Stability of Water Waves
Author(s): R. S. MacKay and P. G. Saffman
Source: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 406, No. 1830 (Jul. 8, 1986), pp. 115-125
Published by: The Royal Society
Stable URL: http://www.jstor.org/stable/2397813
Accessed: 09/08/2009 07:19

Proc. R. Soc. Lond. A **406**, 115–125 (1986)
Printed in Great Britain

Stability of water waves

BY R. S. MACKAY[1] AND P. G. SAFFMAN[2]

[1] *Mathematics Institute, University of Warwick, Coventry CV4 7AL, U.K.*
[2] *Applied Mathematics, California Institute of Technology, Pasadena, California* 91125, *U.S.A.*

(*Communicated by E. C. Zeeman, F.R.S. – Received* 31 *January* 1986)

We apply some general results for Hamiltonian systems, depending on the notion of signature of eigenvalues, to determine the circumstances under which collisions of imaginary eigenvalue for the linearized problem about a travelling water wave of permanent form are avoided or lead to loss of stability, up to non-degeneracy assumptions. A new superharmonic instability is predicted and verified.

INTRODUCTION

Following the studies by Stokes (1847), many authors have investigated periodic uniformly travelling water waves. Various instabilities are known and understood, but there remain many questions about their stability. In particular, as one varies parameters, for example the wave steepness, there are sometimes apparent crossings of eigenvalues for the linearized problem about the wave (see, for example, Longuet-Higgins 1978*a*; McLean *et al.* 1981; McLean 1982; Chen & Saffman 1985). On closer inspection these often turn out to be either 'avoided crossings' (figure 1) or 'bubbles of instability' (figure 2). In this paper we apply results for general Hamiltonian systems that explain why crossing of eigenvalues is rare, and that predict one or other of the above possibilities depending on the 'signatures' of the eigenvalues.

Let $\zeta(x, y, t)$ be the vertical displacement of the surface of an inviscid fluid of infinite depth and constant density ρ above point (x, y) at time t, $\varphi(x, y, z, t)$ be the velocity potential at height z for an irrotational flow moving under the influence of a uniform gravitational field g and surface tension T, and $\psi(x, y, t)$ be the velocity potential on the surface.

For periodic uniformly travelling waves propagating at velocity c_μ in the x-direction, with period L in x, and independent of the transverse direction y, we have

$$(\zeta, \psi)(x, y, t) = (\zeta_\mu, \psi_\mu)(\xi),$$
$$(\zeta_\mu, \psi_\mu)(\xi+L) = (\zeta_\mu, \psi_\mu)(\xi),$$
$$\xi = x - c_\mu t.$$

They come in families, as indicated by the parameter μ, which could represent amplitude. In particular, there is a family connected to the zero solution, say at $\mu = 0$, with

$$c_0 = (g/k + Tk/\rho)^{\frac{1}{2}},$$

where $k = 2\pi/L$, which can be taken as another free parameter.

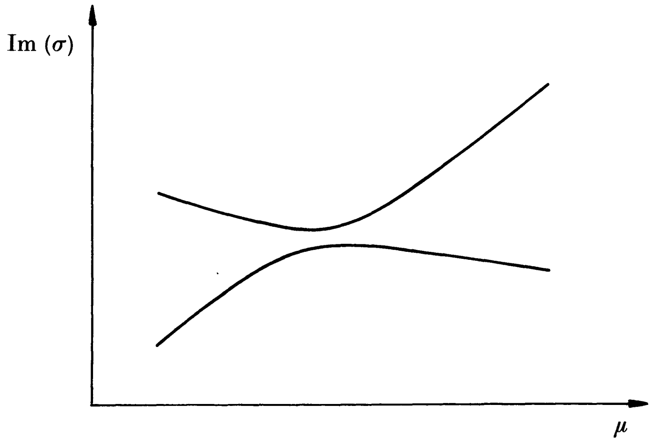

FIGURE 1. An avoided crossing.

