A Realistic Model of the Wind-Induced Ekman Boundary Layer

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ABSTRACT

A physically realistic and general model for the vertical eddy viscosity in a homogeneous fluid is proposed. For an infinitely deep ocean the vertical eddy viscosity increases linearly with depth from a value of zero at the free surface. Based on this model a general theory is developed for the drift current resulting from a time-varying surface shear stress. Explicit expressions are given for the temporal development of the drift current in the vicinity of the free surface and for the steady-state response to a suddenly applied uniform shear stress. The steady-state solution predicts the effective Ekman layer depth to be proportional to the square root of the wind shear stress and reproduces the experimentally observed logarithmic velocity deficit near the free surface. The angle between the surface drift current and the wind stress is found to be somewhat smaller (of the order 10°) than predicted by Ekman's classical solution. For the unsteady response to a suddenly applied wind stress the present model predicts a much shorter response time than that found by Fredholm based on a constant vertical eddy viscosity assumption. The application of the proposed vertical eddy viscosity model to finite depth conditions, including the effects of slope currents, is outlined.

1. Introduction

In Ekman's (1905) classical study of wind-driven currents a constant vertical eddy viscosity was assumed. For the steady wind-driven current in an infinite homogeneous ocean the assumption of a constant vertical eddy viscosity leads to an angle between the wind shear stress and the surface current of $\pi/4$, a value generally considered to be on the high side. In shallower waters, where bottom friction comes into play, the assumption of a constant eddy viscosity in conjunction with a no-slip condition at the bottom leads to unrealistically low velocities as recently pointed out by Murray (1975), who circumvented this problem by introducing the somewhat artificial concept of a slip velocity at the bottom boundary. The shortcomings of a constant vertical eddy viscosity assumption have long been recognized and the proper parameterization of the vertical eddy viscosity was recently identified by Reid (1975) as one of the major problems in the analysis of wind-driven currents.

In relatively shallow water, when the depth h is smaller than or comparable to the thickness of the frictional layer, more realistic models than that of a constant vertical eddy viscosity have been proposed. Fjeldstad (1929), based on an analysis of field data, suggested that the vertical eddy viscosity ν_T was proportional to the $\frac{3}{4}$ power of the distance z_b from the bottom, i.e., $\nu_T \propto z_b^{\frac{3}{4}}$. This model was recently employed by Murray (1975) in a study of nearshore wind-driven currents. Thomas (1975) suggested a physically more realistic version of Fjeldstad's model by taking $\nu_T = \kappa |u_{*b}| z_b$, where κ is von Kármán's constant (0.4) and $|u_{*b}|$ is the shear velocity based on the absolute value of the bottom shear stress, i.e., $|u_{*b}| = (|\tau_b|/\rho)^{\frac{1}{2}}$ with ρ being the fluid density. Thomas' model has several physically pleasing features. The magnitude of the vertical eddy viscosity depends on the flow itself and on the bottom boundary roughness k_b through $|u_{*b}|$, and it leads to the classical logarithmic velocity profile in the vicinity of the bottom. However, the added physical realism of Thomas' model is not achieved without a cost in terms of added computational difficulties. Thus, since the solution for the wind-driven flow depends on the value assigned to ν_T and since ν_T itself depends on the flow characteristics, a rather time-consuming iterative solution procedure must be adopted. This additional computational complexity apparently caused Witten and Thomas (1976) to abandon this model in favor of an explicit model for the vertical eddy viscosity. The eddy viscosity models referred to above are all restricted to application in shallow water. If, despite this limitation, they were adopted for deep water conditions they would effectively correspond to a constant vertical eddy viscosity assumption.

The purpose of this paper is to present a more realistic and more general model for the vertical eddy viscosity than those previously advanced in the context of wind-driven ocean currents. The proposed model is simply that the vertical eddy viscosity is assumed to increase linearly with vertical distance from a sheared boundary, i.e., $\nu_T=\kappa|u_*|z$, where $|u_*|$ is the shear velocity and z the distance from the sheared boundary. In the vicinity of the bottom $z_b=0^+$. This model is identical to that proposed by Thomas (1975). Near the free surface, however, where $z_b=h^-$, the eddy viscosity given by the proposed model varies according to $\nu_T=\kappa|u_{*s}|(h-z_b)$, with $|u_{*s}|$ being the shear velocity based on the surface shear stress. The model for the vertical eddy viscosity proposed here may be shown to agree with the model of steady turbulent shear flows proposed by Reid (1957). It leads to a logarithmic velocity profile near the bottom, as does Thomas' model, and a logarithmic velocity deficit in the vicinity of the free surface. The latter feature, which is absent in Thomas' model, has been observed experimentally in steady, turbulent Couette flow by Reichardt (1959). Furthermore, Shemdin (1972) found the wind-driven current in a laboratory wind wave facility, i.e., in the presence of surface waves, to exhibit a logarithmic velocity deficit in the vicinity of the free surface. These observed features, which are reproduced by the proposed model for the vertical eddy viscosity, are taken to support the physical realism of the proposed model. In addition, the proposed model may be applied in deep as well as in shallow water and is therefore considered to be of a more general nature than previous models. It is noted that the present model, when applied to the case of infinite water depth, is identical to the model proposed by Ellison (1956) in the context of the atmospheric boundary layer.

