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Ray paths and caustics on a slightly oblate ellipsoid

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We investigate the ray paths from a point-source S on a slightly oblate ellipsoidal shell. The caustics are found to form a 4-star, i.e. a regular, 4-cusped hypocycloid, centred on the point antipodal to S. The length-scale of the 4-star varies as $\epsilon \cos^2 \lambda$, where ϵ is the eccentricity and λ is the latitude of the antipodal point.

1. INTRODUCTION

On 20 January 1988 Walter Munk gave a seminar at The Institute of Geophysics and Planetary Physics, UCSD, on the subject of long-range sound propagation in the ocean. In the course of his seminar he posed the following problem.

Given a source of sound at some point S in a uniform, thin, perfectly spherical shell of fluid. All the sound rays from this source will of course converge in a focus at the opposite point S′ on the sphere. But now suppose that the shell, instead of being perfectly spherical, is a slightly oblate ellipsoid of revolution having a small ellipticity ω, say. What then is the pattern of rays in the neighbourhood of the antipodal point on the ellipsoid?

Munk's problem is one in pure ray optics. Apart from the question of how far the situation compares with the real ocean or ionosphere, or to the Earth's crust, the problem itself has a general interest. Although the ray paths are equivalent to geodesics in the surface of an ellipsoid – a problem treated in classical works on differential geometry (see, for example, Forsythe 1920) – what is sought here is the *envelope* of the rays from a given point, a problem that does not appear to have been studied in the literature. The application of conventional techniques such as in Forsythe (1920), §§93–98, would lead to somewhat intractable expressions for the ray caustics. On the other hand we shall show in the following that when the ellipticity is small, quite elementary arguments lead us to a simple and striking result.

2. LENGTH OF THE RAY PATHS

Let P be some point on the ellipsoid in the neighbourhood of the point S′ antipodal to the source S (see figures 1 and 2). We assume there is a ray path (on the ellipsoid) between S and P. If ST denotes the tangent to this ray at the source S, there will be a plane p containing both ST and the centre O of the ellipsoid. The plane p will intersect the ellipsoid in an ellipse e passing through both S and S′. Now the ray path will lie nearly parallel to e, that is to say the directions of the

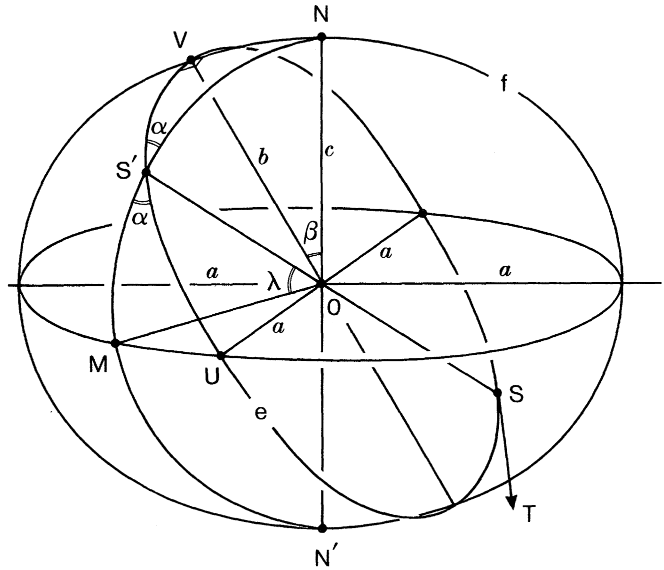

FIGURE 1. Coordinates and notation.

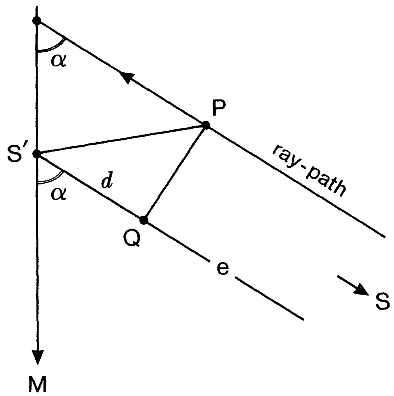

FIGURE 2. Neighbourhood of the antipode S′.

two curves will differ by an angle of order ϵ at most, throughout their length. Their separation also will be of order ϵ. Therefore the length of the ray-path SP will differ by $O(\epsilon^2)$ from the length of the arc SQ of the ellipse e, where Q is the foot of the perpendicular drawn from P to e (see figure 2).

