35–50.

Liu, A-K., and S. H. Davis, 1977: Viscous attenuation of mean drift in water waves. *J. Fluid Mech.*, **81**, Part 1, 63–84.

Longuet-Higgins, M. S., 1953: Mass transport in water waves. *Phil. Trans. Roy. Soc. London*, **A245**, 535–581.

Pierson, W. J., 1962: Perturbation analysis of the Navier-Stokes equations in Lagrangian form with selected linear solutions. *J. Geophys. Res.*, **67**, 3151–3160.

——, and L. Moskowitz, 1964: A proposed spectral form for fully developed wind seas based on the similarity theory of S. A. Kitaigorodskii. *J. Geophys. Res.*, **69**, 5181–5190.

Pollard, R. T., 1970: Surface waves with rotation: An exact solution. *J. Geophys. Res.*, **75**, 5895–5898.

Russell, R. C. H., and J. D. C. Osorio, 1957: An experimental investigation and drift profiles in closed channels. *Proc. Sixth Conf. Coastal Engineering*, Miami, ASCE, 171–193.

Stolzenbach, K. D., O. S. Madsen, E. E. Adams, A. M. Pollack and C. K. Cooper, 1977: A review and evaluation of basic techniques for predicting the behavior of surface oil slicks. Dept. of Civil Engineering, MIT, R. M. Parsons Laboratory Tech. Rep. No. 222.

Ünlüata, U., and C. C. Mei, 1970: Mass transport in water waves. *J. Geophys. Res.*, **75**, 7611–7617.

Ursell, F., 1950: On the theoretical form of ocean swell on a rotating earth. *Mon. Not. Roy. Astro. Soc.* (*Geophys. Suppl.*), **6**, No. 1, 1–8.

U. S. Army, 1973: *Shore Protection Manual*, Vol. 1. Coastal Engineering Res. Center, Fort Belvoir, VA (see Chap. 3).