The model for the vertical eddy viscosity is, of course, limited to the idealized conditions of a homogeneous ocean. It is applied here to the response of an infinitely deep, homogeneous ocean of infinite lateral extent to a time-varying spatially uniform shear stress. Approximate expressions are derived for the temporal development of the pure drift current for a suddenly applied shear stress. The limit of this solution for large times is shown to be identical to Ellison's (1956) solution for the atmospheric boundary layer. The steady response is compared to the classical Ekman solution and reveals the angle between surface shear stress and velocity to be approximately 10° as compared to the 45° predicted by Ekman (1905). The temporal development of the surface current resulting from a suddenly applied shear stress is compared with Fredholm's solution as given by Ekman (1905). The present solution shows the response to be nearly instantaneous when compared to the slow approach to steady-state conditions exhibited by Fredholm's solution. The differences between the present and previous solutions as well as the implications of these differences are discussed in some detail. The problems associated with the application of the proposed model for the vertical eddy viscosity in the general case of finite depth and including the effects of a slope current are outlined.

2. General analysis

For an infinitely deep, homogeneous ocean of infinite lateral extent the linearized form of the horizontal momentum equations may be written

$$\frac{\partial w}{\partial t}+ifw=\frac{\partial}{\partial \hat{z}}\left(\nu_T\frac{\partial w}{\partial \hat{z}}\right), \tag{1}$$

where

$$w=u+iv \tag{2}$$

is the complex horizontal velocity in the $(\hat{x},\hat{y})$ plane of the Cartesian coordinate system, $f=2\omega_e\sin\phi$ is the Coriolis parameter, ω_e and ϕ being the radian frequency of earth's rotation and the latitude, respectively, and $\hat{z}$ is the vertical coordinate chosen positive upward. The right-hand side of (1) represents the contribution of frictional forces on horizontal planes, and the terms expressing the horizontal force components associated with a spatially varying atmospheric pressure and free surface elevation are omitted to be consistent with the assumption of an ocean of infinite lateral extent.

Introducing the more convenient vertical coordinate $\hat{z}=-z$ which is positive downward and taking $z=0$ in the free surface, the proposed model for the vertical eddy viscosity ν_T reads

$$\nu_T=\kappa u_* z, \tag{3}$$

in which $\kappa=0.4$ is von Kármán's constant and $u_*=(|\tau_s|/\rho)^{\frac{1}{2}}$ is the shear velocity based on a representative value of the magnitude of the surface shear stress $|\tau_s|$. In problems where $|\tau_s|$ may be considered constant, such as in the problem of a suddenly applied constant surface shear, there is little ambiguity in the value to be assigned to u_*. For the flow resulting from a time-varying surface shear stress the subsequent analysis necessitates the use of a time-independent value of u_* in (3). The use of a representative value of $|\tau_s|$ to define the magnitude of the vertical eddy viscosity must in this case reflect the intended application of the results.

With the shear stress on horizontal planes expressed in complex form as $\tau=\tau_{\hat{x}}+i\tau_{\hat{y}}$ and defined in the usual manner in the right-handed $(\hat{x},\hat{y},\hat{z})$ coordinate system, we have

$$\frac{\tau}{\rho}=\frac{\tau_{\hat{x}}}{\rho}+i\frac{\tau_{\hat{y}}}{\rho}=\nu_T\frac{\partial w}{\partial \hat{z}}=-\nu_T\frac{\partial w}{\partial z}. \tag{4}$$

For the general problem of a flow starting from rest, i.e.,

$$w=0 \quad \text{for} \quad t\leqslant 0, \tag{5}$$

and driven by a time-varying, spatially uniform surface shear stress $\tau_s(t)$, Eqs. (3) and (4) provide one of the necessary boundary conditions:

$$\frac{\tau_s(t)}{\rho}=\frac{\tau_{s,\hat{x}}(t)}{\rho}+i\frac{\tau_{s,\hat{y}}(t)}{\rho}=-\kappa u_* z\frac{\partial w}{\partial z};\quad z\rightarrow 0,\quad t\geqslant 0. \quad (6)$$

The remaining boundary condition to be satisfied by the solution of (1) is that of a vanishing motion with depth, i.e.,

$$w\rightarrow 0 \quad \text{as} \quad z\rightarrow\infty. \quad (7)$$

Since the governing equation (1) is linear and the coefficients independent of time, the use of Laplace transforms, defined by

$$\bar{w}=\mathcal{L}\{w\}=\int_0^\infty e^{-st}w(t)dt, \quad (8)$$

is convenient. Taking the Laplace transform of (1) and invoking the initial condition (5), the governing equation becomes

$$(s+if)\bar{w}=\frac{\partial}{\partial z}\left(\kappa u_* z\frac{\partial \bar{w}}{\partial z}\right), \quad (9)$$

with the boundary conditions

$$-\kappa u_* z\frac{\partial \bar{w}}{\partial z}=\mathcal{L}\left\{\frac{\tau_{s,\hat{x}}(t)}{\rho}+i\frac{\tau_{s,\hat{y}}(t)}{\rho}\right\} \quad \text{as} \quad z\rightarrow 0 \quad (10)$$

and

$$\bar{w}\rightarrow 0 \quad \text{as} \quad z\rightarrow\infty. \quad (11)$$

Introducing the dimensionless vertical coordinate

$$\xi=\frac{z(s+if)}{\kappa u_*}, \quad (12)$$

Eq. (9) may be written

$$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\bar{w}}{\partial\xi}\right)-\bar{w}=0, \quad (13)$$

whose general solution (Hildebrand, 1965) is

$$\bar{w}=AI_0(2\sqrt{\xi})+BK_0(2\sqrt{\xi}), \quad (14)$$

where A and B are arbitrary constants and I_0 and K_0 are the zeroth-order modified Bessel functions of the first and second kind, respectively.