3. Neighbourhood of the antipode

Consider points P on the ellipsoid within a distance of order ϵ from the antipode S′ (figure 2). To order ϵ these points may be regarded as lying in a plane, the tangent plane to the ellipsoid at S′. The trajectory SP will not generally pass through S′. However, it will be parallel (to lowest order) to the ellipse e which shares the same tangent as ST as S. Thus SP and SQS′ will make approximately equal angles α with the meridian S′M through S′.

The line PQ normal to e will be tangent to the wave-front through P. At the same time, the distances SP (along the ray trajectory) and SQ (along the ellipse) are equal, to order ϵ. We call this distance s.

From figure 2 we see that the distance d from Q to S′ satisfies

$$d+s = C(\alpha), \tag{3.1}$$

where C denotes the half-circumference of the ellipse e. Clearly C is a function of both α and of the latitude λ of S′. This function we shall now calculate.

4. Calculation of $C(\alpha)$

In figure 1, NN′ is the axis of symmetry of the ellipsoid, and a and c denote the lengths of the equatorial and polar semi-axes; λ denotes the latitude of the antipode S′. The ellipse e passes through S and S′ and makes an angle α with the meridian NS′. The major semi-axis of e is clearly the line OU in which the plane of e intersects the equatorial plane; OU has length a. The minor axis of e is the line OV in which the ellipse meets the plane through O perpendicular to OM. Let this have length b, say.

To find b, note that in the spherical triangle NVS′, the angles NS′V and NVS′ are α and $\frac{1}{2}\pi$ respectively, while NS′ is $\frac{1}{2}\pi-\lambda$. So by the sine-rule the angle NOV is β where

$$\sin\beta = \cos\lambda\sin\alpha. \tag{4.1}$$

Now the plane ONV intersects the ellipsoid in an ellipse f whose semi-axis are a and c (see figure 3). Hence we have

$$(b^2/a^2)\sin^2\beta+(b^2/c^2)\cos^2\beta = 1 \tag{4.2}$$

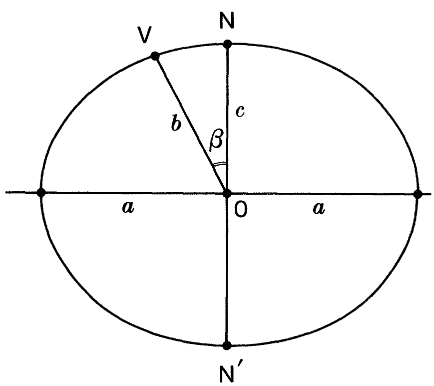

Figure 3. The plane ONV.

and so

$$b = a(\sin^2\beta+(a^2/c^2)\cos^2\beta)^{-\frac{1}{2}}. \tag{4.3}$$

Since

$$c = a(1-\epsilon), \tag{4.4}$$

where ϵ is the ellipticity of the spheroid this reduces to

$$b = a(1-\epsilon\cos^2\beta) \tag{4.5}$$

correct to order ϵ.