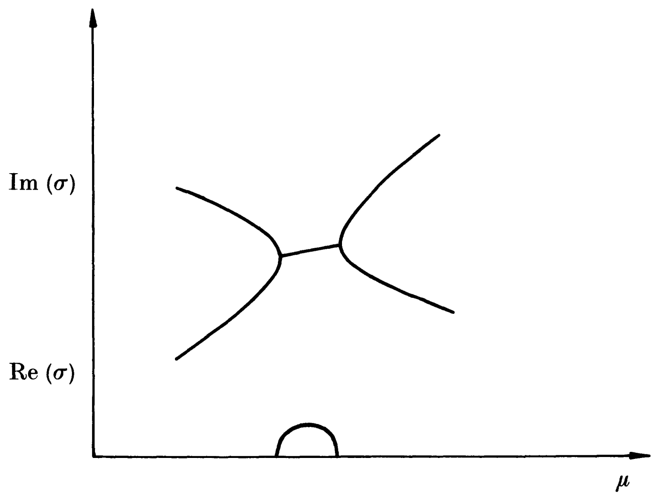

FIGURE 2. A bubble of instability.

To examine the linear stability of a periodic uniformly travelling wave, one considers the linearized equations for infinitesimal perturbations, in a frame travelling with the wave. Since the equations are periodic in ξ and independent of y, they leave invariant the subspaces $\Omega_{p,q}$, p, $q \in \mathbb{R}$, of functions of the form

$$e^{i(p\xi+qy)}\,(\Delta\zeta,\ \Delta\psi)\,(\xi),$$

with $\Delta\zeta$, $\Delta\psi$ L-periodic (note that $\forall m \in \mathbb{Z}$, $\Omega_{p+mk,\,q} = \Omega_{pq}$). Thus we can treat the subspaces $\Omega_{p,q}$ independently, although for real solutions we should take them in pairs $\Omega_{p,q}$, $\Omega_{-p,-q}$. The eigenvalues of the linearized problem are the values of σ such that there is a non-trivial solution with time-dependence $e^{\sigma t}$ (an eigenmode).

The spectrum is easy to evaluate when $\mu = 0$. As is well known (see, for example, Whitham 1974), the general solution of the linearized equations about the rest state can be written as a sum over $p', q \in \mathbb{R}$ of the following modes:

$$\varphi = \psi_0 \, e^{i(p'x+qy-\omega t)} \, e^{\kappa z},$$

$$\zeta = \zeta_0 \, e^{i(p'x+qy-\omega t)},$$

where

$$\kappa = (p'^2+q^2)^{\frac{1}{2}},$$

$$-i\omega\zeta_0 = \kappa\psi_0,$$

$$i\omega\psi_0 = (g+T\kappa^2/\rho)\,\zeta_0,$$

$$\omega = s\Omega(\kappa),\; s = \pm 1,$$

$$\Omega(\kappa) = (g\kappa + T\kappa^3/\rho)^{\frac{1}{2}}.$$

Thus in a frame moving at velocity c_0 in the x-direction, and writing

$$p' = p+mk,$$

the eigenmodes in $\Omega_{p,q}$ are

$$(\Delta\zeta, \Delta\psi) = (\kappa, \; -is\,\Delta(\kappa))\, e^{imk\xi}$$

for $m \in \mathbb{Z}$, $s = \pm 1$, with eigenvalues

$$\sigma^s_m = -i(s\Omega(\kappa) - c_0 p').$$

All the eigenvalues lie on the imaginary axis, so we say that the rest state is *spectrally stable*.

As μ is increased, the eigenvalues can move around. For a wave that is symmetric about some crest or trough, the linearized equations are invariant under $t \to -t$, $\delta\psi \to -\delta\psi$, so the spectrum must be invariant under $\sigma \to -\sigma$. Actually, this conclusion follows even for non-symmetric waves, from the Hamiltonian nature of the system. Thus an eigenvalue cannot leave the imaginary axis unless it is accompanied by its reflection in the imaginary axis. So a necessary condition for a periodic uniformly travelling wave to lose spectral stability is that two eigenvalues on the imaginary axis collide.

This is not a sufficient condition, however. There are cases in which two eigenvalues collide but do not fall off the imaginary axis. In this paper we give a stronger necessary condition for instability, based on the Hamiltonian nature of the system, and we analyse the typical behaviour of eigenvalues near collisions.

Hamiltonian formulation

Following Zakharov (1968), the water wave equations can be written in Hamiltonian form:

$$\partial\zeta/\partial t = \delta H/\delta\psi, \quad \partial\psi/\partial t = -\delta H/\delta\zeta,$$

where H is the average total energy per unit horizontal area. We can adapt this for a frame travelling at velocity $\boldsymbol{c}$ by using

$$H = K + V - \boldsymbol{c}\cdot\boldsymbol{P},$$

where K is the kinetic energy in the rest frame, $\boldsymbol{P}$ the momentum in the rest frame (uniquely defined in water of infinite depth without mean shear), and V the potential energy (gravitational plus capillary). A uniformly travelling wave is an equilibrium of this system in the frame travelling with the wave.