By virtue of the exponential behavior of I_0 for large values of the argument it follows from (11) that $A=0$ in (14), thus leaving us with

$$\bar{w}=BK_0(2\sqrt{\xi}). \quad (15)$$

The constant B in (15) is determined from the free surface stress condition (10), which in terms of the dimensionless vertical coordinate ξ may be written as

$$-\kappa u_*\xi\frac{\partial\bar{w}}{\partial\xi}=\kappa u_*\sqrt{\xi}BK_1(2\sqrt{\xi})=\mathcal{L}\left\{\frac{\tau_{s,\hat{x}}}{\rho}+i\frac{\tau_{s,\hat{y}}}{\rho}\right\}$$

$$=\mathcal{L}\left\{\frac{\tau_s}{\rho}\right\} \quad \text{as} \quad \xi\rightarrow 0. \quad (16)$$

Introducing the asymptotic expansion of the first-order modified Bessel function of the second kind K_1 for small values of the argument, we obtain from (16)

$$B=\frac{2}{\kappa u_*}\mathcal{L}\left\{\frac{\tau_s}{\rho}\right\} \quad (17)$$

and the Laplace transform of the solution to the stated problem has been found as

$$\bar{w}=\frac{2}{\kappa u_*}\mathcal{L}\left\{\frac{\tau_s}{\rho}\right\}K_0(2\sqrt{\xi}). \quad (18)$$

To invert this Laplace transform it is recognized that

$$K_0(2\sqrt{\xi})=K_0\{[(4z/\kappa u_*)(s+if)]^{\frac{1}{2}}\} \quad (19)$$

may be inverted by use of the table of Laplace transforms presented in Abramowitz and Stegun [(1972) Chap. 29, Eqs. (29.2.14) and (29.3.120)]. The result of this inversion may be written in terms of the function to which (19) is the Laplace transform, i.e.,

$$\mathcal{L}\{\tfrac{1}{2}e^{-ift}t^{-1}e^{-z/\kappa u_* t}\}=K_0(2\sqrt{\xi}). \quad (20)$$

Inserting (20) in (18) it is readily seen that the use of the convolution theorem yields the general solution

$$w=\frac{1}{\kappa u_*}\int_0^t\frac{\tau_{s,\hat{x}}(t-\beta)+i\tau_{s,\hat{y}}(t-\beta)}{\rho}e^{-if\beta}\frac{1}{\beta}e^{-z/\kappa u_*\beta}d\beta \quad (21)$$

for the response of an infinite homogeneous ocean to a time-varying, spatially uniform surface shear stress. The main assumption involved in obtaining the above solution, in addition to those made in the problem formulation itself, is that of a constant value of u_* based on a representative value of the surface shear stress, $|\tau_s|=(\tau_{s,\hat{x}}^2+\tau_{s,\hat{y}}^2)^{\frac{1}{2}}$. This assumption is not unique to the present formulation of the problem and is made also when the solution is found based on an assumed constant vertical eddy viscosity. In this respect it may be worthwhile to point out that a spatially varying but time-independent vertical eddy viscosity has been employed successfully by Kajiura (1964, 1968) in an analysis of turbulent oscillatory boundary layers.

3. Response to a suddenly applied constant surface stress

To investigate in more detail the nature of the general solution obtained in the previous section the response to a suddenly applied constant surface shear stress in the $\hat{y}$ direction is considered here, i.e., we take

$$\tau_s=\begin{cases}0, & t\leqslant 0\\ i\tau_{s,\hat{y}}, & t\geqslant 0\end{cases} \quad (22)$$

for which

$$u_*=\left(\frac{\tau_{s,\hat{y}}}{\rho}\right)^{\frac{1}{2}}. \quad (23)$$

For this problem, which corresponds to the problem solved by Fredholm based on a constant eddy viscosity assumption (Ekman, 1905), Eq. (21) becomes

$$w=u+iv=i\frac{u_*}{\kappa}\int_0^t \frac{1}{\beta}e^{-if\beta}e^{-z/\kappa u_*\beta}d\beta. \qquad (24)$$

The solution obtained here is remarkably similar in its appearance to Fredholm's solution [Ekman (1905) Eqs. (11)]. The most striking difference between the two solutions is in their behavior in the vicinity of the free surface, i.e., as $z\rightarrow 0$. Taking $z=0$ Fredholm's solution simplifies to Fresnel integrals, which are convergent, whereas the imaginary part of the present solution is a divergent cosine integral. This behavior of the present solution is, of course, a consequence of the assumed variation of the vertical eddy viscosity, in particular the vanishing of ν_T as the free surface is approached. A similar peculiarity is exhibited by the classical solution for turbulent flow over a rough boundary (Schlichting, 1960) in the context of which the problem is resolved by satisfying the no-slip condition a distance $z_{0b}=k_b/30$ above the theoretical bottom, where k_b is the equivalent sand roughness of the boundary. In analogy with the turbulent flow over a rough boundary we may therefore consider the surface velocity w_s obtained from the present solution to be the velocity evaluated from (21) or (24) corresponding to a value of $z=z_{0s}=k_s/30$, where k_s is the equivalent sand roughness of the free surface.