Now consider any ellipse (such as e) whose semi-axis are a and t where

$$b = a - \Delta a. \tag{4.6}$$

If r and θ denote radial coordinates in the plane of e, the radial distance r is given by

$$r = \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)^{-\frac{1}{2}} \tag{4.7}$$

and hence the half-circumference of the ellipse is

$$C = \int_0^\pi r\,\mathrm{d}\theta = a\int_0^\pi \left[1 - \left(\frac{a^2}{b^2} - 1\right)\sin^2\theta\right]^{-\frac{1}{2}} \mathrm{d}\theta \tag{4.8}$$

(a well-known result). When $\Delta a/a$ is small this reduces to

$$C = \int_0^\pi (a - \Delta a \sin^2\theta)\,\mathrm{d}\theta = \pi(a - \tfrac{1}{2}\Delta a). \tag{4.9}$$

But from (4.5) and (4.6)

$$\Delta a = \epsilon a \cos^2\beta. \tag{4.10}$$

Therefore from (4.1) we find altogether

$$C(\alpha) = \pi a - \tfrac{1}{2}\pi\epsilon a(1 - \cos^2\lambda \sin^2\alpha). \tag{4.11}$$

5. The wave fronts

At any given instant, the path length s in equation (3.1) is a constant, independent of α. So, on combining (3.1) and (4.11) we find

$$d = A - B\cos 2\alpha, \tag{5.1}$$

where A and B are constants independent of α given by

$$\left.\begin{aligned} A &= \pi a - \tfrac{1}{2}\pi\epsilon a(1 - \tfrac{1}{2}\cos^2\lambda) - s, \\ B &= \tfrac{1}{4}\pi\epsilon a \cos^2\lambda. \end{aligned}\right\} \tag{5.2}$$

From equation (5.1) we derive the whole solution.

In the special case $A = 0$, equation (5.1) reduces to

$$d = -B\cos 2\alpha \tag{5.3}$$

an equation which has a very simple interpretation, as follows.

Suppose that a circle Σ rolls, without slipping, inside another circle Σ' having four times its diameter B, as in figure 4. A point P on the circumference of the rolling circle lies originally at D, say. The arc lengths EP and ED are equal, by definition. So if μ denotes the angle ES′D, the angle EFP is 4μ. Hence the angle EGP is 2μ, and the perpendicular S′H from the centre S′ to the line PG is given by

$$d = B\sin 2\mu. \tag{5.4}$$

Now since E is the instantaneous centre of rotation of Σ, the tangent to the locus of P is normal to EP, and is, therefore, the line PG, since EG is a diameter of Σ.

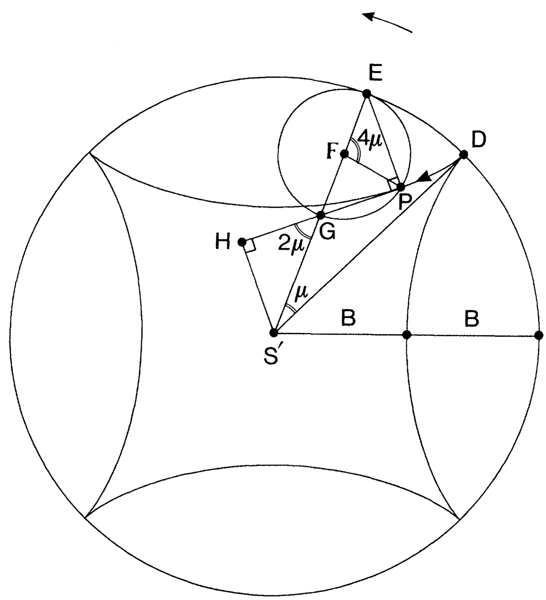

FIGURE 4. The critical wave-front ($A = 0$).

So we see that the perpendicular from S′ to the tangent at P satisfies the same equation as (5.3), provided we identify μ with $(\alpha - \frac{3}{4}\pi)$.

We find then that in the case $A = 0$, the wave-front is a four-cusped hypocycloid, that is, the locus of a point on a circle which rolls inside another of four times it diameter. We shall call this a 4-star.

The above result can also be derived analytically as follows.