There is a well-developed theory that describes how the eigenvalues of an equilibrium Φ_{E} of a Hamiltonian system with Hamiltonian $H(\Phi)$ can move as parameters change (for a review see MacKay 1986). If σ is a pure imaginary non-zero eigenvalue for the linearized problem about the equilibrium, then the second derivative of the energy $\mathrm{D}^2H_{\Phi_{\mathrm{E}}}(\delta\Phi, \delta\Phi)$ is a non-degenerate quadratic form on the real invariant space I_σ associated to the pair of eigenvalues σ, $-\sigma$. If the eigenvalues are simple, then the quadratic form is definite, either positive or negative. This is called the *signature* of the eigenvalue σ. It is preserved as parameters vary as long as the eigenvalue does not collide with another one. Then the fundamental result is:

THEOREM 1. *If two pure imaginary simple eigenvalues of the same signature collide at a point other than zero, then they cannot leave the imaginary axis.*

The proof is basically that when the signatures are the same, energy conservation prevents perturbations from growing arbitrarily large. Thus a necessary condition for loss of spectral stability is collision of two eigenvalues of opposite signature or at zero.

This result has appeared in many places under various guises. It goes back essentially to Weierstrass (1858); see also Wintner (1935) and the appendix to Moser (1958). Nayfeh & Mook (1979) treat the problem of the stability of a rotating oscillator by using the same idea. In the context of waves, the importance of the signatures (= the sign of the 'small signal energy') has been appreciated for quite some time (see, for example, Sturrock (1958), Hasegawa (1975) and references therein, and Cairns (1979)), but has always been limited to situations near the flat state, involving the dispersion relation. We give here applications for which the equilibrium state Φ_{E} is not the flat state and thus the usual theory of negative energy waves does not apply.

An important question for applications is what are the typical behaviours of eigenvalues near collisions. Williamson (1936) worked out normal forms for all cases of linear Hamiltonian systems with multiple eigenvalues, and Galin (1975) (for a summary in English, see Arnol'd (1978), Appendix 6) computed their codimensions and miniversal unfoldings. Galin also gave bifurcation diagrams for all cases of codimension less than or equal to 2. The cases of interest here, namely, existence of a double pure imaginary eigenvalue with diagonal Jordan normal form, turn out to be codimension 3, and the bifurcation diagrams have been worked out by MacKay (1986). The results show that crossing of eigenvalues is exceptional in one-parameter families, and typically unfolds under perturbation into an avoided crossing in the case of eigenvalues of the same signature, and into a bubble of instability in the case of opposite signature. Thus all we have to do to explain the observations is to calculate the signatures of the appropriate eigenvalues.

One should be aware, however, that one is not guaranteed to see the typical case. Further calculations are necessary to check that various coefficients are non-zero.

Symmetries may force some of them to be zero. For example, for the linearized equations about uniformly travelling periodic waves, modes belonging to different $\Omega_{p,q}$ do not interact. More seriously, many of the waves considered are symmetric about some crest or trough. This restricts the linearized equations to a special class of Hamiltonian systems known as 'reversible'. In the space of linear reversible Hamiltonian systems existence of a double pure imaginary eigenvalue with diagonal Jordan normal form is only codimension 2, but its unfolding still gives rise to avoided crossings or bubbles of instability in typical one-parameter families, according to the signatures (Jiménez & MacKay 1986).

The case of multiple eigenvalues at zero is also covered by the theory, but we shall not discuss it here as it does not depend on signature, though it does include the Benjamin–Feir (1967) instability.

There are more related results. For example, if one adds small positive definite dissipation, then the pure imaginary eigenvalues of positive signature move into the left half plane (damped), whereas those of negative signature move into the right half plane (unstable).

A parallel theory has also been developed for stability of periodic orbits of Hamiltonian systems (see Moser (1958), Appendix 29 of Arnol'd & Avez (1968) and references therein, Yakubovich & Starzhinskii (1975), and Howard & MacKay (1986)).

Calculation of signatures

Let us evaluate the signatures for the water wave problem when $\mu = 0$. By continuity, the signature of an eigenvalue will be conserved as far as its first collision with another eigenvalue, as μ is increased. The definition requires us to consider the energy on the *real* invariant space associated to the pair of eigenvalues σ, $-\sigma$. This means that we have to add the complex conjugates to the previous expressions for φ and ζ.