The preceding argument, which is supported by the experimental findings of Reichardt (1959), removes the apparent singularity of our solution. It leaves us, however, with the rather unpleasant problem of estimating the value of the equivalent sand roughness k_s of the free surface. The equivalent sand roughness of a sea surface has been studied to some extent in the context of the atmospheric boundary layer. These results in conjunction with Shemdin's (1973) observation that the equivalent sand roughness of a free surface was of the same order whether the free surface was approached from above, $k_{s,\mathrm{air}}$, or from below, k_s, may be used as a guide for estimating k_s. Analyzing wind velocity profiles above a sea surface Ruggles (1970) found the equivalent sand roughness of the sea surface to be essentially constant and of the order $k_{s,\mathrm{air}}\approx 4$ cm for wind speeds W_{10}, measured 10 m above the still water level, ranging from 3 to 10 m s^{-1}. In a similar study Wu (1969) found by reanalyzing wind data from both laboratory and field experiments that $k_{s,\mathrm{air}}\approx 8$ cm for wind speeds in excess of 15 m s^{-1}. Although several problems regarding the sea surface roughness remain unresolved, including a discrepancy between the values of $k_{s,\mathrm{air}}$ obtained by Wu (1969) and Ruggles (1970) for $W_{10}<10$ m s^{-1} and the equivalency of $k_{s,\mathrm{air}}$ and k_s, the above discussion does provide an order-of-magnitude estimate of k_s. As we shall see the results obtained from (21) and (24) are relatively insensitive to the actual value assigned to k_s, so long as its order of magnitude is known.

Inspection of (24) shows that we may write the equation in the form

$$w=u+iv=\frac{u_*}{\kappa}\int_0^{ft}\frac{\sin\alpha+i\cos\alpha}{\alpha}e^{-\zeta/\alpha}d\alpha, \qquad (25)$$

in which

$$\alpha=f\beta \qquad (26)$$

is the nondimensional time and

$$\zeta=\frac{zf}{\kappa u_*} \qquad (27)$$

is the nondimensional vertical coordinate.

Eq. (27) identifies the characteristic vertical length scale l of the problem to be

$$l=\frac{\kappa u_*}{f}. \qquad (28)$$

By taking $\kappa=0.4$ and $u_*=0.04(\rho_a/\rho)^{\frac{1}{2}}W_{10}$ as found by Ruggles (1970) with the ratio of air to fluid density, $\rho_a/\rho=1/840$, Eq. (28) shows that

$$l\approx\frac{3.66W_{10}}{\sin\phi}, \qquad (29)$$

where l is in meters if W_{10} is in meters per second. The magnitude of l is indicative of the depth of frictional influence, as will be discussed later. In the present context (29) is established merely to show that the parameter ζ in (25) safely may be considered small in the uppermost meters of the ocean.

With the assumption that $\zeta\ll 1$, approximate expressions for (25) may be obtained in terms of tabulated functions. For example, we have from (25)

$$v=\frac{u_*}{\kappa}\int_0^{ft}\frac{\cos\alpha}{\alpha}e^{-\zeta/\alpha}d\alpha$$

$$=\frac{u_*}{\kappa}\left[\int_0^{\alpha_1}\frac{\cos\alpha}{\alpha}e^{-\zeta/\alpha}d\alpha+\int_{\alpha_1}^{ft}\frac{\cos\alpha}{\alpha}e^{-\zeta/\alpha}d\alpha\right]. \qquad (30)$$

Now with $\zeta\ll 1$ and choosing $\zeta<\alpha_1<1$, we may expand $\cos\alpha=1-\alpha^2/2$ in the first and $e^{-\zeta/\alpha}=1-\zeta/\alpha$ in the second integral of the right-hand side of (30). Retaining only the leading term in the expansions and omitting the algebraic manipulations, we obtain

$$v=\frac{u_*}{\kappa}\left[\mathrm{E}_1\left(\frac{\zeta}{\alpha_1}\right)-\mathrm{Ci}(\alpha_1)+\mathrm{Ci}(ft)\right],\quad ft>\alpha_1=0.1, \qquad (31)$$

in which

$$\mathrm{E}_1\left(\frac{\zeta}{\alpha_1}\right)=\int_{\zeta/\alpha_1}^{\infty}\frac{e^{-\beta}}{\beta}d\beta \qquad (32)$$

and

$$\mathrm{Ci}(\alpha_1) = -\int_{\alpha_1}^{\infty} \frac{\cos\beta}{\beta} d\beta \tag{33}$$

are tabulated exponential integrals (Abramowitz and Stegun, 1972, Chap. 5). By retaining the omitted terms in the expansions of $\cos\alpha$ and $e^{-\zeta/\alpha}$ utilized in obtaining (31), an error bound may be obtained; the choice of $\alpha_1 = 0.1$ made here ensures us that the error in v is less than $25\,\zeta u_*$.

In a similar manner from (25) we obtain

$$u \approx \frac{u_*}{\kappa}\{\mathrm{Si}(ft) + \zeta[\mathrm{Ci}(\zeta) - \mathrm{Ci}(ft)]\}, \quad ft > \zeta, \tag{34}$$

in which

$$\mathrm{Si}(ft) = \int_0^{ft} \frac{\sin\beta}{\beta} d\beta \tag{35}$$

is the tabulated sine integral. The error term in (34) is $O(\zeta)$.