The wave front is the envelope of the lines PQ as the angle α varies. To find this, take rectangular coordinates (ξ, η) in the tangent plane (figure 2) with the origin at S′ and the ξ-axis corresponding to $\alpha = 0$. The equation of the line PQ is

$$\xi \cos\alpha + \eta \sin\alpha = d = A - B\cos 2\alpha \tag{5.5}$$

by (5.1). A point on the envelope, which is the intersection of two adjacent lines, is found by differentiating (5.3) with respect to the parameter α. This gives

$$-\xi \sin\alpha + \eta \cos\alpha = 2B \sin 2\alpha. \tag{5.6}$$

Equations (5.3) and (5.4) may be solved for the coordinates (ξ, η) of P. Taking first the case $A = 0$ we find

$$\left.\begin{aligned} \xi &= -B(\cos 2\alpha \cos\alpha + 2\sin 2\alpha \sin\alpha), \\ \eta &= -B(\cos 2\alpha \sin\alpha - 2\sin 2\alpha \cos\alpha), \end{aligned}\right\} \tag{5.7}$$

or more simply

$$\left.\begin{aligned} \xi &= \tfrac{1}{2}B(\cos 3\alpha - 3\cos\alpha), \\ \eta &= \tfrac{1}{2}B(\sin 3\alpha + 3\sin\alpha). \end{aligned}\right\} \tag{5.8}$$

Taking now the general case when $A \neq 0$, we see from (5.3) and (5.4) that we must add to ξ and η the terms $A\cos\alpha$ and $A\sin\alpha$ respectively. The effect of this is that the wave-front no longer has 4-fold symmetry. When $|A| < 3B$ it has four cusps; when $|A| > 3B$ it has no cusps. Some examples of the wave fronts are shown in figure 5.

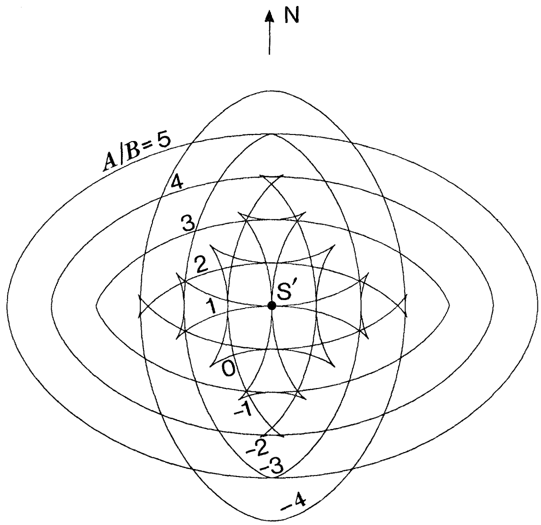

FIGURE 5. The pattern of ray-fronts when $A \neq 0$.

6. The ray caustics

The caustics are the envelope of the ray paths, i.e. the envelope of the lines SP in figure 2. These lines are normals to all the wave fronts. So to calculate the envelope we may choose any particular wave-front. Let us choose the critical front corresponding to $A = 0$.

Now the ray paths correspond to the lines EP in figure 4, which are normals to the wave-front at the point P. We note that the angle which the tangent PG makes with the fixed direction S′D is $-\mu$. Thus, as the point E travels around the circle Σ' with angular velocity $\dot{\mu}$, say, the tangent PG, and hence the normal PE rotate in the opposite direction, with angular velocity $-\dot{\mu}$. Our problem, then is to find the envelope of a line (such as EP) which passes through a point rotating around a circle with constant angular velocity while itself rotating about this point with the opposite angular velocity.

In fact, our problem is already solved. For the point G also rotates around the circle Σ'' centre S′ and radius B with constant angular velocity $\dot{\mu}$, while the line GP rotates in the opposite direction with angular velocity $-\dot{\mu}$. We know the envelope of GP: the 4-star shown in figure 4. By similarity, it follows that the envelope of the ray paths PE is also a 4-star, but of twice the size, that is to say of outer radius $4B$. As the point E approaches D the normals become transverse to the radius vector S′D. Hence the orientation of the new 4-star is rotated $\frac{1}{4}\pi$ relative to the first one.