Averaged over unit area in x, y, we have the standard results for the energy and momentum in the rest frame (to second order)

$$K = \int \tfrac{1}{2}\rho\varphi(\partial\varphi/\partial z)\,\mathrm{d}x\,\mathrm{d}y = \rho\kappa|\psi_0|^2,$$

$$V = \int (\tfrac{1}{2}\rho g\zeta^2 + T\nabla^2\zeta)\mathrm{d}x\,\mathrm{d}y = (g+T\kappa^2/\rho)\,|\zeta_0|^2,$$

$$P_x = \int \rho\zeta(\partial\varphi/\partial x)\,\mathrm{d}x\,\mathrm{d}y = \mathrm{i}p'\rho(\psi_0\,\zeta_0^* - \psi_0^*\,\zeta_0).$$

Substitute

$$\psi_0 = -\mathrm{i}\omega\zeta_0/\kappa.$$

Then

$$E = K + V - c_0 P_x,$$

$$= 2\rho\omega(\omega - c_0 p')\,|\zeta_0|^2.$$

The disturbance is moving with speed ω/p' in the rest frame. *Thus the energy in a moving frame of a disturbance to the steady state is negative if it is going in the same* *direction as the moving frame but slower, and positive if it is going faster or in the opposite direction.*

Actually, it was not necessary to calculate the energies and momentum to see this. It follows from the Lagrangian formulation that a disturbance moving with speed c' in the x-direction has kinetic energy

$$K = \tfrac{1}{2}P_x c',$$

and hence its energy in a frame moving with speed c is

$$E = 2K(1-c/c') + (V-K).$$

This reduces to the same results on remembering that $V-K$ is of third order in the amplitude of the disturbance from the flat state.

Given T/ρ, g, and k (and $c_0 = \Omega(k)/k$), we can calculate the wave velocity for the eigenvalues σ_m^s, and hence their signatures. For example, for $T = 0$, $p = 0$, $q = 0$ (i.e. two-dimensional superharmonic disturbances to gravity waves), we get the eigenvalues

$$\sigma_m^s = \mathrm{i}(m - s|m|^{\frac{1}{2}})\,(gk)^{\frac{1}{2}},$$

with wave speed

$$\omega/p' = sc_0|m|^{\frac{1}{2}}/m.$$

Thus σ_m^s has positive signature for $sm \leqslant -1$, and negative signature for $sm \geqslant 2$. Note that the cases $sm = 0$ and 1 have $\sigma = 0$, and correspond to change of the mean level and to horizontal translation, respectively.

Numerical verification

The predictions of the theory from such calculations of the signature are supported perfectly by numerical results for three-dimensional disturbances (i.e. with arbitrary p and q) to gravity waves ($T = 0$) (McLean *et al.* 1981; McLean 1982), and to capillary waves ($g = 0$) (Chen & Saffman 1985), on water of infinite depth. In the first case, all collisions found for $\mu = 0$ were between eigenvalues of opposite signature and led to loss of spectral stability for $\mu > 0$. In the second case, collisions were found between eigenvalues of opposite signatures and between eigenvalues of the same signature; for $\mu > 0$, the former led to loss of spectral stability, while the latter became avoided crossings.

A non-trivial example which predicts a new instability is provided by superharmonic two-dimensional instabilities of Stokes waves. In figure 3 we show the values of $\mathrm{Im}\,(\sigma_m^s)$ ($= m - s|m|^{\frac{1}{2}}$ at $\mu = 0$) as functions of μ (in this case the wave steepness ka) calculated by Longuet-Higgins (1978*a*). We have normalized to $g = 1$ and $k = 1$, and our notation is slightly different. There are four collisions in this figure, three of opposite signature and one at zero.

The collision marked (i) at $\mu = 0$ between modes $(m, s) = (1, -1)$ and $(4, 1)$ does not lead to loss of stability; the eigenvalues avoid each other. This appears to be an exception, but in fact is not, as will be explained below. The collision (ii) at $\mu \approx 0.2456$, $\mathrm{Im}\,(\sigma) \approx 3.3533$, we found on close examination appears to be actually a bubble of instability. Figure 4 shows results of detailed numerical cal-