For $\zeta \ll 1$ the asymptotic behavior of $E_1(\beta) = -\mathrm{Ci}(\beta) = -\gamma - \ln\beta$ as $\beta \rightarrow 0$ may be introduced in (31) and (34) to give

$$u + iv \approx \frac{u_*}{\kappa}[\tfrac{1}{2}\pi + \zeta \ln\zeta + i(-2\gamma - \ln\zeta)]$$

$$+ \frac{u_*}{\kappa}[\mathrm{Si}(ft) - \tfrac{1}{2}\pi - \zeta\,\mathrm{Ci}(ft) + i\,\mathrm{Ci}(ft)], \tag{36}$$

in which $\gamma = 0.577...$ is Euler's constant. In the limit $ft \rightarrow \infty$ we have $\mathrm{Si}(ft) \rightarrow \pi/2$ and $\mathrm{Ci}(ft) \rightarrow 0$ and the second bracketed term in (36) vanishes. The steady-state response is therefore expressed by the first bracketed term in (36).

The exact steady-state solution may be found from (24) by taking the limit as $t \rightarrow \infty$. This procedure is rather time-consuming and involves numerous changes of variables and contour integration to obtain an expression which by use of Abramowitz and Stegun [(1972, Eq. (9.6.25)] may be shown to be identical to

$$u + iv = i\frac{2u_*}{\kappa}[\mathrm{ker}(2\sqrt{\zeta}) + i\,\mathrm{kei}(2\sqrt{\zeta})]$$

$$= i\frac{2u_*}{\kappa}K_0(2\sqrt{\zeta}e^{i\pi/4}), \tag{37}$$

in which ker and kei are the zeroth-order Kelvin functions. It is, however, relatively simple to obtain (37), which is identical to Ellison's (1956) solution for the atmospheric boundary layer, by returning to (1) and solving this equation for $\partial/\partial t = 0$. It is reassuring to find that the asymptotic expansion of (37) for small values of $2\sqrt{\zeta}$ to $O(\zeta)$ is identical to the steady-state solution obtained from (36).

4. Discussion of the results

As discussed in the preceding section the value of the surface drift current is obtained from the general solutions corresponding to a value of $z = z_{0s} = k_s/30$. With the order of magnitude of k_s being 5 cm and with l given by (29), it is evident that $\zeta_{0s} = z_{0s}/l \ll 1$ so that the steady surface drift obtained from (36) is

$$w_s = u_s + iv = \frac{u_*}{\kappa}\left[\frac{\pi}{2} + i\left(-1.15 + \ln\frac{30l}{k_s}\right)\right]. \tag{38}$$

Combining (36) and (38) it is seen that the velocity deficit, $w_s - w$, in the vicinity of the free surface is logarithmic as observed by Shemdin (1972). The value of the deflection angle θ_s between the surface shear stress and the steady surface drift current is found from (38) to be given by

$$\tan\theta_s = \frac{\pi/2}{-1.15 + \ln(30l/k_s)}, \tag{39}$$

where the deflection is to the right on the Northern Hemisphere.

With l given by (29) the sensitivity of the predicted surface velocities and deflection angles to the value assigned to k_s is seen from Table 1 to be relatively insignificant. Thus, a change of the estimated value of k_s by a factor of 2 changes the nondimensional surface current, $\kappa v_s/u_*$, as well as the deflection angle by only about 10%. The most striking difference between the results presented in Table 1 and the classical results of Ekman's (1905) theory is the much smaller value of the deflection angle. Observations of oil spill trajectories, either real (Smith, 1968) or simulated (Teeson *et al.*, 1970), have consistently shown values of the deflection angle of the order 10° or less and are therefore taken to support the present results. In the context of oil slick trajectories, which initially were the motivation for the present study, it is also interesting to note that the commonly employed rule of thumb that the speed of advection of an oil slick is 3% of the wind speed, W_{10} follows from the results presented in Table 1. Thus, with the magnitude of the surface current $|w_s| = (u_s^2 + v_s^2)^{\frac{1}{2}}$ being essentially equal to v_s and adopting Ruggles' (1970) result $u_* = 0.04(\rho_a/\rho)^{\frac{1}{2}}W_{10}$, it follows that

$$|w_s| \approx \frac{u_*}{\kappa}\left(\frac{\kappa v_s}{u_*}\right) = (\rho_a/\rho)^{\frac{1}{2}}\left(0.1\frac{\kappa v_s}{u_*}\right)W_{10}. \tag{40}$$

Inspection of the values of $\kappa v_s/u_*$ presented in Table 1 and taking $\rho_a/\rho = 1/840$ show $|w_s| = 0.03W_{10}$ to provide a reasonable approximation to (40). The preceding result should not be interpreted to support the "3% rule" which clearly oversimplifies the problem. The effect of oil slicks on the water surface in reducing the apparent surface roughness (Barger *et al.*, 1970) is not

TABLE 1. Values of surface drift current and deflection angle for various wind speeds and surface roughness.

	W_{10}(m s^{-1})									
	5		10		15		20		30	
$k_s \sin\phi$ (cm)	$\kappa v_s/u_*$	θ_s^0	$\kappa v_s/u_*$	θ_s^0	$\kappa v_s/u_*$	θ_s^0	$\kappa v_s/u_*$	θ_s^0	$\kappa v_s/u_*$	θ_s^0
10	7.47	11.9	8.16	10.9	8.57	10.4	8.85	10.1	9.26	9.6
5	8.16	10.9	8.85	10.1	9.26	9.6	9.55	9.3	9.95	9.0
2.5	8.85	10.1	9.55	9.3	9.95	9.0	10.24	8.7	10.65	8.4

accounted for by the 3% rule, nor is the possible existence of a geostrophic current at large depths.