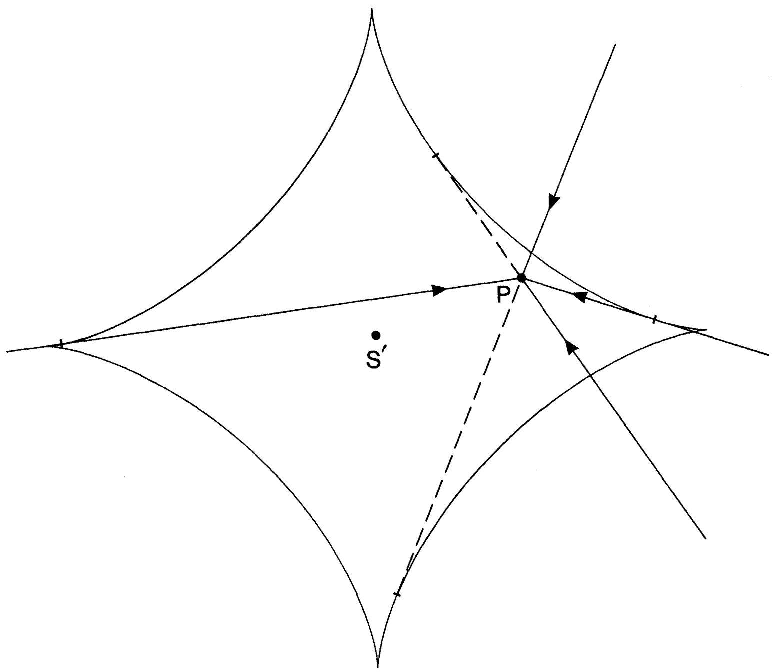

FIGURE 6. The form of the caustics near S′.

We have then the following result: the ray caustics form an asteroid or 4-star, with centre at S′ and outer radius

$$4B = \pi \epsilon a \cos^2 \lambda \tag{6.1}$$

the four cusps being oriented due north, south, east and west of S′. The inner radius is $2B$. The total length of the curve is $6 \times 2^{\frac{1}{2}}\pi B$ and it encloses an area $|12\pi B^2|$.

7. The number of ray paths

A 4-star is a curve of the fourth class, that is to say from a general point P inside the curve, 4 tangents to the curve may be drawn. Each of these corresponds to a possible ray path from S to P. If P lies on the curve, two of these rays coincide, except that if P is at one of the cusps all four rays are coincident. If P lies outside the curve, there are only two rays from S to P.

When P lies close to S on the ellipsoid there are of course two rays SP, one relatively short and the other encircling the ellipsoid almost once.

In this discussion we have included only those rays whose path length is less than the circumference of the ellipsoid. If longer rays are considered it is clear that near the source S there is another caustic, in the form of a 4-star having twice the size of the first caustic 4-star at S′. Again, from rays circling the ellipsoid one-and-a-half times, we find another caustic near S′ with three times the original size, i.e. having outer radius $12B$; and so on.

8. Applications

Some caution should be exercised in applying the above results to any situation where the medium of the shell is not strictly uniform. Nevertheless it may be worth noting that on the surface of the Earth, a quadrant NM is about 10^4 km, and the ellipticity is about 1:300. Hence in equation (5.2) we have

$$B = 17\cos^2\lambda \text{ km}, \tag{8.1}$$

where λ is the latitude. At latitude 45°, for instance, $B = 8.3$ km, and the outer radius of the caustic is 33 km.

Equation (8.1) shows that the size of the caustic is greatest at the Equator, and that it tends to zero strongly at the north and south poles, where the length scale varies as the square of the polar distance.

It would make an attractive experiment if an ellipsoidal shell could be constructed out of glass, or some other translucent material. A light source introduced into the shell might then produce a visible 4-star caustic near the antipole, surrounded by fainter 4-stars of size 3, 5, 7, etc., times the first. If the light source were moved around, the dimensions of the pattern would vary accordingly to the square of the cosine of the latitude.

References

Forsythe, A. R. 1920 *Lectures on the differential geometry of curves and surfaces.* (287 pages.) Cambridge University Press.

Lemaire, J. 1929 *Hypocycloides et epicyloides.* (295 pages.) Paris: Libraire Vuibert.