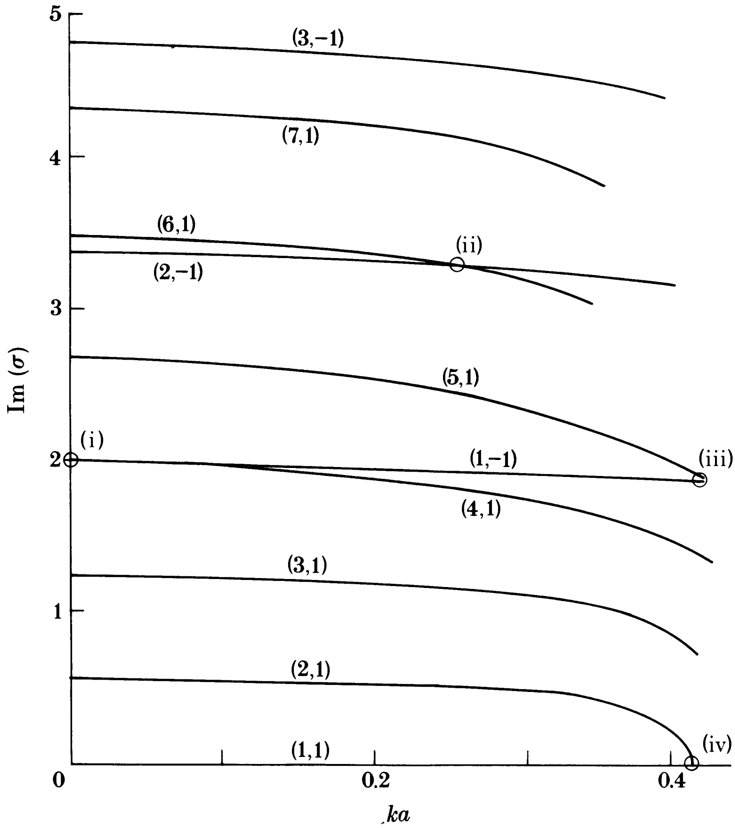

FIGURE 3. Results of Longuet-Higgins (1978*a*) for eigenvalues σ labelled by (m, s) plotted against wave amplitude for superharmonic two-dimensional disturbances. Circles (i), (ii) and (iii) mark intersections of eigenvalues of opposite signature. The intersection at zero (iv) is the Tanaka instability. Values of Im (σ) at $ka = 0$ are given by $m - s|m|^{\frac{1}{2}}$.

culations in the neighbourhood of the crossing (cf. Chen & Saffman 1985, figure 1). The instability can be interpreted as a fourth-order resonance, i.e. proportional to $(ka)^4 \approx 10^{-4}$, so the smallness of the growth rate and of the range of unstable steepnesses is not unexpected. These results were obtained by using two entirely different codes and satisfy the usual tests of reliability. The values shown in the figure are due to Mr J. A. Zufiria.

The value of *ka* at crossing (iii) is too large for our codes to demonstrate instability reliably, but it is believed to be present for reasons to be given shortly. Crossing (iv) at $\mu \approx 0.4292$, Im$(\sigma) = 0$, is the Tanaka (1983) instability; see Saffman (1985) for a simple proof of Tanaka's empirical finding that this occurs when the rest frame wave energy $K+V$ is a maximum. Note, however, that it is not the least steep superharmonic instability; crossing (ii) is a less steep one.

These results are implicit in the work of McLean *et al.* (1981). They studied numerically the $\mu > 0$ stability diagram in the p, q plane. A sketch of typical results is shown in figure 5. The instabilities were separated into two classes: class I when the collisions were between modes $(p+M, q)$ and $(p-M, q)$, and class II