To examine the variation of the steady drift current with depth the solution given by (37) is plotted in Fig. 1. The velocity vector is indicated at increments of $\zeta^{1/2}=(z/l)^{1/2}$ of 0.1, with the surface current given by (38) rather than corresponding to $\zeta=0$. The hodograph shown in Fig. 1 is based on a value of $\kappa v_s/u_*=10$ but is, except for the location of the point indicating the surface current, quite general. The extremely rapid decrease and rotation of the drift current with depth, a consequence of the logarithmic velocity deficit near the surface, is noted. With l given by (29) it is seen that $l=O(100\text{ m})$ for $W_{10}\approx 20$ m s^{-1} and the results presented in Fig. 1 indicate that for $z=0.01\ l\approx 1$ m ($\sqrt{\zeta}=0.1$) the velocity is only approximately one-third of its value at the surface with a deflection angle of 25° as compared to $\theta_s=9°$. It is also evident from Fig. 1 that there is practically no motion at a depth corresponding to $\zeta=1$, i.e., at $z=l$, which shows that l indeed is a measure of the extent of frictional influence as previously mentioned. In this respect it is worthwhile noting

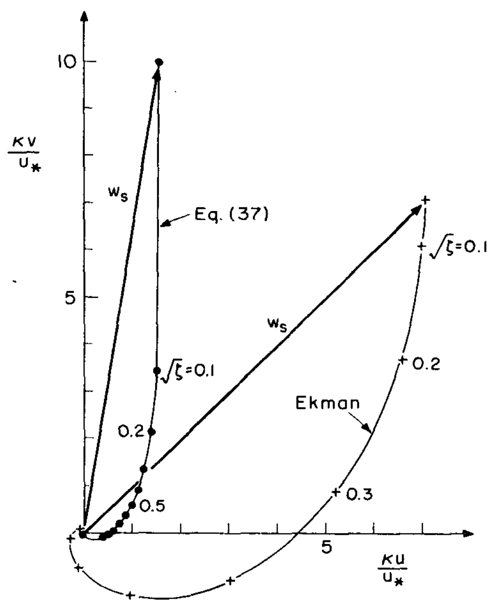

FIG. 1. Vertical velocity structure of a pure drift current in an infinitely deep homogeneous ocean of infinite lateral extent, comparing the turbulent Ekman spiral (●) and the classical Ekman spiral (+).

that (29) yields estimates of l comparable to empirical formulas for this quantity given in Neumann and Pierson (1966), for example.

For comparison the classical Ekman spiral is also shown in Fig. 1. From Ekman's (1905) solution we have that the magnitude of the surface drift current is $w_s=u_*^2/(\nu_e f)^{1/2}$, where ν_e is the constant value of the vertical eddy viscosity. Requiring that the surface velocity be the same for the two solutions leads to a determination of ν_e, which not surprisingly is similar to formulas quoted by Neumann and Pierson (1966). With this formula for ν_e the characteristic vertical scale of Ekman's solution becomes

$$a^{-1}=\left(\frac{2\nu_e}{f}\right)^{\frac{1}{2}}=\sqrt{2}\frac{u_*}{\kappa v_s}\frac{\kappa u_*}{f}=\sqrt{2}\frac{u_*}{\kappa v_s}l. \qquad (41)$$

For $\kappa v_s/u_*=10$, as chosen for the turbulent Ekman spiral, the proportionality of a^{-1} and l enables us to present the classical Ekman spiral in the nondimensional form used in Fig. 1. Aside from the difference in the value of the deflection angle at the surface the much more rapid decrease of the drift current with depth predicted by the present theory is noted. This feature may be of considerable importance in the design of offshore pile-supported structures.

Despite the considerable differences between the details of the velocity structure predicted by the present theory and that of Ekman, the two solutions share a common feature. The total mass transport predicted by the present theory is found from (37) to be

$$q_{\hat{x}}+iq_{\hat{y}}=\frac{2u_*}{\kappa}\int_0^\infty[-\mathrm{kei}(2\sqrt{\zeta})+i\,\mathrm{ker}(2\sqrt{\zeta})]dz$$

$$=\frac{u_* l}{\kappa}\int_0^\infty[-\beta\,\mathrm{kei}\beta+i\beta\,\mathrm{ker}\beta]d\beta=\frac{u_*^2}{f}, \qquad (42)$$

which is identical to the result obtained from Ekman's theory, as it should be since this result is independent of ν_T.