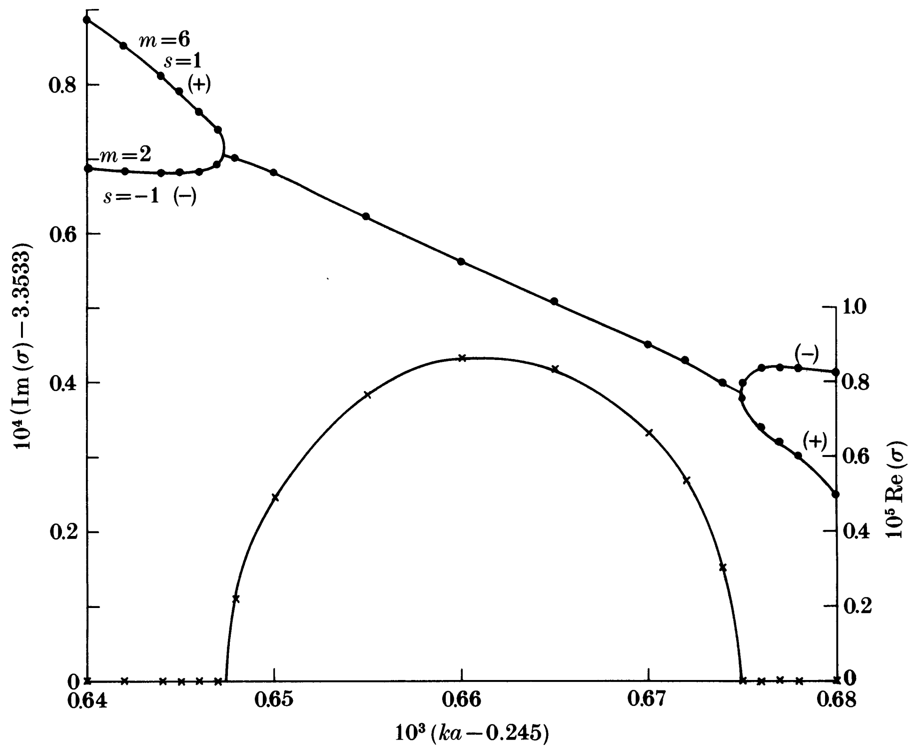

FIGURE 4. Eigenvalues in a neighbourhood of crossing (ii) of figure 3, showing the instability region suggested by the apparent intersection of modes of opposite signature. Symbol in parenthesis shows signature. Note that the signature of the more rapid disturbance has changed across the instability, confirming an instability and not an avoided crossing.

between $(p+M, q)$ and $(p-M-1, q)$. The sketch shows bands of instability for the class I, $M = 2$ and the class II, $M = 1$. (The class I, $M = 1$ collision (not shown) is the Benjamin–Feir long-wave modulational instability and is not treated here.) The solid lines show the collision loci for $\mu = 0$. Note that these intersect the p-axis (two-dimensional disturbances) at $p = 3$ and $p = \frac{17}{4}$ respectively. As μ increases these broaden into bands that get wider and move to the left.

Consider first the class II, $M = 1$. It moves away from $p = 3$, $q = 0$, so there is apparent stabilization of the (i) crossing, but only because the analysis was restricted to superharmonic disturbances. In fact this collision leads to instability. As μ increases, the band widens vertically and intersects the p-axis at $p = \frac{1}{2}$ when $\mu \approx 0.4050$, giving rise to a subharmonic wavelength doubling instability discovered by Longuet-Higgins (1978*b*). As μ continues to increase, the intersection with the p-axis gets larger and meets $p = 1$ when $\mu \approx 0.4292$. This is the Tanaka instability of crossing (iv).

Consider now the class I, $M = 2$ results. As μ increases, this band moves to the left and for $0.245645 < \mu < 0.245675$ includes $p = 4$. The wave numbers of the colliding modes are $4+2$ and $4-2$, i.e. 6 and 2, and this is the (ii) intersection in

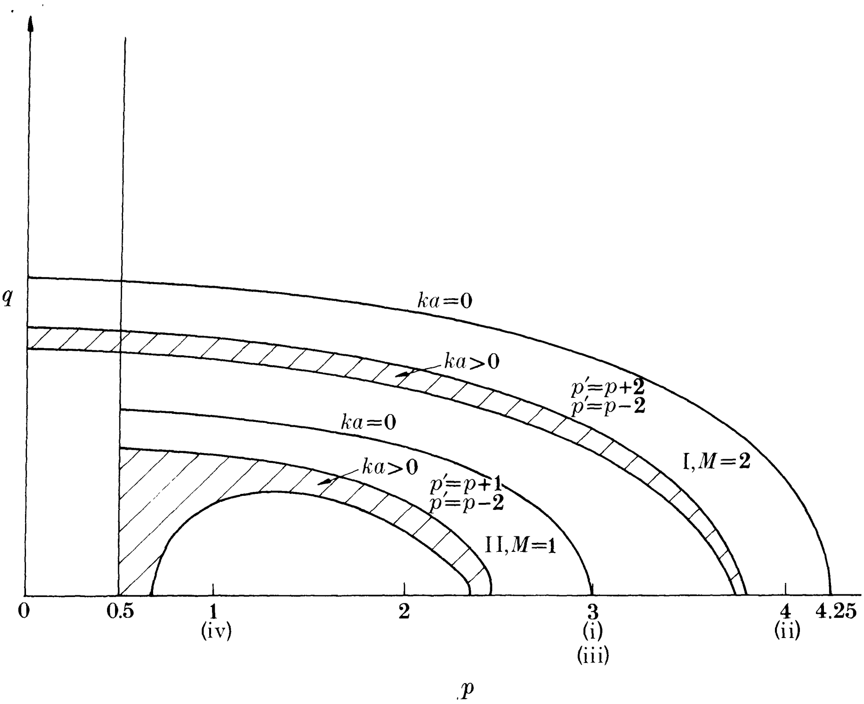

FIGURE 5. Sketch of instability bands in the p, q plane for gravity waves. Solid lines show collisions for $ka = 0$; shaded regions show values of p, q for which modes are unstable. The two-dimensional superharmonic results can be understood in terms of movement of the bands through integer values on the p-axis.