In order to compare the unsteady response to a suddenly applied surface shear stress, Eqs. (31) and (34) are plotted in Fig. 2 corresponding to a value of the steady surface velocity $\kappa v_s/u_*=10$, i.e., Fig. 2 represents the development of the surface current whose

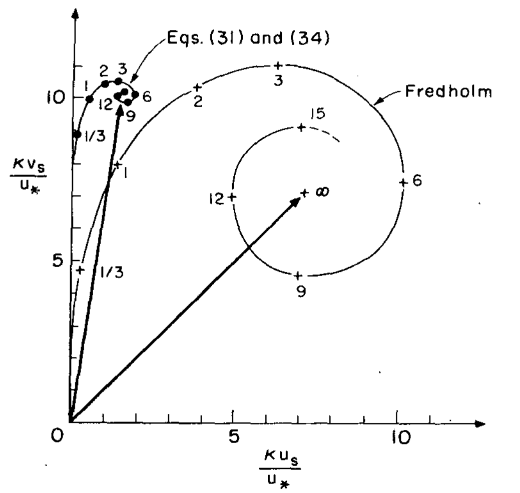

FIG. 2. The temporal development of the surface drift current due to a suddenly applied uniform surface shear stress. Time from time of application is indicated in pendulum hours. The present eddy viscosity model is shown by solid circles and Fredholm's classical solution by plus signs.

steady-state solution was the one presented in Fig. 1. For comparison Fredholm's solution which is based on an assumed constant value of the vertical eddy viscosity is also shown. The temporal development is shown in terms of pendulum hours and the striking difference is the rapidity with which the present solution attains its steady-state value. Whereas Fredholm's solution very slowly approaches the steady state, the present theory shows that the steady state is approximately reached within 3 pendulum hours. This extremely rapid response predicted by the present theory suggests that unsteadiness in many problems may be neglected and a solution based on a quasi-steady analysis, i.e., assuming steady conditions to be reached immediately, therefore may lead to meaningful results.

A simple physical explanation for the much faster response predicted by the present model may be found by comparing the order of magnitude of the vertical eddy viscosity of the present model and the constant value ν_e assumed in the classical model. With ν_T being assumed to vary linearly with depth, an equivalent constant value of the vertical eddy viscosity corresponding to the present model would be ν_T evaluated at $z=l/2$. Thus, the present model corresponds to an equivalent constant value of $\nu_{T,e}=\kappa^2 u_*^2/(2f)$. This value may be shown to be considerably larger than the constant value ν_e obtained by requiring that the surface current be of the order of 3% of the wind speed (by a factor of the order 50). Since the difference between the instantaneous velocity vector during the unsteady response and the steady-state velocity (according to the classical analysis) is proportional to $\nu_e^{-\frac{1}{2}}$, it is evident that the present model, with its much larger apparent eddy viscosity, approaches steady state more rapidly.

The nature of the unsteady response at some finite depth below the free surface is readily envisioned by examining (36). With the first term on the right-hand side of (36) expressing the steady-state response, the second term is identified as expressing the manner in which steady state is approached. This term depends only weakly on the value of ζ so long as $\zeta \ll 1$ and the manner in which steady state is approached is therefore that exhibited by the surface current development shown in Fig. 2. Since the magnitude of the steady-state velocity decreases rapidly with depth the approach of steady-state conditions will appear somewhat slower at greater depths.

6. Concluding remarks

The results obtained in the previous sections for the pure drift current in an infinite homogeneous ocean were obtained based on a proposed model for the vertical eddy viscosity. This proposed model, which simply assumes that the vertical eddy viscosity increases linearly with distance from a sheared boundary is particularly simple to apply to the problem of the response of an infinitely deep ocean to a prescribed surface shear stress when it is assumed that the drift current vanishes at large depths. In a more general analysis the effects of a spatially varying atmospheric pressure p_a and free surface elevation $\hat{z}=\eta=\eta(\hat{x},\hat{y},t)$ should be included.

For this more general problem formulation the governing equation replacing (1) reads

$$\frac{\partial w}{\partial t}+ifw=-P+\frac{\partial}{\partial \hat{z}}\left(\nu_T\frac{\partial w}{\partial \hat{z}}\right), \qquad (43)$$

in which

$$P=\frac{1}{\rho}\frac{\partial p_a}{\partial \hat{x}}+g\frac{\partial \eta}{\partial \hat{x}}+i\left(\frac{1}{\rho}\frac{\partial p_a}{\partial \hat{y}}+g\frac{\partial \eta}{\partial \hat{y}}\right) \qquad (44)$$

is the term giving rise to the slope current.

When the effects of a varying atmospheric pressure and free surface elevation are included in the analysis the current no longer vanishes at large depths. This, in turn, gives rise to the development of a bottom boundary layer in addition to the surface boundary layer treated in detail in the present paper. Thus, in the general case and assuming, for simplicity, steady-state conditions Eq. (43) reads for the present vertical eddy viscosity model

$$ifw_s=-P+\frac{\partial}{\partial z_s}\left(\kappa|u_{*s}|z_s\frac{\partial w_s}{\partial z_s}\right), \quad z_s\leqslant z_m, \qquad (45a)$$

$$ifw_b=-P+\frac{\partial}{\partial z_b}\left(\kappa|u_{*b}|z_b\frac{\partial w_b}{\partial z_b}\right), \quad z_b\leqslant h\leqslant z_m, \qquad (45b)$$

in which z_s is the vertical coordinate, denoted by z in the main body of this paper, i.e., $z_s=0$ in the free surface $\hat{z}=\eta\approx 0$ and positive downward and z_b is zero at the bottom, $\hat{z}=-h$, and positive upward. $|u_{*s}|$ and $|u_{*b}|$ are the shear velocities corresponding to the magnitudes of the surface shear stress and the bottom shear stress, respectively.