figure 3. As μ continues to increase, the band crosses $p = 3$ when μ is approximately 0.4; the colliding modes are then $3+2$ and $3-2$, i.e. 5 and 1, and this is the collision (iii).

Alternative calculation of signature

There is an alternative definition of signature (Moser 1958), which is equivalent to the one given here (MacKay 1986) and simpler to calculate. A Hamiltonian system can be expressed in the form

$$\mathrm{d}\Phi/\mathrm{d}t = J\mathrm{D}H_{\Phi},$$

where Φ describes the state at time t, H is the Hamiltonian, $\mathrm{D}H_{\Phi}$ is the derivative of the Hamiltonian at Φ, and J is an isomorphism between linear forms and tangent vectors, induced by a symplectic form ω_2 (see Arnol'd 1978, §37). For the water wave problem, $\Phi(t)$ is $(\zeta, \psi)(x, y, t)$ and

$$\omega_2\{(\delta\zeta_1, \delta\psi_1), (\delta\zeta_2, \delta\psi_2)\} = \int (\delta\psi_1\,\delta\zeta_2 - \delta\psi_2\,\delta\zeta_1)\,\mathrm{d}x\,\mathrm{d}y.$$

Given a pair of pure imaginary non-zero eigenvalues σ, $-\sigma$, choose $\mathrm{Im}\,(\sigma) > 0$. Let E_σ be the corresponding eigenspace. Then

$$\mathrm{i}\omega_2(\delta\Phi,\, \delta\Phi^*), \quad \delta\Phi \in E_\sigma$$

is a non-degenerate, real quadratic form, and has the same signature as $\mathrm{D}^2H(\delta\Phi,\, \delta\Phi)$ on I_σ. Equivalently one can use

$$-\mathrm{i}\omega_2(\delta\Phi,\, \delta\Phi^*),\; \delta\Phi \in E_{-\sigma}.$$

So let us evaluate the signatures this way. Averaged over unit area,

$$\mathrm{i}\int (\delta\psi\delta\zeta^* - \delta\psi^*\delta\zeta)\,\mathrm{d}x\,\mathrm{d}y = 2sk\Omega(k),$$

which has the sign of s. So the signature is the sign of $s\,\mathrm{Im}\,(\sigma)$. This agrees with the previous result, since ω has the sign of s and $\mathrm{Im}\,(\sigma)$ has the sign of $(\omega - c_0\,p')$.

We used this method to calculate the signature of the modes that cross at (ii). The results show that the signatures of the two modes, ordered by the value of $\mathrm{Im}\,(\sigma)$, are exchanged. This confirms the instability shown in figure 4, as the signature would be unchanged in an avoided crossing, though this argument does not exclude the possibility that the eigenvalues pass through each other without interaction.

Conclusion

We have shown that for a uniformly travelling periodic water wave to lose spectral stability, it is necessary that there be for the linearized problem about it a collision of eigenvalues of opposite signature or at zero. For waves that can be traced back to zero amplitude, the signature of an eigenvalue is negative if when traced back to zero amplitude the corresponding disturbance moves in the same direction as the wave but slower, and positive if it moves faster or in the opposite direction. If the signatures are the same we predict an avoided crossing; if they are opposite we predict a bubble of instability.

It will be interesting to confirm the predictions of this paper for cases with both T and g non-zero by numerical calculation, when rich phenomena are expected. Also, the case of finite depth is expected to show interesting properties. Another physical problem to which these ideas may well be of use is the resonance of Kelvin waves on a straight vortex filament in a weak straining field (Moore & Saffman 1975; Tsai & Widnall 1976).

We thank the London Mathematical Society and the Science and Engineering Research Council for sponsoring the 1984 Durham Bifurcation Theory Symposium, which stimulated this work. P.G.S. also thanks the Office of Naval Research (N00014-79-C-0412, NR062-639) for support, and R.S.M. the Nuffield Foundation for travel expenses to discuss this work. Last, but not least, thanks to Peta McAllister for typing the paper.