It is evident from the steady-state solution for the surface boundary layer presented and discussed in the previous sections that the characteristic vertical length scales of (45a) and (45b) are $l_s=\kappa|u_{*s}|/f$ and $l_b=\kappa|u_{*b}|/f$, respectively. Thus, for a water depth $h\gg l_s+l_b$ the depth is effectively infinite and the solution of (45a) and (45b) is readily found to consist of a surface boundary layer, a frictionless layer in which the flow is geostrophic, i.e., $w=w_s=w_b=-P/if$, and a bottom boundary layer. Hence, for $h\gg l_s+l_b$ it follows that the choice of the matching location $z_s=h-z_b=z_m$ is immaterial. In fact, the solution for the bottom boundary layer becomes identical to the solution obtained by Ellison (1956) for the atmospheric boundary layer.

In the general case of a finite depth, namely, $h\leqslant O(l_s+l_b)$, the preceding reasoning that the extent of the surface influence is l_s and that of the bottom is l_b, suggests the following general model for the vertical eddy viscosity:

$$\nu_T=\begin{cases}\kappa|u_{*s}|z_s, & z_s\leqslant\dfrac{|u_{*s}|}{|u_{*s}|+|u_{*b}|}h\\[2ex] \kappa|u_{*b}|z_b, & z_b\leqslant\dfrac{|u_{*b}|}{|u_{*s}|+|u_{*b}|}h\end{cases}, \quad (46)$$

This general model for the vertical eddy viscosity will reproduce the observed features of turbulent shear flows, i.e., the logarithmic velocity deficit near the free surface and the classical logarithmic velocity profile near solid boundaries. It is quite general, in that it may be applied in shallow as well as in deep water, and it is sufficiently simple to apply to be practical. In this respect it is noted that the problems associated with the application of (46) are similar to those associated with Thomas' (1975) model, i.e., the value of the bottom shear stress must first be estimated and a solution based on (46) must be obtained. Based on this solution an updated and improved estimate of $|u_{*b}|$ is obtained and the ultimate solution is approached in an iterative manner. This procedure is, of course, not trivial in the general case but is necessary if the details of the vertical velocity profile are to be resolved in a physically realistic manner.

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REFERENCES

Abramowitz, M., and I. A. Stegun, 1972: *Handbook of Mathematical Functions. NBS Appl. Math. Ser.*, No. 55, 1945 pp.

Barger, W. R., W. D. Garret, E. L. Mollo-Christensen and K. W. Ruggles, 1970: Effects of an artificial sea slick upon the atmosphere and the ocean. *J. Appl. Meteor.*, **9**, 396–400.

Ekman, V. W., 1950: On the influence of earth's rotation on ocean currents. *Ark. Math. Astron. Fys.*, **2**, 1–53.

Ellison, T. H., 1956: Atmospheric turbulence. *Surveys in Mechanics*, G. K. Batchelor and R. M. Davis, Eds., Cambridge University Press, 400–430.

Fjeldstad, J. E., 1929: Ein Beitrag zur Theorie der Winderzeugten Meeresströmungen. *Beitr. Geophys.*, **23**, 237–247.

Hildebrand, F., 1965: *Calculus for Applications.* Prentice Hall, 646 pp.

Kajiura, K., 1964: On the bottom friction in an oscillatory current. *Bull. Earthquake Res. Inst., Univ. Tokyo*, **42**, 147–174.

——, 1968: A model of the bottom boundary layer in water waves. *Bull. Earthquake Res. Inst. Univ. Tokyo*, **46**, 75–123.

Murray, S. P., 1975: Trajectories and speeds of wind-driven currents near the coast. *J. Phys. Oceanogr.*, **5**, 347–360.

Neumann, G., and W. J. Pierson, 1966: *Principles of Physical Oceanography.* Prentice Hall, 545 pp.

Reichardt, H., 1959: Gesetzmässigkeiten der geradlinigen Turbulenten Couette-Strömung. Mitteilungen aus dem Max-Planck-Institute fur Strömungsforschung und der Aerodynamischen Versuchsanstalt, No. 22, Göttingen, 1–45.

Reid, R. O., 1957: Modification of the quadratic bottom-stress law for turbulent channel flow in the presence of surface wind-stress. U. S. Army Corps of Engineers, Beach Erosion Board, Tech. Memo. No. 93, 33 pp.

——, 1975: Analytical and numerical studies of ocean circulation. *Rev. Geophys. Space Phys.*, **13**, 606–609.

Ruggles, K. W., 1970: The vertical mean wind profile over the ocean in light to moderate winds. *Jr. Appl. Meteor.*, **9**, 389–395.

Schlichting, H., 1960: *Boundary Layers*, 4th ed. McGraw-Hill, 647 pp.

Shemdin, O. H., 1972: Wind-generated current and phase speed of wind waves. *J. Phys. Oceanogr.* **2**, 411–419.

——, 1973: Modelling of wind-induced currents. *J. Hydraulic Res.*, **11**, 281–297.

Smith, J. E., 1968: *Torrey Canyon Pollution and Marine Life.* Cambridge University Press, 196 pp.

Teeson, D., F. M. White and H. Schenck, 1970: Studies of the simulation of drifting oil by polyethylene sheets. *Ocean Eng.*, **2**, 1–11.

Thomas, J. H., 1975: A theory of steady wind-driven currents in shallow water with variable eddy viscosity. *J. Phys. Oceanogr.*, **5**, 136–142.

Witten, A. J., and J. H. Thomas, 1976: Steady wind-driven currents in a large lake with depth-dependent eddy viscosity. *J. Phys. Oceanogr.*, **6**, 85–92.

Wu, J., 1969: Wind stress and surface roughness of air-sea interface. *J. Geophys. Res.*, **74**, 444–454.