REFERENCES

Arnol'd, V. I. 1978 *Mathematical methods of classical mechanics*. Heidelberg and Berlin: Springer-Verlag.

Arnol'd, V. I. & Avez, A. 1968 *Ergodic problems of classical mechanics*. Benjamin.

Benjamin, T. B. & Feir, J. E. 1967 *J. Fluid Mech.* **27**, 417.

Cairns, R. A. 1979 The role of negative energy waves in some instabilities of parallel flows. *J. Fluid Mech.* **92**, 1–14.

Chen, B. & Saffman, P. G. 1985 Three-dimensional stability and bifurcation of capillary and gravity waves on deep water. *Stud. appl. Math.* **72**, 125–147.

Galin, D. M. 1975 Versal deformations of linear Hamiltonian systems. (In Russian.) *Trudy Sem. I. G. Petrovskogo Vyp.* **1**, 63–74.

Hasegawa, A. 1975 *Plasma instabilities and non-linear effects*. Heidelberg and Berlin: Springer-Verlag.

Howard, J. E. & MacKay, R. S. 1986 Linear stability of symplectic maps. (In preparation.)

Jiménez, J. & MacKay, R. S. 1986 Linear stability of vortex streets. (In preparation.)

Longuet-Higgins, M. S. 1978*a* The instabilities of gravity waves of finite amplitude in deep water. I. Superharmonics. *Proc. R. Soc. Lond.* A **360**, 471–488.

Longuet-Higgins, M. S. 1978*b* The instabilities of gravity waves of finite amplitude in deep water. II. Subharmonics. *Proc. R. Soc. Lond.* A **360**, 489–505.

MacKay, R. S. 1986 Stability of equilibria of Hamiltonian systems. In *Nonlinear phenomena and chaos* (ed. S. Sarkar). Bristol: Adam Hilger. (In the press.)

McLean, J. W. 1982 Instabilities of finite-amplitude water waves. *J. Fluid Mech.* **114**, 315–330.

McLean, J. W., Ma, Y. C., Martin, D. U., Saffman, P. G. & Yuen, H. C. 1981 Three-dimensional instability of finite-amplitude water waves. *Phys. Rev. Lett.* **46**, 817–820.

Moore, D. W. & Saffman, P. G. 1975 The instability of a straight vortex filament in a strain field. *Proc. R. Soc. Lond.* A **346**, 413–425.

Moser, J. K. 1958 New aspects in the theory of stability of Hamiltonian systems. *Communs pure appl. Math.* **11**, 81–114.

Nayfeh, A. H. & Mook, D. T. 1979 *Nonlinear oscillations*. New York and London: Wiley.

Saffman, P. G. 1985 The superharmonic instability of finite amplitude water waves. *J. Fluid Mech.* **159**, 169–174.

Stokes, G. G. 1847 On the theory of oscillatory waves. *Trans. Camb. phil. Soc.* **8**, 441–473.

Sturrock, P. A. 1958 Kinematics of growing waves. *Phys. Rev.* **112**, 1488–1503.

Tanaka, M. 1983 The stability of steep gravity waves. *J. phys. Soc. Japan* **52**, 3047–3055.

Tsai, C.-Y. & Widnall, S. E. 1976 The stability of short waves on a straight vortex filament in a weak externally imposed strain field. *J. Fluid Mech.* **73**, 721–733.

Weierstrass, K. 1858 Über ein die homogenen Funktionen zweiten Grades betroffendes Theorem, nebst Anwendung desselben auf die Theorie der kleinen Schwingungen. In *Weierstrass: mathematische Werke*, vol. 1, pp. 233–246. (Berlin, 1894; reprinted in 1967 by Olms & Johnson, Hildesheim and New York).

Whitham, G. B. 1974 *Linear and nonlinear waves*. New York and London: Wiley.

Williamson, J. 1936 On the algebraic problem concerning the normal forms of linear dynamical systems. *Am. J. Math.* **58**, 141–163.

Wintner, A. 1935 On the linear conservative dynamical systems. *Ann. Mat. pura appl.* **13**, 105–112.

Yakubovich, V. A. & Starzhinskii, V. M. 1975 *Linear differential equations with periodic coefficients*, vols. 1 and 2. London and New York: Wiley.

Zakharov, V. E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. appl. Mech. tech. Phys.* **2**, 190–194.